Study of torsional vibrations in an initially stressed composite poroelastic cylinders

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Abstract. This paper investigates torsional vibrations in an initially stressed composite poroelastic cylinder in the framework of Biot's theory of wave propagation in poroelastic solids. Poroelastic composite cylinder consists of two concentric cylindrical layers made of different poroelastic materials. The governing equations are formulated from the Biot's incremental deformation theory. The non-dimensional frequency is computed as a function of ratio of thickness to wavelength. The limiting cases of a poroelastic solid cylinder and poroelastic hollow cylinder are discussed. The results are presented graphically for two poroelastic composite cylinders and then compared with the published results.

1. Introduction

The studies of wave propagation in poroelastic solids have many applications in various fields such as Seismology, Soil-mechanics, Bio-mechanics, Civil engineering, and Mechanical engineering. From the real time experiences, one may find that buildings, bridges and some manmade structures consist two or more material that could be combined to take advantage of the good characteristics of each of the materials. In the frame work of Biot's theory [1], the effect of boundaries on torsional vibrations in a poroelastic composite cylinder is reported in several papers [2-3]. Axially symmetric vibrations of composite poroelastic cylinder are investigated by Malla Reddy and Tajuddin [4]. The study of torsional vibrations of an elastic solid is important in several applications such as transmission of power through shafts with flanges at the ends as integral parts of the shafts. The other use of torsional vibrations is measurement of the shear modulus of elastic solids. The basic literature on the propagation of elastic waves is given by Ewing et.al. [5]. Torsion waves of an elastic composite infinite circular solid rod of two different materials are studied by Armenkas [6]. Investigation of axially symmetric wave propagation in a two layered elastic cylinder is made by Whitter and Jones [7]. Torsional vibrations in a poroelastic cylinder are reported in several papers [8-10]. The problem related to pre-stressed elastic solids has been a subject of continued interest due to its importance in the said areas. A detailed discussion about the theory of elastic medium under initial stress is given in Biot's incremental theory [11]. Torsional wave propagation in an initially stressed elastic cylinder is studied by Dey and Dutta [12]. In the paper [12], governing equations are formulated for the Biot's incremental deformation theory. In the said paper, the velocities of torsional wave propagation due to the presence of initial stress are calculated for different extension ratios. Selim [13] investigated torsional wave propagation in dissipative elastic solid cylinder subjected to initial stress. In the paper [13], the effect of damping on the propagation of torsional waves

in incompressible cylinder of infinite length is discussed and it is proved that the damping of the medium has strong effect on the propagation of torsional waves. Propagation of Rayleigh waves in an initially stressed incompressible half space under a rigid layer are studied by Dey et.al. [14]. It has been shown that Rayleigh waves cannot propagate in an isotropic medium without tensile initial stress [14]. Plane strain deformation of an initially stressed orthotropic elastic medium is studied using eigen value approach [15]. Surface wave propagation in an initially stressed transversely isotropic thermoelastic solid is studied by Baljeet Singh and Renu [16]. Torsional surface waves in an initially stressed anisotropic poroelastic layer over a semi-infinite heterogeneous half space with linearly varying rigidity and density due to irregularity at the interface is studied in [17]. The propagation of plane waves is investigated in a general anisotropic elastic medium in the presence of initial stress [18]. Love wave propagation in a porous rigid layer lying over an initially stressed half space is studied in [19]. In the paper [19], the propagation of Love waves in fluid saturated, anisotropic, porous rigid layer over a pre stressed, non-homogeneous, elastic half space is investigated. In all the said papers, initial stress is not considered in composite poroelastic solid cylinder. The presence of initial stresses in solid materials can have a substantial effect on their subsequent response to applied loads that is very different from the corresponding response in the absence of initial stresses. In the present paper, torsional vibrations in the composite poroelastic solid cylinder of infinite extent consists of an inner solid circular cylinder of one material bounded by and bounded to a circular core made of another poroelastic material under initial stress are investigated. The non-dimensional frequency is computed as a function of ratio of thickness to wavelength for thin and thick coating.

The rest of the paper is organized as follows. In section 2, governing equations and solution of the problem are discussed. Boundary conditions and frequency equation are presented in section 3. Particular cases are given in section 4. Numerical results are discussed in section 5. Finally, conclusion is given in section 6.

2. Governing equations and solution of the problem

Let (r, θ, z) be the cylindrical polar coordinates. Consider a composite concentric isotropic infinite poroelastic solid cylinder with inner and outer radii r_1 and r_2, respectively, subjected to initial stress. The $z-$axis coincides with the axis of the cylinder. The substrate is a circular solid cylinder with radius r_1 and the coating is a thick-walled hollow cylinder having thickness $h(= r_2 - r_1)$. The equations of motion under initial compression stress $\sigma_{zz} = -p$ along the axis of the cylinder are [11]:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} - p\frac{\partial \omega_\theta}{\partial z} = \frac{\partial^2}{\partial t^2}(\rho_{11}u_1 + \rho_{12}U_1) + b\frac{\partial}{\partial t}(u_1 - U_1),$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + p\frac{\partial \omega_r}{\partial z} = \frac{\partial^2}{\partial t^2}(\rho_{11}u_2 + \rho_{12}U_2) + b\frac{\partial}{\partial t}(u_2 - U_2),$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rz}}{r} - p(\frac{\partial \omega_\theta}{\partial r} - \frac{\partial \omega_r}{\partial \theta}) = \frac{\partial^2}{\partial t^2}(\rho_{11}u_3 + \rho_{12}U_3) + b\frac{\partial}{\partial t}(u_3 - U_3),$$

$$Q\frac{\partial e}{\partial r} + R\frac{\partial \varepsilon}{\partial r} = \frac{\partial^2}{\partial t^2}(\rho_{12}u_1 + \rho_{22}U_1) - b\frac{\partial}{\partial t}(u_1 - U_1),$$

$$\frac{Q}{r}\frac{\partial e}{\partial r} + \frac{R}{r}\frac{\partial \varepsilon}{\partial r} = \frac{\partial^2}{\partial t^2}(\rho_{12}u_2 + \rho_{22}U_2) - b\frac{\partial}{\partial t}(u_2 - U_2),$$

$$Q\frac{\partial e}{\partial z} + R\frac{\partial \varepsilon}{\partial z} = \frac{\partial^2}{\partial t^2}(\rho_{12}u_3 + \rho_{22}U_3) - b\frac{\partial}{\partial t}(u_3 - U_3),$$

$$(1)$$

where (u_1, u_2, u_3) and (U_1, U_2, U_3) are the displacement components of solid and fluid respectively, $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}, \sigma_{r\theta}$, and $\sigma_{\theta z}$ are the stress components, ρ_{ij} are mass coefficients,

b is the dissipative coefficient and ω_r, ω_θ are rotational components given by,

$$\omega_r = \tfrac{1}{2}\left(\tfrac{\partial u_3}{\partial \theta} - \tfrac{\partial u_2}{\partial z}\right),$$

$$\omega_\theta = \tfrac{1}{2}\left(\tfrac{\partial u_1}{\partial z} - \tfrac{\partial u_3}{\partial r}\right). \tag{2}$$

The stress component σ_{ij} and fluid pressure s [1] are

$$\sigma_{ij} = 2Ne_{ij} + (Ae + Q\varepsilon)\delta_{ij}, (i,j = 1,2,3),$$

$$s = Qe + R\varepsilon. \tag{3}$$

In the above, e_{ij} are strain components given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), (i,j = 1,2,3).$$

In the case of torsional vibrations, the equations of motion is reduced to the following equations:

$$\tfrac{\partial \sigma_{r\theta}}{\partial r} + \tfrac{\partial \sigma_{\theta z}}{\partial z} + \tfrac{2\sigma_{r\theta}}{r} - \tfrac{p}{2}\tfrac{\partial^2 u_2}{\partial z^2} = \tfrac{\partial^2}{\partial t^2}(\rho_{12}u_2 + \rho_{22}U_2) + b\tfrac{\partial}{\partial t}(u_2 - U_2),$$

$$0 = \tfrac{\partial^2}{\partial t^2}(\rho_{12}u_2 + \rho_{22}U_2) - b\tfrac{\partial}{\partial t}(u_2 - U_2). \tag{4}$$

In the case of harmonic wave solution $e^{ik(z-ct)}$, the solution for circumferential displacements in the cylinder become,

$$_ju_2(r,z,t) = {}_jf(r)e^{ik(z-ct)}, \ _jU_2(r,z,t) = {}_jF(r)e^{ik(z-ct)}, \ j = 1,2. \tag{5}$$

In Eq. (5), the quantities with subscripts 1 refer to the substrate while 2 refers to the coating. c is the torsional wave velocity, k is wavenumber, i is complex unity, and t is time. Substituting Eq. (5) and Eq. (3) in Eq. (4), one obtains

$$N\tfrac{d^2\,_jf}{dr^2} - (N - \tfrac{p}{2})k^2\,_jf + \tfrac{N}{r}\tfrac{d\,_jf}{dr} - \tfrac{N}{r^2}\,_jf = -kc(_jf\rho_{11} + {}_jF\rho_{12}) - bikc(_jf - {}_jF),$$

$$0 = -k^2c^2(_jf\rho_{12} + {}_jF\rho_{12}) + bikc(_jf - {}_jF). \tag{6}$$

Solutions of Eq. (6) are

$$_2f(r) = C_1 J_1(_2qr) + C_2 Y_1(_2qr),$$

$$_1f(r) = C_3 J_1(_1qr). \tag{7}$$

In Eq. (7),

$$_jq^2 = \frac{1}{_jN}(k^2c^2\,_j\rho_{11} + k^2c^2\,_j\rho_{12}(\tfrac{-_j\rho_{12}kc + ib}{-_j\rho_{22}kc + ib}) + bikc(1 - (\tfrac{-_j\rho_{12}kc + ib}{-_j\rho_{22}kc + ib})) - k^2(_jN - \tfrac{p}{2})), j = 1,2.$$

The non-zero stresses both for the substrate and the coating are

$$_2\sigma_{r\theta} = {}_2N\,_2q(C_1 J_2(_2qr) + C_2 Y_2(_2qr))e^{ik(z-ct)},$$

$$_1\sigma_{r\theta} = {}_1N\,_1q(C_3 J_2(_1qr)e^{ik(z-ct)}. \tag{8}$$

3. Boundary conditions and frequency equation

The boundary conditions for stress free outer surface $r = r_2$ and perfect bonding between the substrate and the coating at the interface $r = r_1$ are

$$_2\sigma_{r\theta} = 0 \quad at \quad r = r_2$$

$$_2\sigma_{r\theta} = \,_1\sigma_{r\theta} \quad at \quad r = r_1, \tag{9}$$

$$_2u_2 = \,_1u_2 \quad at \quad r = r_1.$$

Substitution of Eqs. (7), (8) and (5) in Eq. (9) gives three homogeneous equations in three unknowns $C_1, C_2,$ and C_3. A non-trivial solution can be obtained if the determinant of the coefficient matrix vanishes. Accordingly, one obtains the following frequency equation:

$$\mid A_{ij} \mid = 0, \quad (i,j = 1,2,3) \tag{10}$$

where,

$$A_{11} = \frac{-2\,_2N}{r_2}J_1(_2qr_2) + \,_2N\,_2qJ_0(_2qr_2), \quad A_{12} = \frac{-2\,_2N}{r_2}Y_1(_2qr_2) + \,_2N\,_2qY_0(_2qr_2),$$

$$A_{13} = 0, \quad A_{23} = \frac{-2\,_1N}{r_1}J_1(_1qr_1) - \,_1N\,_1qJ_0(_1qr_1), \tag{11}$$

$$A_{31} = J_1(_2qr_1), \quad A_{32} = Y_1(_2qr_1), \quad A_{33} = -J_1(_1qr_1),$$

$$A_{21}, A_{22} \text{ are similar expressions as } A_{11}, A_{12} \text{ with } r_2 \text{ replaced by } r_1.$$

4. Particular cases

The composite poroelastic cylinder will be reduced to the poroelastic solid cylinder and the poroelastic hollow cylinder under some special cases.

4.1. Poroelastic solid cylinder

When the poroelastic parameters of the substrate and coating are of same material, then the composite cylinder will be reduced to solid cylinder of one material. Setting $_2N = \,_1N = N$, $_2q = \,_1q = q$. Then the frequency equation Eq. (10) reduces to

$$J_2(qr_2) = 0, \tag{12}$$

which is the frequency equation of torsional vibrations in poroelastic solid cylinder similar to that of [10].

4.2. Poroelastic hollow cylinder

When the material constants of a substrate vanish, the composite poroelastic cylinder will become a hollow poroelastic cylinder. Setting $_1N = 0$, $_2N = N$, $_2q = q$, $_1q = \,_2q$. at the interface $r = r_1$, then the frequency equation Eq. (10) reduces to

$$J_2(qr_2)Y_2(qr_1) - J_2(qr_1)Y_2(qr_2) = 0, \tag{13}$$

which is the frequency equation of torsional vibrations in poroelastic hollow cylinder similar to that of [10].

5. Numerical results

Due to presence of dissipative nature of the solids, waves are attenuated. Attenuation presents some difficulty in the definition of phase velocity. If the dissipation coefficient is non-zero, the wavenumber, densities are complex. Consequently velocities of dilatational waves and shear waves are complex valued. Finally, frequency equations will be complex valued and implicit. Therefore, b is made to be zero so that frequency equation will be real valued and the roots will be obtained easily that explicitly give phase velocity. Even if b is zero, problem would be poroelastic in nature as the coefficients A, N, Q, R would not vanish. The frequency equation Eq. (10) is investigated by introducing the non-dimensional quantities given below:

$$d_1 = \frac{2\rho_{11}}{1\rho}, \quad d_2 = \frac{2\rho_{12}}{1\rho}, \quad d_3 = \frac{2\rho_{22}}{1\rho}, \quad a_4 = \frac{2N}{1H},$$

$$g_1 = \frac{1\rho_{11}}{1\rho}, \quad g_2 = \frac{1\rho_{12}}{1\rho}, \quad g_3 = \frac{1\rho_{22}}{1\rho}, \quad b_4 = \frac{1N}{1H},$$

$$_1\rho = {}_1\rho_{11} + 2\,{}_1\rho_{12} + {}_1\rho_{22}, \quad {}_1H = {}_1 P + 2\,{}_1Q + {}_1R,$$

$$\delta = \frac{h}{L}, \quad c = \frac{\omega}{k}, \quad \Omega = \frac{\omega h}{1c_0} \quad g = \frac{r_2}{r_1}.$$

(14)

In the Eq.(14), h is the thickness of the cylinder and L is wavelength, ω is the frequency, ${}_1c_0$ is reference velocities $({}_1c_0^2 = {}_1N_1\rho^{-1}), \Omega$ is the non-dimensional frequency. For the numerical process, two types of composite poroelastic cylinders are considered, namely composite cylinder-I and composite cylinder-II. Composite cylinder-I consists of the substrate made of sandstone saturated with water [20] and the coating made of sandstone saturated with kerosene [21]; while in composite cylinder-II, the substrate is made of sandstone saturated with kerosene and the coating with sandstone saturated with water. The physical parameters of the said materials pertaining to Eq. (10) are given in Table 1.

Table 1. Material parameters

Material parameter	Composite cylinder-I	Composite cylinder-II
d_1	0.887	0.891
d_2	-0.001	0
d_3	0.099	0.125
a_4	0.123	0.780
g_1	0.887	0.901
g_2	0	-0.001
g_3	0.123	0.101
b_4	0.412	0.780

Employing the non-dimensional quantities in Eq.(10), one obtains an implicit relation between non-dimensional frequency (Ω) ratio of thickness to wavelength, ratio of radii and initial stress. The numerical values are depicted in figures 1-2. Variation of non-dimensional frequency as a function of ratio of thickness to wavelength for composite cylinder-I and composite cylinder-II is computed in the cases of thin coating and thick coating when initial stress is one. The numerical results are compared with the published results (when initial stress is zero, Ref. [3]) in the case of thin coating ($g = 1.01$) and thick coating ($g = 4$) for composite cylinder-I and composite cylinder-II. From figures 1 and 2, it is observed that the frequency values of composite cylinder II, in general, greater than that of composite cylinder-I. Also, it is observed that the frequency values lower when initial stress is present in the solid. From the numerical results, one can infer that the frequency depends on the presence of initial stress.
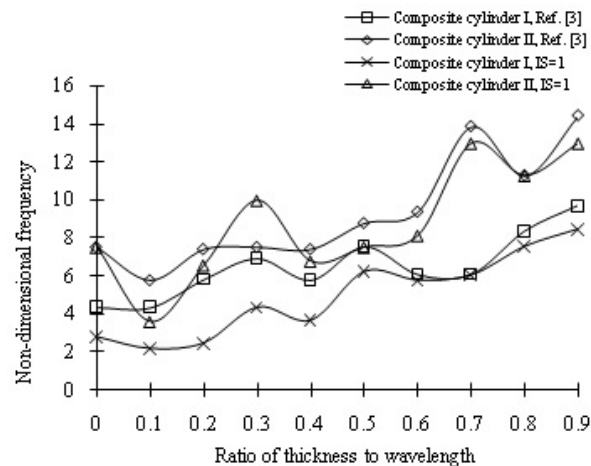
Figure 1. Variation of non-dimensional frequency with ratio of thickness to wave length in the case of thin coating ($g = 1.01$).
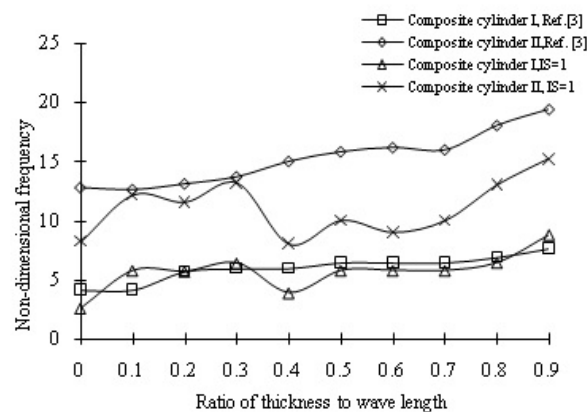

Figure 2. Variation of non-dimensional frequency with ratio of thickness to wave length in the case of thick coating ($g = 4$).

6. Conclusion

Torsional vibrations in composite poroelastic solid cylinder in the presence of initial stress are investigated. The governing equations are formulated from Biot's incremental deformation theory. Non-dimensional frequency against ratio of thickness to wavelength is computed for two composite poroelastic cylinders. From the paper, it is clear that the frequency of waves depends on the initial stress present in the solid.

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