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On the Separation Theorem of Stochastic Systems in the Case Of Continuous Observation Channels with Memory

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Abstract. The article proves the separation theorem for optimal control of stochastic systems in the case when an observed continuous-time process possess memory of arbitrary ration relating to a state vector.

1. Introduction

The separation theorem [1] is the basis for a theory of optimal control by incompletely observable stochastic systems. Being fundamental theoretical results, it ensured solving of a number of important practical problems in a wide range of areas [2, 3]. With the results [4, 5], we provided a generalized separation theorem in the case when observations have memory of arbitrary ratio, i. e. they depend not only on current values but also on an arbitrary number of previous values of the system state vector.

Used notations: $P\{\cdot\}$ is event probability; $E\{\cdot\}$ is the mathematical expectation; normal (Gaussian) density with parameters a and b by $N\ \{a;\ b\}$; $|\cdot|$ and $tr[\cdot]$ are determinant and trace of matrix; D^{-1} is the inversion matrix of D; D^T denotes transpose of a matrix or a vector; $D>0$ $(D\geq 0)$ is positive (non-negative) definite matrix.

2. The problem statement

On a certain probability space $(\Omega, F, \mathbf{F}=(F_t)_{t\geq 0}, \mathbf{P})$ [6] the unobservable n-dimensional process x_t (state vector of system) and the observable l- dimensional process z_t with continuous time are determined by the stochastic differential equations (Ito's differential rule)

$$dx_t = f(t, x_t, u_t)dt + \Phi_1(t, x_t)dw_t, \quad t\in[0, \mathrm{T}], \tag{1}$$

$$dz_t = h(t, x_t, x_{\tau_1}, \cdots, x_{\tau_N}, z)dt + \Phi_2(t)dv_t, \tag{2}$$

where u_t is $r-$dimensional control vector, $0<\tau_N<\tau_{N-1}<\cdots<\tau_1<t_m\leq t$, and $\tau_k=const$, $k=\overline{1;N}$. It is assumed: 1) w_t and v_t are r_1-dimensional and r_2-dimensional standard Wiener processes; 2) x_0, w_t, v_t are jointly independent; 3) $f(\cdot)$, $h(\cdot)$, $\Phi_1(\cdot)$, $\Phi_2(\cdot)$ are continuous functions for all arguments; 4) $Q(\cdot)=\Phi_1(\cdot)\Phi_1^T(\cdot)>0$, $R(\cdot)=\Phi_2(\cdot)\Phi_2^T(\cdot)>0$; 5) original density is set $p_0(x)=\partial P\{x_0\leq x\}/\partial x$.

Task: to find control u_t^0 that provides optimum condition, on the class of $F_t^{z,\eta}$-measurable functional, where $u_t=u_t\left[z_0^t\right]$, $z_0^t=\{z_s;0\leq s\leq t\}$

$$J=E\left\{b(\mathrm{T},x_\mathrm{T})+\int_\tau^\mathrm{T}\Lambda(s,x_s,u_s)ds\middle|p_0(x)\right\}\to\min_{\{u_{t_0}^\mathrm{T}\}},\tag{3}$$

where $\tau>\tau_1$, $u_{t_0}^T=\{u_s;\tau\leq s\leq \mathrm{T}\}$, $b(\cdot)\geq 0$, $\Lambda(\cdot)>0$.

To solve the set task, let us apply the method of sufficient coordinates [7], assuming that there is F_t^z - measurable process $\lambda_t\left[z_0^t\right]$ that fully characterizes posterior density

$$p_t(x)=\partial P\left\{x_t\leq x\middle|z_0^t\right\}/\partial x\tag{4}$$

of x_t system state vector, on the one hand. On the other hand, it can be found by means of $p_t(x)$.

Remark 1. We consider that the optimal control dates from moment $t_0>\tau_1$. An arbitrary F_t^z-measurable process is used as u_t on the interval $t\in[0,\tau]$.

3. Preliminary results

In accordance with the sufficient coordinates method, we introduce Bellman function

$$S(t,\lambda)=\min_{\{u_\tau^T\}}E\left\{b(\mathrm{T},x_\mathrm{T})+\int_t^\mathrm{T}\Lambda(t,x_{t'},u_{t'})dt'\middle|\lambda_t=\lambda\right\}.\tag{5}$$

Theorem 1. Let the following conditions be met:

1) process λ_t is Markov diffusion process with characteristics

$$a(t,\lambda)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}E\{\Delta\lambda_t|\lambda_t=\lambda\},\tag{6}$$

$$D(t,\lambda)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}E\left\{[\Delta\lambda_t][\Delta\lambda_t]^T\middle|\lambda_t=\lambda\right\},\tag{7}$$

where $\Delta\lambda_t=\lambda_{t+\Delta t}-\lambda_t$.

2) Let process $\{x_t;\lambda_t\}$ be Markov process with transition probability density

$$p\left(t',x',\lambda'\middle|t,x,\lambda\right)=\partial^2P\left\{x_{t'}\leq x',\lambda_{t'}\leq\lambda'\middle|x_t=x,\lambda_t=\lambda\right\}/\partial x'\partial\lambda'.\tag{8}$$

Then, Bellman equation for $S(t,\lambda)$ has the form

$$\min_{\{u\}}\left\{\frac{\partial S(t,\lambda)}{\partial t}+L_{t,\lambda}^*[S(t,\lambda)]+E\{\Lambda(t,x_t,u_t)|\lambda_t=\lambda\}\right\}=0,\tag{9}$$

$$S(t,\lambda)|_{t=\mathrm{T}}=E\{b(\mathrm{T},x_\mathrm{T})|\lambda_\mathrm{T}=\lambda\},\tag{10}$$

where $L^*_{t,\lambda}[\cdot]$ denotes inverse Kolmogorov operator that corresponds to process λ_t [6], i.e.

$$L^*_{t,\lambda}[S(t,\lambda)] = a^T(t,\lambda)\frac{\partial S(t,\lambda)}{\partial \lambda} + \frac{1}{2}tr\left[D(t,\lambda)\frac{\partial^2 S(t,\lambda)}{\partial \lambda^2}\right], \tag{11}$$

and the smallest value of performance criterion has the form $J^0 = S(\tau,\lambda)$.

Proof. Let

$$p_t(x|\lambda) = \partial P\{x_t \le x|\lambda_t = \lambda\}/\partial x. \tag{12}$$

Then, by expanding operator $E\{\cdot\}$ into (), and considering condition2), we obtain

$$S(t,\lambda) =$$
$$= \min_{\{u_t^T\}}\left\{\int p_t(x|\lambda)\left\{\int_t^{\mathrm{T}} \Lambda(t',x',u)p(t',x',\lambda'|t,x,\lambda)dx'd\lambda'dt' + \int b(\mathrm{T},x')p(t',x',\lambda'|t,x,\lambda)dx'd\lambda'\right\}dx\right\}, \tag{13}$$

where $u_t^T = \{u_s; t \le s \le T\}$.

Lemma 1. Accurate to $o(\Delta t)$, the function $S(t,\lambda)$ satisfies Δt - recurrence equation

$$S(t,\lambda) = \min_{\{u\}}\left\{\int S(t+\Delta t,\lambda'')p(t+\Delta t,\lambda''|t,\lambda)d\lambda'' + (\Delta t)\int \Lambda(t,x,u)p_t(x|\lambda)dx + o(\Delta t)\right\}, \tag{14}$$

where

$$p(t+\Delta t,\lambda''|t,\lambda) = \partial P\{\lambda_{t+\Delta t} \le \lambda''|\lambda_t = \lambda\}/\partial \lambda'' \tag{15}$$

is transition probability density of Markov process λ_t.

Proof. Expressing interval $[t,\mathrm{T}]$ as $[t,\mathrm{T}] = [t,t+\Delta t] + [t+\Delta t,\mathrm{T}]$, we obtain from (13)

$$S(t,\lambda) = \min_{\{u_t^{t+\Delta t}\}}\min_{\{u_{t+\Delta t}^T\}}\{S_1(t,t+\Delta t) + S_2(t+\Delta t,\mathrm{T})\}, \tag{16}$$

$$S_1(t,t+\Delta t) = \int p_t(x|\lambda)\left\{\int_t^{t+\Delta t} \Lambda(t',x',u)p(t',x',\lambda'|t,x,\lambda)dx'd\lambda'dt'\right\}dx, \tag{17}$$

$$S_2(t+\Delta t,\mathrm{T}) =$$
$$= \int p_t(x|\lambda)\left\{\int_{t+\Delta t}^{\mathrm{T}} \Lambda(t',x',u)p(t',x',\lambda'|t,x,\lambda)dx'd\lambda'dt' + \int b(\mathrm{T},x')p(t',x',\lambda'|t,x,\lambda)dx'd\lambda'\right\}dx, \tag{18}$$

So far as $p(t',x',\lambda'|t,x,\lambda) = \delta(x'-x,\lambda'-\lambda)$, when $\Delta t \downarrow 0$, it follows from (17), that

$$S_1(t,t+\Delta t) = (\Delta t)\int \Lambda(t,x,u)p_t(x|\lambda)dx + o(\Delta t). \tag{19}$$

Markov process transition probability density $\{x_t,\lambda_t\}$ (see condition 2) satisfies the Chapman-Kolmogorov equation [8]

$$p(t',x',\lambda'|t,x,\lambda) = \int p(t',x',\lambda'|t+\Delta t,x'',\lambda'')p(t+\Delta t,x'',\lambda''|t,x,\lambda)dx''d\lambda''. \tag{20}$$

With regard to (12), (15)

$$\int p(t+\Delta t,x'',\lambda''|t,x,\lambda)p_t(x|\lambda)dx = p_{t+\Delta t}(x''|\lambda'')p(t+\Delta t,\lambda''|t,\lambda), \tag{21}$$

Then, it follows from (18), (20), (21) that

$$S_2(t+\Delta t,\mathrm{T})=$$
$$=\int p\left(t+\Delta t,\lambda''|t,\lambda\right)\left\{\int p_{t+\Delta t}\left(x''|\lambda''\right)\left\{\int_{t+\Delta t}^{\mathrm{T}}\Lambda\left(t',x',u\right)p\left(t',x',\lambda'|t,x,\lambda\right)dx'd\lambda'dt'+\right.\right. \tag{22}$$
$$\left.\left.+\int b\left(\mathrm{T},x'\right)p\left(t',x',\lambda'|t+\Delta t,x'',\lambda''\right)dx'd\lambda'\right\}dx''\right\}d\lambda''.$$

With regard to (19), it follows from (16), that

$$S(t,\lambda)=\min_{\{u\}}\left\{\min_{\{u_{t+\Delta t}^{\mathrm{T}}\}}S_2(t+\Delta t,\mathrm{T})+(\Delta t)\int\Lambda(t,x,u)p_t(x|\lambda)dx+o(\Delta t)\right\}. \tag{23}$$

Substituting (22) in (23) with regard to (13) leads to (14). Lemma 1 is proved.
We expand

$$S\left(t+\Delta t,\lambda''\right)=S(t+\Delta t,\lambda)+\left(\frac{\partial S(t+\Delta t,\lambda)}{\partial\lambda}\right)^T\Delta\lambda+\frac{1}{2}\Delta\lambda^T\frac{\partial^2 S(t+\Delta t,\lambda)}{\partial\lambda^2}\Delta\lambda+o\left([\Delta\lambda]^2\right) \tag{24}$$

Then, with regard to (6), (7), (15) it follows, that

$$\int S(t+\Delta t,\lambda)p\left(t+\Delta t,\lambda''|t,\lambda\right)d\lambda''=S(t+\Delta t,\lambda), \tag{25}$$

$$\int\left(\frac{\partial S(t+\Delta t,\lambda)}{\partial\lambda}\right)^T\Delta\lambda p\left(t+\Delta t,\lambda''|t,\lambda\right)d\lambda''=\left(\frac{\partial S(t+\Delta t,\lambda)}{\partial\lambda}\right)^T E\{\Delta\lambda_t|\lambda_t=\lambda\}=$$
$$=(\Delta t)\int\left(\frac{\partial S(t+\Delta t,\lambda)}{\partial\lambda}\right)^T a(t|\lambda)+o(\Delta t), \tag{26}$$

$$\int\Delta\lambda^T\frac{\partial^2 S(t+\Delta t,\lambda)}{\partial\lambda^2}\Delta\lambda p\left(t+\Delta t,\lambda''|t,\lambda\right)d\lambda''=tr\left[\frac{\partial^2 S(t+\Delta t,\lambda)}{\partial\lambda^2}E\left\{[\Delta\lambda_t][\Delta\lambda_t]^T|\lambda(t)=\lambda\right\}\right]=$$
$$=(\Delta t)tr\left[\int\frac{\partial^2 S(t+\Delta t,\lambda)}{\partial\lambda^2}D(t|\lambda)\right]+o(\Delta t), \tag{27}$$

$$\int o\left([\Delta\lambda]^2\right)p\left(t+\Delta t,\lambda''|t,\lambda\right)d\lambda''=E\left\{o\left([\Delta\lambda]^2\right)\Big|\lambda_t=\lambda\right\}=o(\Delta t). \tag{28}$$

Substitution of (24) in (14) with regard to the latter formulae and (11) leads to the relation ($b^T g=g^T b$, $tr[BD]=tr[DB]$)

$$S(t,\lambda)=\min_u\left\{S(t+\Delta t,\lambda)+(\Delta t)L_{t,x}^*[S(t+\Delta t,\lambda)]+(\Delta t)\int\Lambda(t,x,u)p_t(x|\lambda)dx+o(\Delta t)\right\} \tag{29}$$

In (29) proceeding to the limit with $\Delta t\to 0$ and considering (12), we obtain equation (9). Boundary condition (10) and the expression for J^0 arise from (3), (5). Theorem 1 is proved.

Affirmation 1. Let

$$f(\cdot)=F(t)x_t+B(t)u_t,\quad \Phi_1(\cdot)=\Phi_1(t),\ p_0(x)=N\{x;\mu_0,\Gamma_0\}, \tag{30}$$

$$h(\cdot)=H_0(t)x_t+\sum_{k=1}^{N}H_k(t)x_{\tau_k}, \tag{31}$$

$$\mu(t) = E\{x_t \mid z_0^t\}, \tag{32}$$

$$\mu(\tau_k, t) = E\{x_{\tau_k} \mid z_0^t\}, \quad k = \overline{1; N}, \tag{33}$$

$$\Gamma(t) = E\left\{[x_t - \mu(t)][x_t - \mu(t)]^T \middle| z_0^t\right\}, \tag{34}$$

$$\Gamma_{kk}(\tau_k, t) = E\left\{\left[x_{\tau_k} - \mu(\tau_k, t)\right]\left[x_{\tau_k} - \mu(\tau_k, t)\right]^T \middle| z_0^t\right\}, \tag{35}$$

$$\Gamma_{0k}(\tau_k, t) = E\left\{[x_t - \mu(t)]\left[x_{\tau_k} - \mu(\tau_k, t)\right]^T \middle| z_0^t\right\} \tag{36}$$

$$\Gamma_{ki}(\tau_k, \tau_i, t) = E\left\{\left[x_{\tau_k} - \mu(\tau_k, t)\right]\left[x_{\tau_i} - \mu(\tau_i, t)\right]^T \middle| z_0^t\right\} \tag{37}$$

$$k = \overline{1; N}, \quad i = \overline{2; N}, \quad i > k. \tag{38}$$

Then, for a posteriori density (4) property

$$p_t(x) = \mathrm{N}\{x; \mu(t), \Gamma(t)\} \tag{39}$$

The density parameters are specified by the equations

$$d\mu(t) = [F(t)\mu(t) + B(t)u_t]dt + \tilde{H}_0^T(t)R^{-1}(t)d\tilde{z}_t, \tag{40}$$

$$d\mu(\tau_k, t) = \tilde{H}_k^T(t)R^{-1}(t)d\tilde{z}_t, \tag{41}$$

$$d\Gamma(t)/dt = F(t)\Gamma(t) + \Gamma(t)F^T(t) - \tilde{H}_0^T(t)R^{-1}(t)\tilde{H}_0(t) + Q(t), \tag{42}$$

$$d\Gamma_{kk}(\tau_k, t)/dt = -\tilde{H}_k^T(t)R^{-1}(t)\tilde{H}_k(t), \tag{43}$$

$$d\,\Gamma_{0k}(\tau_k, t)/dt = F(t)\Gamma_{0k}(\tau_k, t) - \tilde{H}_0^T(t)R^{-1}(t)\tilde{H}_k(t), \tag{44}$$

$$d\Gamma_{ki}(\tau_k, \tau_i, t)/dt = -\tilde{H}_k^T(t)R^{-1}(t)\tilde{H}_i(t) \tag{45}$$

where

$$d\tilde{z}(t) = dz(t) - \left[H_0(t)\mu(t) - \sum_{j=1}^{N} H_j(t)\mu(\tau_j, t)\right]dt, \tag{46}$$

$$\tilde{H}_0(t) = H_0(t)\Gamma(t) + \sum_{j=1}^{N} H_j(t)\Gamma_{0j}^T(\tau_j, t), \tag{47}$$

$$\tilde{H}_k(t) = H_k(t)\Gamma_{kk}(\tau_k, t) + \sum_{j \neq k}^{N} H_j(t)\Gamma_{kj}^T(\tau_k, \tau_j, t). \tag{48}$$

This relation arises from [4].

Lemma 2. A sufficient coordinates vector is an optimal in a mean-root-square sense estimate $\mu(t)$ of the process x_t, i.e. $\lambda_t\left[z_0^t\right] = \mu(t)$, which is a Markov diffusion process with the characteristics (see (9))

$$a(t, \mu) = F(t)\mu + B(t)u, \quad D(t, \mu) = \tilde{H}_0^T(t)R^{-1}(t)\tilde{H}_0(t), \tag{49}$$

Proof. Since $\Gamma(t)$, according to (42), does not depend on z_t, then it follows from (39), that $\lambda_t = \mu(t)$. According to [6], process $\tilde{z}_t$, a differential of which is given by (46), and for which $\tilde{Z}_t = \left(\tilde{z}_t, F_t^z\right)$, is the Wiener process with

$$E\left\{\tilde{z}_t \tilde{z}_t^T \middle| F_t^z\right\} = \int_0^t R(s)ds. \tag{50}$$

Then, Markov property $\mu(t)$ and formulae (49) arise from (41), (50).

Affirmation 2. The coupled processes $\{x_t; \mu(t)\}$ are a Markov diffusion process.

Proof. The statement is proved by (1), (31), (41) with regard to Lemma 2.

Remark 2. Since the conditions of Theorem 1 are satisfied for $\lambda_t = \mu(t)$, then $S(t,\lambda) = S(t,\mu)$, and it follows from (3)-(5), that

$$\min_u \left\{ \frac{\partial S(t,\mu)}{\partial t} + L^*_{t,\mu}[S(t,\mu)] + E\left\{\Lambda(t, x_t, u_t) \middle| z_0^t\right\} \right\} = 0, \tag{51}$$

$$S(t,\mu)\big|_{t=\mathrm{T}} = E\left\{b(\mathrm{T}, x_\mathrm{T}) \middle| z_0^T\right\}, \tag{52}$$

$$L^*_{t,\mu}[S(t,\mu)] = a^T(t,\mu)\frac{\partial S(t,\mu)}{\partial \mu} + \frac{1}{2} tr\left[D(t,\mu)\frac{\partial^2 S(t,\mu)}{\partial \mu^2}\right], \tag{53}$$

where $a(t,\mu)$ and $D(t,\mu)$ have the form (49).

Substitution of $E\left\{\cdot \middle| z_0^t\right\}$ for $E\left\{\cdot \middle| \lambda_t = \lambda\right\}$ results from F_t^z measurability of the process $\mu(t)$ (see (41), (46), (50)).

4. Main results

Theorem 2.

Let

$$\Lambda(\cdot) = x_t^T L(t) x_t + u_t^T N(t) u_t, \quad b(\cdot) = x_\mathrm{T}^T S_\mathrm{T} x_\mathrm{T}, \tag{54}$$

where $L(t)$, $N(t)$, S_T are symmetric matrices, and $L(t) \geq 0$, $S_\mathrm{T} \geq 0$, $N(t) > 0$. Hence, the Bellman equation (51) and boundary condition (52) take the following form

$$\begin{gathered}\min_u \left\{ \frac{\partial S(t,\mu)}{\partial t} + [F(t)\mu + B(t)u]^T \frac{\partial S(t,\mu)}{\partial \mu} + \frac{1}{2} tr\left[\tilde{H}_0^T(t) R^{-1}(t) \tilde{H}_0(t) \frac{\partial^2 S(t,\mu)}{\partial \mu^2}\right] + \right. \\ \left. + \mu^T L(t)\mu + u^T N(t) u + tr[L(t)\Gamma(t)] \right\} = 0,\end{gathered} \tag{55}$$

$$S(t,\mu)\big|_{t=\mathrm{T}} = tr[S_T \Gamma(\mathrm{T})] + \mu^T S_\mathrm{T} \mu. \tag{56}$$

Proof. From (54), with regard to (37) and F_t^z-measurability u_t, we obtain

$$\begin{aligned} E\left\{\Lambda(t,x_t,u_t)\middle| z_0^t\right\} &= E\left\{x_t^T L(t) x_t + u_t^T L(t) u_t \middle| z_0^t\right\} = E\left\{x_t^T L(t) x_t \middle| z_0^t\right\} + u_t^T L(t) u_t = \\ &= \mu_t^T L(t)\mu_t + tr[L(t)\Gamma(t)] + u_t^T L(t) u_t. \end{aligned} \tag{57}$$

Analogous

$$E\left\{b(\mathrm{T},x_{\mathrm{T}})\middle|z_0^T\right\}=E\left\{x_{\mathrm{T}}^T S_{\mathrm{T}} x_{\mathrm{T}}\middle|z_0^T\right\}=\mu^T(\mathrm{T})S_{\mathrm{T}}\mu(\mathrm{T})+tr[S_{\mathrm{T}}\Gamma(\mathrm{T})]. \tag{58}$$

Substitution of (49), (53), (57), in (51) results in (55), and substitution of (58) in (52) leads to (56). The theorem is proved.

From now on, the t-derivative will be identified by the point at the top.

Theorem 3 (The separation theorem). The optimal control u_t^0 has the form

$$u_t^0=-N^{-1}(t)B^T(t)S(t)\mu(t), \tag{59}$$

where an optimal in a mean-root-square sense estimate $\mu(t)$ of the state vector x_t is defined by filter equations (40)-(48), where $u_t=u_t^0$, the matrix $S(t)$ is defined by matrix differential Riccati equation

$$\dot{S}(t)=-F^T(t)S(t)-S(t)F(t)+S(t)B(t)N^{-1}(t)B^T(t)S(t)-L(t), \tag{60}$$

with boundary equation

$$S(\mathrm{T})=S_{\mathrm{T}}, \tag{61}$$

and the smallest value J^0 of quality criterion has the following form

$$J^0=\mu^T(t_0)S(t_0)\mu(t_0)+tr[S_{\mathrm{T}}\Gamma(\mathrm{T})]+\int_{t_0}^{\mathrm{T}} tr[L(t)\Gamma(t)]dt+\int_{t_0}^{t} tr\left[\tilde{H}_0(t)R^{-1}(t)\tilde{H}_0(t)S(t)\right]dt \tag{62}$$

Proof. Taking u-derivative from the left-hand side of (55), we obtain the equation to compute the optimal control

$$B^T(t)\frac{\partial S(t,\mu)}{\partial\mu}+2N(t)u=0. \tag{63}$$

Hence, we obtain the expression for optimal control by means of Bellman function in the following form

$$u_t^0=-(1/2)N^{-1}(t)B^T(t)[\partial S(t,\mu)/\partial\mu]. \tag{64}$$

Substituting (64) in (55), we obtain a second-order partial equation for Bellman function in the following form

$$\begin{gathered}\frac{\partial S(t,\mu)}{\partial t}+\mu^T F(t)\frac{\partial S(t,\mu)}{\partial\mu}-\frac{1}{4}\left(\frac{\partial S(t,\mu)}{\partial\mu}\right)^T B(t)N^{-1}(t)B^T(t)\frac{\partial S(t,\mu)}{\partial\mu}+\\+\mu^T L(t)\mu+tr[L(t)\Gamma(t)]+\frac{1}{2}tr\left[\tilde{H}_0^T(t)R^{-1}(t)\tilde{H}_0(t)\frac{\partial^2 S(t,\mu)}{\partial\mu^2}\right]=0.\end{gathered} \tag{65}$$

We solve the equation (65) by using the separation of variables method in the form

$$S(t,\mu)=l(t)+\mu^T S(t)\mu, \tag{66}$$

where $l(t)$ -unknown scalar function, and $S(t)$ - unknown matrix $(n\times n)$-function with imposed symmetry condition. Thereafter

$$\frac{\partial S(t,\mu)}{\partial t}=l(t)+\mu^T\dot{S}(t)\mu,\quad \frac{\partial S(t,\mu)}{\partial\mu}=2S(t)\mu,\quad \frac{\partial^2 S(t,\mu)}{\partial\mu^2}=2S(t). \tag{67}$$

As long as $S(t)$ is imposed with symmetry condition, then, with regard to (67) and the fact that $b = b^T = (1/2)(b + b^T)$ is true for scalar *b*, we obtain

$$\mu^T F^T(t)\frac{\partial S(t,\mu)}{\partial \mu} = 2\mu^T F^T(t)S(t)\mu = \mu^T F^T(t)S(t)\mu + \mu^T S(t)F(t)\mu. \tag{68}$$

Substituting (67), (68) in (65), we have the following formula

$$\begin{aligned} &\dot{l}(t) + \mu^T \dot{S}(t)\mu + \mu^T F^T(t)S(t)\mu + \mu^T S(t)F(t)\mu - \mu^T S(t)B(t)N^{-1}(t)B^T(t)S(t)\mu + \\ &+ \mu^T L(t)\mu + tr[L(t)\Gamma(t)] + tr\left[\tilde{H}_0^T(t)R^{-1}(t)\tilde{H}_0(t)S(t)\right] = 0 \end{aligned} \tag{69}$$

Next, according to the method of separation of variables, we set the coefficients in (69) equal, provided powers μ are equal. Then, we have the equation (60) for $S(t)$, and for $l(t)$ we obtain the following equation

$$\dot{l}(t) = -tr[L(t)\Gamma(t)] - tr\left[\tilde{H}_0^T(t)R^{-1}(t)\tilde{H}_0(t)S(t)\right]. \tag{70}$$

According to (66)

$$S(t,\mu)\big|_{t=\mathrm{T}} = l(\mathrm{T}) + \mu^T S(\mathrm{T})\mu. \tag{71}$$

The boundary condition (61) for equation (60) follows from comparison (56) and (71), while the boundary condition for (70) has the form

$$l(t)\big|_{t=\mathrm{T}} = tr[S_{\mathrm{T}}\Gamma(\mathrm{T})]. \tag{72}$$

According to Theorem 1, $J^0 = S(t_0, \lambda(t_0))$, and $\lambda = \mu$, thus, it follows from (66), that

$$J^0 = l(t_0) + \mu^T(t_0)S(t_0)\mu(t_0). \tag{73}$$

The solution to equation (70) with boundary condition (72) has the form

$$l(t) = tr[S_{\mathrm{T}}\Gamma(\mathrm{T})] + \int_t^{\mathrm{T}} tr[L(\tau)\Gamma(\tau)]d\tau + \int_t^{\mathrm{T}} tr\left[\tilde{H}_0^T(\tau)R^{-1}(\tau)\tilde{H}_0(\tau)S(\tau)\right]d\tau. \tag{74}$$

When $t = t_0$, substitution of (74) in (73) results in (62). Applying (67) in (64), we get (59).

5. Conclusions

Comparing results of Theorem 3 with separation Theorem for memoryless sources in the classical case [1], we come to the conclusion that the expression of optimal control has the same form (59). The difference is the following: in the classical case, Kalman filter estimates the state vector, while in the case considered above, it is the filter (40)-(48), that estimates not only filtering parameters $\mu(t)$ for the current value of state vector x_t, but also estimates interpolation $\mu(\tau_k, t)$ for previous values of the state vector x_{τ_k}, $k = \overline{1; N}$. Thus, the expression for smallest values of quality criterion J^0 changes.

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