On fault detection filters design with unitary transfer function matrices

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Abstract. An adaptation of unitary system principle in fault detection filter design for continuous-time linear MIMO systems is presented in the paper. The conformation is based on an enhanced fault residual transfer function matrix with unitary construction and offers the key advantages on providing high residual sensitivity with respect to faults. Reflecting the emplacement of singular values in unitary construction, an associated structure of linear matrix inequalities with built-in structured constraints is outlined to verify the filter stability. The proposed design conditions are verified by the numerical illustrative example.

1. Introduction

A conventional control for complex systems may results in unsatisfactory performances in the event of system component malfunctions. To overcome these weaknesses, different approaches to control system design are developed to tolerate component malfunctions and to maintain acceptable performances of the system with faults. These control structures are known as fault-tolerant control (FTC) systems and force the ability to accommodate component failures. In that sense, research in FTC is subject of a wide range of publications, reflecting faults effect on control structure reconfiguration [4], [14], fault estimation [16], [20], and fault residuals generation and analysis [7], [25]. The ideal approach is to construct disturbance-decoupled residuals, with responsiveness to the faults, as shown in [6].

To scale up accuracy of fault detection, it is eligible to craft residuals with high sensitivity to faults under robustness to disturbances. One of the options is the use of $\mathrm{H}_\infty/\mathrm{H}_-$ optimization principle [8], [9]. The restriction of these methods is mainly the necessity of existence of a full rank direct-feed external gain matrix from faults to residuals [21], which limits them to be used only to residuals revealing actuator faults. Another approach, based on an unitary system property, is proposed in [22], [23] where optimization is realized inherently in the sense that if the singular values of a unitary system are assigned as the magnitude frequency response of a first-order transfer function then H_∞ is the maximum and H_- is the minimum of the magnitude of the transfer function. To apply this, Riccati equation based formulation was proposed to design a stable unitary fault transfer function approximation.

Reflecting this approach in unitary systems design, the issue of this paper is to simplify the design conditions, mainly because solutions of the introduced Riccati equation form for this singular task often fail. Searching for accurate parameter matrices of the fault residual state-space description and to ensure the constructed unitary model closely approximates the actual fault residual transfer function matrix, the proposed extended adaptation of the results given in [13] presents design task which is non-singular with unitary conditions reaching approximately the theoretical limits for the prescribed set of singular values. To analyze stability of the observer-based residual filter, the proposed conditions use standard arguments and require to solve only LMIs with one built-in prescribed constraint, depending on the system output and fault input matrix structures.

The paper is organized as follows. Placed immediately after Introduction, Sec. 2 presents the problem statement and Sec. 3 summaries in basic preliminaries the auxiliary lemmas on the issue of the design task. The enhanced structure of unitary fault residual transfer function matrix, as well as the set of LMIs describing its stability, is theoretically explained in Sec 4. An example is provided to demonstrate the proposed approach in Sec. 5 and, finally, Sec. 6 draws some concluding remarks.

Used notations are conventional so that $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denote transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, $\boldsymbol{X} = \boldsymbol{X}^T > 0$ means that $\boldsymbol{X}$ is a symmetric positive definite matrix, the symbol $\boldsymbol{I}_n$ marks the n-th order unit matrix, $\rho(\boldsymbol{X})$ and $rank(\boldsymbol{X})$ indicate the eigenvalue spectrum and rank of a square matrix $\boldsymbol{X}$, $\boldsymbol{Y}^{\perp}$ designates the orthogonal complement to a rank-deficient matrix $\boldsymbol{Y}$, $\sigma_i(\boldsymbol{Z})$ labels the i-th singular value of matrix $\boldsymbol{Z}$, $I\!R$ denotes the set of real numbers and $I\!R^n$, $I\!R^{n\times r}$ refer to the set of all n-dimensional real vectors and $n \times r$ real matrices, respectively.

2. The Problem Statement

The systems under consideration are linear MIMO continuous-time dynamic systems represented as follows

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{F}\boldsymbol{f}(t) + \boldsymbol{E}\boldsymbol{d}(t), \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t), \tag{2}$$

where $\boldsymbol{q}(t) \in I\!R^n$, $\boldsymbol{u}(t) \in I\!R^r$, $\boldsymbol{y}(t) \in I\!R^m$ are vectors of the state, input and output variables, respectively, $\boldsymbol{f}(t) \in I\!R^p$ is fault vector and $\boldsymbol{d}(t) \in I\!R^{r_d}$ is vector of disturbance. The real matrices $\boldsymbol{A} \in I\!R^{n\times n}$, $\boldsymbol{B} \in I\!R^{n\times r}$, $\boldsymbol{C} \in I\!R^{m\times n}$, $\boldsymbol{F} \in I\!R^{n\times p}$, $\boldsymbol{E} \in I\!R^{n\times r_d}$ are finite values, satisfying the rank conditions $rank(\boldsymbol{F}) = p$, $rank(\boldsymbol{C}) = m$, $p = m$, $p < n$. Moreover, it is supposed that the matrix $\boldsymbol{V} = \boldsymbol{C}\boldsymbol{F}$ is regular matrix such that $\boldsymbol{V} \in I\!R^{m\times m}$.

Problem of the interest is an unitary representation of the fault residual filter for the system with the square transfer function matrix of unknown fault input and the residuals. Note, such construction of an unitary systems to given linear system, with respect to the singular values of the system transfer function matrix, is non unique task also for square linear systems [13], [22].

3. Basic Preliminaries

3.1. Unitary Linear Systems

If $\mathcal{H}$ and $\mathcal{E}$ be Krein spaces [2], a continuous linear transformation

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix} : \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{E} \end{matrix} \rightarrow \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{E} \end{matrix} \tag{3}$$

is called a linear system. The underlying Krein space $\mathcal{H}$ is called the state space and the auxiliary Krein space $\mathcal{E}$ is called the coefficient space or the external space [19]. The transformation $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ is the main, input and output transformation, respectively, the operator $\boldsymbol{D}$ is called the external operator. A linear system is said to be unitary if the above matrix is unitary, and the transfer function $\boldsymbol{G}(s)$ of the linear system is defined by

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\,. \tag{4}$$

If $\boldsymbol{A}$ has no imaginary eigenvalues then $\boldsymbol{G}(j\omega)$ is defined for all $\omega \in I\!R$ and the singular values of the transfer function matrix $\boldsymbol{G}(s)$, evaluated on the imaginary axis, are $\sigma_i(\boldsymbol{G}(j\omega))$, where the i-th singular value of the complex matrix $\boldsymbol{G}(j\omega)$ is the nonnegative square-root of the i-th largest eigenvalue of $\boldsymbol{G}(j\omega)\boldsymbol{G}^*(j\omega)$. The H_∞ norm of the transfer matrix $\boldsymbol{G}(s)$ is [5]

$$\|\boldsymbol{G}\|_\infty = \sup_{\omega \in I\!R} \sigma_1(\boldsymbol{G}(j\omega))\,. \tag{5}$$

Some reflections can be found, e.g., in [3], [10] and [24].

Definition 1 *[22] A stable linear time-invariant system of m-inputs and m-outputs is defined as a unitary system if the singular values of its transfer function (matrix) $\boldsymbol{G}(s)$ satisfies*

$$\sigma_1 = \sigma_2 = \cdots = \sigma_m, \tag{6}$$

where σ_i is the i-th singular value of $\boldsymbol{G}(s)$.

Defining the unitary system gain matrix equal to $\boldsymbol{V} = \boldsymbol{C}\boldsymbol{F}$, the following system state coordinate transformation can be done.

Lemma 1 *If for the matrix parameters $\boldsymbol{C}$ and $\boldsymbol{F}$ of the system (1), (2)*

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{V} & \boldsymbol{0} \end{bmatrix} \boldsymbol{T}, \quad \boldsymbol{T}\boldsymbol{F} = \begin{bmatrix} \boldsymbol{I}_p \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{V} = \boldsymbol{C}\boldsymbol{F}, \tag{7}$$

then for $m = p$ the transform matrix $\boldsymbol{T} \in I\!R^{n\times n}$ takes the form

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{V}^{-1}\boldsymbol{C} \\ \boldsymbol{F}^{\perp} \end{bmatrix}, \tag{8}$$

where $\boldsymbol{V}^{-1}\boldsymbol{C} \in I\!R^{p\times n}$, $\boldsymbol{F}^{\perp} \in I\!R^{(n-p)\times n}$, respectively, and $\boldsymbol{F}^{\perp}$ is the left orthogonal complement to $\boldsymbol{F}$ [1], [11].

Proof: Writing the first term of (7) as

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{V} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \end{bmatrix} = \boldsymbol{C}\,, \tag{9}$$

it is evident that

$$\boldsymbol{V}\boldsymbol{T}_1 = \boldsymbol{C}, \quad \boldsymbol{T}_1 = \boldsymbol{V}^{-1}\boldsymbol{C}\,. \tag{10}$$

Analyzing the second term of (7), i.e.,

$$\begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \end{bmatrix} \boldsymbol{F} = \begin{bmatrix} \boldsymbol{I}_p \\ \boldsymbol{0} \end{bmatrix}, \tag{11}$$

the following condition results

$$\boldsymbol{T}_2\boldsymbol{F} = \boldsymbol{0}, \quad \boldsymbol{T}_2 = \boldsymbol{F}^{\perp}. \tag{12}$$

Thus, (10), (12) imply (8).

It is easily verified using (9), (12) that

$$\boldsymbol{T}_1\boldsymbol{F} = \boldsymbol{V}^{-1}\boldsymbol{C}\boldsymbol{F} = \boldsymbol{V}^{-1}\boldsymbol{V} = \boldsymbol{I}_p\,, \tag{13}$$

$$\boldsymbol{C}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{V} & \boldsymbol{0} \end{bmatrix}, \tag{14}$$

respectively.

Note, the last equality imply that, in general, $\boldsymbol{T}^{-1}$ can be computed only numerically. This concludes the proof. ■

3.2. Structures of Unitary Fault Transfer Matrices

The basic structures of unitary system transfer matrices are introduced by the following lemmas.

Lemma 2 *For the system (1), (2) with $m = p$ there exist, for a prescribed positive scalar $s_o \in I\!R$, the matrix $\boldsymbol{L}^o \in I\!R^{n\times m}$ such that the fault transfer function matrix takes the form*

$$\boldsymbol{G}_f(s) = \boldsymbol{G}_\Delta(s)\boldsymbol{G}_{f\Delta}(s)\,, \tag{15}$$

where

$$\boldsymbol{G}_{f\Delta}(s) = \boldsymbol{C}\left(s\boldsymbol{I}_n - (\boldsymbol{A} - \boldsymbol{L}^o\boldsymbol{C})\right)^{-1} = \frac{\boldsymbol{V}}{s+s_o}\,, \tag{16}$$

$$\boldsymbol{G}_\Delta(s) = \boldsymbol{I}_m + \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{M}, \tag{17}$$

$$\boldsymbol{L}^o = \begin{bmatrix} s_o\boldsymbol{I}_m + \boldsymbol{A}_{o11} \\ \boldsymbol{A}_{o21} \end{bmatrix}\boldsymbol{V}^{-1}, \quad \boldsymbol{A}_o = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{A}_{o11} & \boldsymbol{A}_{o12} \\ \boldsymbol{A}_{o21} & \boldsymbol{A}_{o22} \end{bmatrix}, \quad \boldsymbol{M} = \boldsymbol{T}^{-1}\boldsymbol{L}^o \tag{18}$$

and $\boldsymbol{T}$ is given by (8).

Proof: (compare [13], [22]) Since

$$\boldsymbol{G}_f(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{F} \tag{19}$$

is the fault transfer function matrix of dimension $m\times p$, then by using (7), (14), (18) the transfer function matrix (19) can be rewritten as

$$\begin{aligned} \boldsymbol{G}_f(s) &= \boldsymbol{C}\boldsymbol{T}^{-1}\boldsymbol{T}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{T}^{-1}\boldsymbol{T}\boldsymbol{F} \\ &= \boldsymbol{C}\boldsymbol{T}^{-1}(\boldsymbol{T}(s\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{T}^{-1})^{-1}\boldsymbol{T}\boldsymbol{F} \\ &= \begin{bmatrix} \boldsymbol{V} & \boldsymbol{0} \end{bmatrix}(s\boldsymbol{I}_n - \boldsymbol{A}_o)^{-1}\begin{bmatrix} \boldsymbol{I}_p \\ \boldsymbol{0} \end{bmatrix}. \end{aligned} \tag{20}$$

Defining the matrix product $\boldsymbol{A}^o = \boldsymbol{T}\boldsymbol{M}\boldsymbol{C}\boldsymbol{T}^{-1}$, where $\boldsymbol{M} \in I\!R^{n\times m}$ is a real matrix, then by exploiting (8), (14) it yields

$$\boldsymbol{A}^o = \boldsymbol{T}\boldsymbol{M}\boldsymbol{C}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{V}^{-1}\boldsymbol{C} \\ \boldsymbol{F}^\perp \end{bmatrix}\boldsymbol{M}\begin{bmatrix} \boldsymbol{V} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{V}^{-1}\boldsymbol{C}\boldsymbol{M}\boldsymbol{V} & \boldsymbol{0} \\ \boldsymbol{F}^\perp\boldsymbol{M}\boldsymbol{V} & \boldsymbol{0} \end{bmatrix} \tag{21}$$

and, accepting the analogy between (18) and (21), it can define

$$\Delta\boldsymbol{A}_o = \boldsymbol{A}_o - \boldsymbol{A}^o = \begin{bmatrix} \boldsymbol{A}_{o11} - \boldsymbol{V}^{-1}\boldsymbol{C}\boldsymbol{M}\boldsymbol{V} & \boldsymbol{A}_{o12} \\ \boldsymbol{A}_{o21} - \boldsymbol{F}^\perp\boldsymbol{M}\boldsymbol{V} & \boldsymbol{A}_{o22} \end{bmatrix}. \tag{22}$$

Setting

$$\boldsymbol{A}_{o11} - \boldsymbol{V}^{-1}\boldsymbol{C}\boldsymbol{M}\boldsymbol{V} = -s_o\boldsymbol{I}_m\,, \quad \boldsymbol{A}_{o21} - \boldsymbol{F}^\perp\boldsymbol{M}\boldsymbol{V} = \boldsymbol{0}\,, \tag{23}$$

where $s_o \in I\!R$ is a prescribed positive real value, and writing (23) as

$$\begin{bmatrix} s_o\boldsymbol{I}_m + \boldsymbol{A}_{o11} \\ \boldsymbol{A}_{o21} \end{bmatrix} = \begin{bmatrix} \boldsymbol{V}^{-1}\boldsymbol{C} \\ \boldsymbol{F}^\perp \end{bmatrix}\boldsymbol{M}\boldsymbol{V} = \boldsymbol{T}\boldsymbol{M}\boldsymbol{V} = \boldsymbol{T}\boldsymbol{T}^{-1}\boldsymbol{L}^o\boldsymbol{V} = \boldsymbol{L}^o\boldsymbol{V}, \tag{24}$$

then, with

$$\boldsymbol{M} = \boldsymbol{T}^{-1}\boldsymbol{L}^o, \tag{25}$$

the following yields

$$\boldsymbol{A}^o = \boldsymbol{T}\boldsymbol{M}\boldsymbol{C}\boldsymbol{T}^{-1} = \boldsymbol{L}^o\boldsymbol{C}\boldsymbol{T}^{-1}, \tag{26}$$

$$\Delta \boldsymbol{A}_o = \boldsymbol{A}_o - \boldsymbol{A}^o = \boldsymbol{T}(\boldsymbol{A} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{T}^{-1} = \boldsymbol{T}\Delta\boldsymbol{A}\boldsymbol{T}^{-1}, \tag{27}$$

where

$$\Delta \boldsymbol{A} = \boldsymbol{A} - \boldsymbol{M}\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{T}^{-1}\boldsymbol{L}^o\boldsymbol{C} \tag{28}$$

and (22) takes the form

$$\Delta \boldsymbol{A}_o = \begin{bmatrix} -s_o\boldsymbol{I}_m & \boldsymbol{A}_{o12} \\ \boldsymbol{0} & \boldsymbol{A}_{o22} \end{bmatrix}. \tag{29}$$

Defining the transfer function matrix $\boldsymbol{G}_{f\Delta}(s)$ as follows

$$\boldsymbol{G}_{f\Delta}(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \Delta\boldsymbol{A})^{-1}\boldsymbol{F}, \tag{30}$$

then with (27) it is

$$\boldsymbol{G}_{f\Delta}(s) = \boldsymbol{C}\boldsymbol{T}^{-1}(s\boldsymbol{I}_n - \boldsymbol{T}\Delta\boldsymbol{A}\boldsymbol{T}^{-1})^{-1}\boldsymbol{T}\boldsymbol{F} = \begin{bmatrix} \boldsymbol{V} & \boldsymbol{0} \end{bmatrix} (s\boldsymbol{I}_n - \Delta\boldsymbol{A}_o)^{-1} \begin{bmatrix} \boldsymbol{I}_p \\ \boldsymbol{0} \end{bmatrix}. \tag{31}$$

Since

$$s\boldsymbol{I}_n - \Delta\boldsymbol{A}_o = \begin{bmatrix} (s+s_o)\boldsymbol{I}_m & -\boldsymbol{A}_{o12} \\ \boldsymbol{0} & s\boldsymbol{I}_{n-m} - \boldsymbol{A}_{o22} \end{bmatrix} \tag{32}$$

and

$$(s\boldsymbol{I}_n - \Delta\boldsymbol{A}_o)^{-1} = \begin{bmatrix} (s+s_o)^{-1}\boldsymbol{I}_m & (s+s_o)^{-1}\boldsymbol{A}_{o12}(s\boldsymbol{I}_{n-m} - \boldsymbol{A}_{o22})^{-1} \\ \boldsymbol{0} & (s\boldsymbol{I}_{n-m} - \boldsymbol{A}_{o22})^{-1} \end{bmatrix}, \tag{33}$$

then, substituting (33) into (31), it can obtain

$$\boldsymbol{G}_{f\Delta}(s) = \begin{bmatrix} \boldsymbol{V} & \boldsymbol{0} \end{bmatrix} (s\boldsymbol{I}_n - \Delta\boldsymbol{A}_o)^{-1} \begin{bmatrix} \boldsymbol{I}_p \\ \boldsymbol{0} \end{bmatrix} = \frac{\boldsymbol{V}}{s+s_o}, \tag{34}$$

which implies (16).

The transfer function (30) together with (28) and (19) can be rewritten as

$$\begin{aligned} \boldsymbol{G}_{f\Delta}(s) &= \boldsymbol{C}(s\boldsymbol{I}_n - \Delta\boldsymbol{A})^{-1}\boldsymbol{F} \\ &= \boldsymbol{C}\left((s\boldsymbol{I}_n - \boldsymbol{A})(\boldsymbol{I}_n + (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{M}\boldsymbol{C})\right)^{-1}\boldsymbol{F} \\ &= \boldsymbol{C}(\boldsymbol{I}_n + (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{M}\boldsymbol{C})^{-1}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{F}, \end{aligned} \tag{35}$$

which gives, by using the equality

$$(\boldsymbol{F}\boldsymbol{C})^{-1}\boldsymbol{F}\boldsymbol{C} = \boldsymbol{I}_n, \tag{36}$$

$$\begin{aligned} \boldsymbol{G}_{f\Delta}(s) &= \boldsymbol{C}(\boldsymbol{I}_n + (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{M}\boldsymbol{C})^{-1}(\boldsymbol{F}\boldsymbol{C})^{-1}\boldsymbol{F}\boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{F} \\ &= \boldsymbol{C}(\boldsymbol{I}_n + (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{M}\boldsymbol{C})^{-1}(\boldsymbol{F}\boldsymbol{C})^{-1}\boldsymbol{F}\,\boldsymbol{G}_f(s), \end{aligned} \tag{37}$$

as well as, after some manipulations,

$$\begin{aligned} \boldsymbol{G}_{f\Delta}(s) &= \boldsymbol{C}\left((\boldsymbol{F}\boldsymbol{C})(\boldsymbol{I}_n + (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{M}\boldsymbol{C})\right)^{-1}\boldsymbol{F}\,\boldsymbol{G}_f(s) \\ &= \boldsymbol{C}\left(\boldsymbol{F}(\boldsymbol{I}_m + \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{M})\boldsymbol{C})\right)^{-1}\boldsymbol{F}\,\boldsymbol{G}_f(s) \\ &= \left(\boldsymbol{I}_m + \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{M}\right)^{-1}\boldsymbol{G}_f(s). \end{aligned} \tag{38}$$

Thus, with the notation (17), then (38) implies (15). This concludes the proof. ■

With existence of such transformation the structure of (20) really means that there exists the subset of transformed state variables whose dynamics is explicitly affected by the fault $\boldsymbol{f}(t)$ and the second one, whose dynamics is not affected explicitly by the fault $\boldsymbol{f}(t)$.

Defined by (6), a linear time-invariant system is considered as unitary if all singular values of its transfer function matrix are equal. Because the construction given in Lemma 2 is not unique, equivalent structures can be used.

Lemma 3 *An equivalent structure of the fault transfer function matrix of the system (1), (2) takes the form*

$$\boldsymbol{G}_f(s) = \boldsymbol{G}^{\circ}_{\Delta}(s)(\boldsymbol{G}^{\circ}_{f\Delta}(s) - \boldsymbol{V}) \,, \tag{39}$$

where

$$\boldsymbol{G}^{\circ}_{f\Delta}(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{N}\boldsymbol{C})^{-1}(\boldsymbol{F} + \boldsymbol{N}\boldsymbol{V}) + \boldsymbol{V} \,, \tag{40}$$

$$\boldsymbol{G}^{\circ}_{\Delta}(s) = \boldsymbol{I}_m + \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{N} \,. \tag{41}$$

If $\boldsymbol{M} = \boldsymbol{N}$ *then*

$$\frac{\boldsymbol{G}^{\circ}_{f\Delta}(s)}{s + s_o + 1} = \frac{\boldsymbol{V}}{s + s_o} = \boldsymbol{G}_{f\Delta}(s) \,. \tag{42}$$

Proof: Considering the associated system (40), it can be written

$$(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{N}\boldsymbol{C})^{-1} = ((s\boldsymbol{I}_n - \boldsymbol{A})(\boldsymbol{I}_n + (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{N}\boldsymbol{C}))^{-1} = \boldsymbol{G}^{\diamond}_{\Delta}(s)(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}, \tag{43}$$

where

$$\boldsymbol{G}^{\diamond}_{\Delta}(s) = (\boldsymbol{I}_n + (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{N}\boldsymbol{C})^{-1}. \tag{44}$$

Therefore, substituting (43) in (40) leads to

$$\boldsymbol{G}^{\circ}_{f\Delta}(s) = \boldsymbol{C}\left(\boldsymbol{G}^{\diamond}_{\Delta}(s)(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}(\boldsymbol{I}_n + \boldsymbol{N}\boldsymbol{C}) + \boldsymbol{I}_n\right)\boldsymbol{F} \tag{45}$$

and it yields

$$\begin{aligned} \boldsymbol{G}^{\circ}_{f\Delta}(s) &= \boldsymbol{C}\left(\boldsymbol{G}^{\diamond}_{\Delta}(s)((s\boldsymbol{I}_n - \boldsymbol{A})^{-1} + (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{N}\boldsymbol{C}) + \boldsymbol{I}_n\right)\boldsymbol{F} \\ &= \boldsymbol{C}\left(\boldsymbol{G}^{\diamond}_{\Delta}(s)((s\boldsymbol{I}_n - \boldsymbol{A})^{-1} + \boldsymbol{G}^{\diamond -1}_{\Delta}(s)\right)\boldsymbol{F} \\ &= \boldsymbol{C}\boldsymbol{G}^{\diamond}_{\Delta}(s)(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{F} + \boldsymbol{V}. \end{aligned} \tag{46}$$

Since, using the equality (36), it can obtain

$$\begin{aligned} \boldsymbol{C}\boldsymbol{G}^{\diamond}_{\Delta}(s) &= \boldsymbol{C}\boldsymbol{G}^{\diamond}_{\Delta}(s)(\boldsymbol{F}\boldsymbol{C})^{-1}\boldsymbol{F}\boldsymbol{C} \\ &= \boldsymbol{C}\left(\boldsymbol{F}\boldsymbol{C}(\boldsymbol{I}_n + (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{N}\boldsymbol{C})\right)^{-1}\boldsymbol{F}\boldsymbol{C} \\ &= \boldsymbol{C}\left(\boldsymbol{F}(\boldsymbol{I}_m + \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{N})\boldsymbol{C}\right)^{-1}\boldsymbol{F}\boldsymbol{C} \\ &= \boldsymbol{G}^{\circ -1}_{\Delta}(s)\boldsymbol{C}, \end{aligned} \tag{47}$$

then the substitution of (47) into (46) gives

$$\boldsymbol{G}^{\circ}_{f\Delta}(s) = \boldsymbol{G}^{\circ -1}_{\Delta}(s)\boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{F} + \boldsymbol{V} = \boldsymbol{G}^{\circ -1}_{\Delta}(s)\boldsymbol{G}_f(s) + \boldsymbol{V}. \tag{48}$$

Thus, (48) implies (39).

Considering that $\boldsymbol{M} = \boldsymbol{N}$ then (17) and (41) implies $\boldsymbol{G}^{\circ}_{\Delta}(s) = \boldsymbol{G}_{\Delta}(s)$ and (15) defines $\boldsymbol{G}_f(s) = \boldsymbol{G}_{\Delta}(s)\boldsymbol{G}_{f\Delta}(s)$. Thus, (39) can be written as

$$\boldsymbol{G}_f(s) = \boldsymbol{G}_{\Delta}(s)(\boldsymbol{G}^{\circ}_{f\Delta}(s) - \boldsymbol{V}) = \boldsymbol{G}_{\Delta}(s)\boldsymbol{G}_{f\Delta}(s) \,, \tag{49}$$

which gives, with respect to (16),

$$\boldsymbol{G}^{\circ}_{f\Delta}(s) = \boldsymbol{G}_{f\Delta}(s) + \boldsymbol{V} = \boldsymbol{V}\left(\frac{1}{s + s_o} + 1\right) = \frac{\boldsymbol{V}}{s + s_o}(s + s_o + 1) \,, \tag{50}$$

which gives (42). This concludes the proof. ■

Note, the relations (42) implies that both transfer functions have the same gain but with different pole. This property gives the possibility to combine two specific gains in the design of unitary fault transfer function matrix by the way specified in the following section. Singular values related properties, such as H_2 norm, H_∞ norm, as well as H_- index, can be determined based on this pole in the FTC structures where the optimization of singular values related properties are of key importance.

4. Enhanced Structure of Unitary Fault Transfer Function Matrix

To exploit the properties all structures presented above, the enhanced form of unitary fault transfer matrix is proposed in the form

$$\boldsymbol{G}_f^\circ(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A} + (\boldsymbol{M} + \boldsymbol{N})\boldsymbol{C})^{-1}\boldsymbol{V}\,, \tag{51}$$

where $\boldsymbol{M}$ is introduced in (18) and $\boldsymbol{N}$ is designed in such way that $\boldsymbol{F} + \boldsymbol{N}\boldsymbol{V} = \boldsymbol{0}$.

To formulate the stability condition of the unitary system, approximated by the equivalent transform function matrix (40), the following theorems are given.

Lemma 4 *The state-space representation of the enhanced structure of transfer function matrix (51) in the form of a closed-loop system is*

$$\dot{\boldsymbol{q}}_d^\circ(t) = \boldsymbol{A}^\circ\boldsymbol{q}_d^\circ(t) + \boldsymbol{B}^\circ\boldsymbol{u}^\circ(t)\,, \tag{52}$$

$$\boldsymbol{y}^\circ(t) = \boldsymbol{C}^\circ\boldsymbol{q}_d^\circ(t)\,, \tag{53}$$

$$\boldsymbol{u}^\circ(s) = -\boldsymbol{K}^\circ\boldsymbol{q}^\circ(t) = -(\boldsymbol{M}^T + \boldsymbol{N}^T)\boldsymbol{q}^\circ(t)\,, \tag{54}$$

where the system constraint is

$$\boldsymbol{R}^\circ = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{S} \\ \boldsymbol{S}^T & -\boldsymbol{R} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{F}\boldsymbol{V}^T \\ \boldsymbol{V}\boldsymbol{F}^T & -\boldsymbol{V}\boldsymbol{V}^T \end{bmatrix} \geq 0 \tag{55}$$

and the matrix parameters are

$$\boldsymbol{A}^\circ = \boldsymbol{A}^T, \quad \boldsymbol{B}^\circ = \boldsymbol{C}^T, \quad \boldsymbol{C}^\circ = \boldsymbol{F}^T, \tag{56}$$

$$\boldsymbol{M}^T = \boldsymbol{L}^{oT}\boldsymbol{T}^{-T}, \quad \boldsymbol{N}^T = -\boldsymbol{V}^{-T}\boldsymbol{F}^T. \tag{57}$$

Thus,

$$\dot{\boldsymbol{q}}_d^\circ(t) = \boldsymbol{A}_{dc}^\circ\boldsymbol{q}_d^\circ(t), \quad \boldsymbol{A}_{dc}^\circ = \boldsymbol{A}^\circ - \boldsymbol{B}^\circ\boldsymbol{K}^\circ\,. \tag{58}$$

$$\boldsymbol{y}^\circ(t) = \boldsymbol{C}^\circ\boldsymbol{q}_d^\circ(t)\,. \tag{59}$$

Proof: Considering that

$$\boldsymbol{F} + \boldsymbol{N}\boldsymbol{V} = \boldsymbol{0} \tag{60}$$

then

$$\boldsymbol{N} = -\boldsymbol{F}\boldsymbol{V}^{-1} \tag{61}$$

and using the Laplace transform property [15], then (51) conditioned by the design constraint (60) implies

$$\widetilde{\boldsymbol{q}}_d(s) = (s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{M}\boldsymbol{C} + \boldsymbol{N}\boldsymbol{C})^{-1}\boldsymbol{V}\widetilde{\boldsymbol{f}}(s) = (s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{M}\boldsymbol{C} - \boldsymbol{F}\boldsymbol{V}^{-1}\boldsymbol{C})^{-1}\boldsymbol{V}\widetilde{\boldsymbol{f}}(s)\,, \tag{62}$$

Writing in the dual state-space form [17], the response (62) reflects a free-response of the system

$$\dot{\boldsymbol{q}}_d^\circ(t) = (\boldsymbol{A}^T - \boldsymbol{C}^T\boldsymbol{M} + \boldsymbol{C}^T\boldsymbol{V}^{-T}\boldsymbol{F}^T)\boldsymbol{V}\boldsymbol{f}(t)\,, \tag{63}$$

$$\boldsymbol{y}^\circ(t) = \boldsymbol{F}^T\boldsymbol{q}_d^\circ(t) \tag{64}$$

and can be interpreted as the response of the closed-loop system (52)-(54), (56), (57). It is evident that this system should be stable under constraint (64) for $\boldsymbol{N}$ of structure given in (61).

Multiplying the right side of (60) by $\boldsymbol{V}^T$ leads to

$$\boldsymbol{F}\boldsymbol{V}^T + \boldsymbol{N}\boldsymbol{V}\boldsymbol{V}^T = \boldsymbol{0}\,, \tag{65}$$

which implies

$$\boldsymbol{N} = -\boldsymbol{F}\boldsymbol{V}^T(\boldsymbol{V}\boldsymbol{V}^T)^{-1}. \tag{66}$$

Since pre-multiplying the right side of (66) by $\boldsymbol{V}\boldsymbol{F}^T$ and then using (60) gives

$$\boldsymbol{F}\boldsymbol{V}^T(\boldsymbol{V}\boldsymbol{V}^T)^{-1}\boldsymbol{V}\boldsymbol{F}^T = -\boldsymbol{N}\boldsymbol{V}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{F}^T \geq 0 \tag{67}$$

and because a dual state-space description (53), (56) implies

$$\boldsymbol{y}^{\circ T}(t)\boldsymbol{y}^{\circ}(t) = \boldsymbol{q}^{\circ T}(t)\boldsymbol{C}^{\circ T}\boldsymbol{C}^{\circ}\boldsymbol{q}^{\circ}(t) = \boldsymbol{q}^{\circ T}(t)\boldsymbol{F}\boldsymbol{F}^T\boldsymbol{q}^{\circ}(t)\,, \tag{68}$$

it is evident that (67) is the constraint given on $\boldsymbol{q}^{\circ}(t)$.

Thus, using the Schur complement property, (67) implies the quadratic constraint (55). This concludes the proof. ■

Theorem 1 *The equivalent system (52)-(57) is stable if there exists a symmetric positive definite matrix* $\boldsymbol{P} \in I\!R^{n\times n}$ *such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0\,, \tag{69}$$

$$\begin{bmatrix} (\boldsymbol{A}-\boldsymbol{M}\boldsymbol{C}+\boldsymbol{F}\boldsymbol{V}^{-1}\boldsymbol{C})\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{M}\boldsymbol{C}+\boldsymbol{F}\boldsymbol{V}^{-1}\boldsymbol{C})^T & \boldsymbol{P}\boldsymbol{C}^T+\boldsymbol{F}\boldsymbol{V}^T \\ \boldsymbol{C}\boldsymbol{P}+\boldsymbol{V}\boldsymbol{F}^T & -\boldsymbol{V}\boldsymbol{V}^T \end{bmatrix} < 0 \tag{70}$$

and the common gain matrix is

$$\boldsymbol{K}^{\circ} = -\boldsymbol{M} - \boldsymbol{N} = -\boldsymbol{M} + \boldsymbol{F}\boldsymbol{V}^{-1}. \tag{71}$$

Proof: Introducing the notation

$$\boldsymbol{q}^{\bullet T}(t) = \begin{bmatrix} \boldsymbol{q}_d^{\circ T}(t) & \boldsymbol{u}^{\circ T}(t) \end{bmatrix}, \tag{72}$$

the Lyapunov function candidate can be considered in the form

$$v(\boldsymbol{q}^{\bullet}(t)) = \boldsymbol{q}_d^{\circ T}(t)\boldsymbol{P}\boldsymbol{q}_d^{\circ}(t) + \int_0^t \boldsymbol{q}^{\bullet T}(v)\boldsymbol{R}^{\circ}\boldsymbol{q}^{\bullet}(v)\mathrm{d}v > 0\,, \tag{73}$$

where $\boldsymbol{P} \in I\!R^{n\times n}$ is a symmetric, positive definite matrix and $\boldsymbol{R}^{\circ} \in I\!R^{(n+m)\times(n+m)}$ is given in (55).

Therefore, the time derivative of (73) can be written as

$$\dot{v}(\boldsymbol{q}^{\bullet}(t)) = \dot{\boldsymbol{q}}^{\circ T}(t)\boldsymbol{P}\boldsymbol{q}^{\circ}(t) + \boldsymbol{q}^{\circ T}(t)\boldsymbol{P}\dot{\boldsymbol{q}}^{\circ}(t) + \boldsymbol{q}^{\bullet T}(t)\boldsymbol{R}^{\circ}\boldsymbol{q}^{\bullet}(t) < 0\,, \tag{74}$$

which, by substituting (52), takes the form

$$\dot{v}(\boldsymbol{q}^{\bullet}(t)) = \boldsymbol{q}^{\bullet T}(t)\boldsymbol{R}^{\circ}\boldsymbol{q}^{\bullet}(t) + (\boldsymbol{A}^{\circ}\boldsymbol{q}_d^{\circ}(t)+\boldsymbol{B}^{\circ}\boldsymbol{u}^{\circ}(t))^T\boldsymbol{P}\boldsymbol{q}_d^{\circ}(t) + \boldsymbol{q}_d^{\circ T}(t)\boldsymbol{P}(\boldsymbol{A}^{\circ}\boldsymbol{q}_d^{\circ}(t)+\boldsymbol{B}^{\circ}\boldsymbol{u}^{\circ}(t)) < 0\,. \tag{75}$$

Then, using (72), the inequality (75) can be written as

$$\dot{v}(\boldsymbol{q}^{\bullet}(t)) = \boldsymbol{q}^{\bullet T}(t)\boldsymbol{P}^{\bullet}\boldsymbol{q}^{\bullet}(t) < 0\,, \tag{76}$$

where

$$\boldsymbol{P}^{\bullet} = \begin{bmatrix} \boldsymbol{P}\boldsymbol{A}^{\circ}+\boldsymbol{A}^{\circ T}\boldsymbol{P} & \boldsymbol{P}\boldsymbol{B}^{\circ}+\boldsymbol{F}\boldsymbol{V}^T \\ \boldsymbol{B}^{\circ T}\boldsymbol{P}+\boldsymbol{V}\boldsymbol{F}^T & -\boldsymbol{V}\boldsymbol{V}^T \end{bmatrix} < 0\,. \tag{77}$$

Constructing the closed-loop system matrix

$$\boldsymbol{A}_{dc}^{\circ} = \boldsymbol{A}^{\circ} - \boldsymbol{B}^{\circ}\boldsymbol{K}^{\circ} = \boldsymbol{A}^{T} - \boldsymbol{C}^{T}(\boldsymbol{M}^{T} + \boldsymbol{N}^{T})\,, \tag{78}$$

which, by using (57), is of the form

$$\boldsymbol{A}_{dc}^{\circ} = \boldsymbol{A}^{T} - \boldsymbol{C}^{T}\boldsymbol{M}^{T} + \boldsymbol{C}^{T}\boldsymbol{V}^{-T}\boldsymbol{F}^{T}, \tag{79}$$

it is evident that

$$\boldsymbol{A}_{dc}^{\circ T}\boldsymbol{P} = (\boldsymbol{A} - \boldsymbol{M}\boldsymbol{C} + \boldsymbol{F}\boldsymbol{V}^{-1}\boldsymbol{C})\boldsymbol{P} \tag{80}$$

and

$$\boldsymbol{P}\boldsymbol{B}^{\circ} = \boldsymbol{P}\boldsymbol{C}^{T}. \tag{81}$$

Thus, replacing $\boldsymbol{A}^{\circ}$ in (77) by $\boldsymbol{A}_{c}^{\circ}$ and using (81), the matrix inequality (77) takes the form (70). This concludes the proof. ■

5. Unitary Fault Detection Filter

Writing the dual form of the state-space description (58), (78) in the direct form, it is obtained

$$\dot{\boldsymbol{q}}^{\circ}(t) = \boldsymbol{A}_{dc}^{\circ T}\boldsymbol{q}^{\circ}(t)\,, \tag{82}$$

$$\boldsymbol{A}_{dc}^{\circ T} = \boldsymbol{A}^{\circ T} - \boldsymbol{K}^{\circ T}\boldsymbol{B}^{\circ T} = \boldsymbol{A} - (\boldsymbol{M} + \boldsymbol{N})\boldsymbol{C}\,. \tag{83}$$

Denoting

$$\boldsymbol{e}(t) = \boldsymbol{q}^{\circ}(t)\,, \quad \boldsymbol{J} = \boldsymbol{M} + \boldsymbol{N}\,, \tag{84}$$

it yields

$$\dot{\boldsymbol{e}}(t) = (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}(t), \tag{85}$$

which is the autonomous equation of the estimation error of Luenberger observer to the nominal noise-free system (1), (2), defined in the form [11]

$$\dot{\boldsymbol{q}}_{e}(t) = \boldsymbol{A}\boldsymbol{q}_{e}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{J}\boldsymbol{C}(\boldsymbol{q}(t) - \boldsymbol{q}_{e}(t))\,, \tag{86}$$

Introducing $\boldsymbol{e}(t) = \boldsymbol{q}(t) - \boldsymbol{q}_{e}(t)$, then with (1), (2) and (86) it yields

$$\begin{aligned} \dot{\boldsymbol{e}}(t) &= \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{F}\boldsymbol{f}(t) + \boldsymbol{E}\boldsymbol{d}(t) - \boldsymbol{A}\boldsymbol{q}_{e}(t) - \boldsymbol{B}\boldsymbol{u}(t) - \boldsymbol{J}\boldsymbol{C}(\boldsymbol{q}(t) - \boldsymbol{q}_{e}(t)) \\ &= (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}(t) + \boldsymbol{F}\boldsymbol{f}(t) + \boldsymbol{E}\boldsymbol{d}(t) \end{aligned} \tag{87}$$

$$\widetilde{\boldsymbol{e}}(s) = (s\boldsymbol{I}_{n} - (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C}))^{-1}(\boldsymbol{F}\widetilde{\boldsymbol{f}}(s) + \boldsymbol{E}\widetilde{\boldsymbol{d}}(s) \tag{88}$$

respectively, where $\widetilde{\boldsymbol{e}}(s)$, $\widetilde{\boldsymbol{f}}(s)$, $\widetilde{\boldsymbol{d}}(s)$ stand for the Laplace transform of the n dimensional observer error vector, the m dimensional fault vector and the p dimensional disturbance vector.

Designing the fault residuals as

$$\boldsymbol{r}(t) = \boldsymbol{V}^{-1}\boldsymbol{C}\boldsymbol{e}(t)\,, \quad \widetilde{\boldsymbol{r}}(s) = \boldsymbol{V}^{-1}\boldsymbol{C}\widetilde{\boldsymbol{e}}(s)\,, \tag{89}$$

then the residual transfer function matrices of the fault and the disturbance are

$$\boldsymbol{G}_{f}(s) = \boldsymbol{V}^{-1}\boldsymbol{C}(s\boldsymbol{I}_{n} - (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C}))^{-1}\boldsymbol{F}\,, \tag{90}$$

$$\boldsymbol{G}_{d}(s) = \boldsymbol{V}^{-1}\boldsymbol{C}(s\boldsymbol{I}_{n} - (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C}))^{-1}\boldsymbol{E}\,. \tag{91}$$

It is evident that with $\boldsymbol{J}$ of the structure (84), and designed by (57), $\boldsymbol{G}_{f}(s)$ is an unitary transfer function matrix with optimized singular values related properties.

6. Illustrative Example

The considered unstable system is represented by the model (1), (2) with the parameters

$$\boldsymbol{A}=\begin{bmatrix} 1.380 & -0.208 & 6.175 & -1.576\\ 0.581 & -4.290 & 0.000 & 0.675\\ 1.067 & 4.273 & -6.654 & 5.893\\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix} 0.000 & 0.000 & 0.000\\ 5.679 & -1.000 & 0.000\\ 1.136 & -3.146 & 1.324\\ 1.136 & 0.000 & 3.496 \end{bmatrix},\ \boldsymbol{E}=\begin{bmatrix} 1.400\\ 1.504\\ 2.233\\ 0.610 \end{bmatrix},$$

$$\boldsymbol{B}=\boldsymbol{F},\ \boldsymbol{C}=\begin{bmatrix} 4 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 \end{bmatrix},\ \boldsymbol{V}=\boldsymbol{C}\boldsymbol{F}=\begin{bmatrix} 1.136 & -3.146 & 1.324\\ 1.136 & 0.000 & 3.496\\ 5.679 & -1.000 & 0.000 \end{bmatrix},\ \det(\boldsymbol{V})\neq 0.$$

The parameters of the matrix $\boldsymbol{T}$ were computed as follows

$$\boldsymbol{V}^{-1}\boldsymbol{C}=\begin{bmatrix} -0.2331 & 0.1833 & -0.0583 & 0.0221\\ -1.3237 & 0.0411 & -0.3309 & 0.1253\\ 0.0757 & -0.0596 & 0.0189 & 0.2789 \end{bmatrix},$$

$$\boldsymbol{F}^{\perp}=\begin{bmatrix} -1.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix},$$

that is

$$\boldsymbol{T}=\begin{bmatrix} -0.2331 & 0.1833 & -0.0583 & 0.0221\\ -1.3237 & 0.0411 & -0.3309 & 0.1253\\ 0.0757 & -0.0596 & 0.0189 & 0.2789\\ -1.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}.$$

Computing (18) and separating the blocks of the matrix $\boldsymbol{A}^{\circ}$ give the results

$$\boldsymbol{A}_{o11}=\begin{bmatrix} -6.1156 & 4.1078 & -0.9991\\ -11.1342 & 19.0383 & -8.0335\\ 8.6811 & -3.7656 & -1.2707 \end{bmatrix},\quad \boldsymbol{A}_{o12}=\begin{bmatrix} -3.8116\\ -21.0651\\ 2.7614 \end{bmatrix},$$

$$\boldsymbol{A}_{o21}=\begin{bmatrix} -4.0432 & 19.2186 & -2.6660 \end{bmatrix},\quad \boldsymbol{A}_{o22}=\begin{bmatrix} -23.3200 \end{bmatrix}$$

and so, choosing $s_o=22$, it was obtained using (18) and (61) that

$$\boldsymbol{L}^{o}=\begin{bmatrix} -2.3040 & 0.5868 & 3.1405\\ -13.0844 & 2.6574 & 0.1252\\ 1.1328 & 5.5004 & 0.2018\\ -6.1750 & 1.5760 & 0.2080 \end{bmatrix},\ \boldsymbol{M}=\begin{bmatrix} 6.1750 & -1.5760 & -0.2080\\ 0.0000 & 0.6750 & 17.7100\\ 15.3460 & 5.8930 & 4.2730\\ 1.3430 & 19.8960 & 4.2730 \end{bmatrix},$$

$$\boldsymbol{N}=\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ -1 & 0 & 0\\ 0 & -1 & 0 \end{bmatrix},\ \boldsymbol{J}=\begin{bmatrix} -6.1750 & 1.5760 & 0.2080\\ 0.0000 & -0.6750 & -18.7100\\ -16.3460 & -5.8930 & -4.2730\\ -1.3430 & -20.8960 & -4.2730 \end{bmatrix},$$

respectively. Thus, constructing the observer system matrix

$$\boldsymbol{A}_e=\boldsymbol{A}-\boldsymbol{J}\boldsymbol{C}=\begin{bmatrix} -23.3200 & 0.0000 & 0.0000 & 0.0000\\ 0.5810 & -23.0000 & 0.0000 & 0.0000\\ -64.3170 & 0.0000 & -23.0000 & 0.0000\\ -5.3240 & 0.0000 & 0.0000 & -23.0000 \end{bmatrix},$$

it is evident that the eigenvalues spectrum of $\boldsymbol{A}_e$ is

$$\rho(\boldsymbol{A}_e)=\{\ -23.32 \quad -23.00 \quad -23.00 \quad -23.00\ \}$$

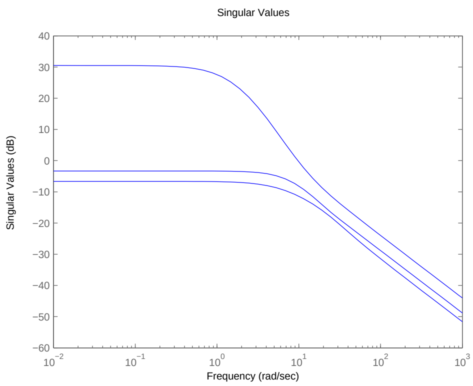

Figure 1. Singular values of the original system.

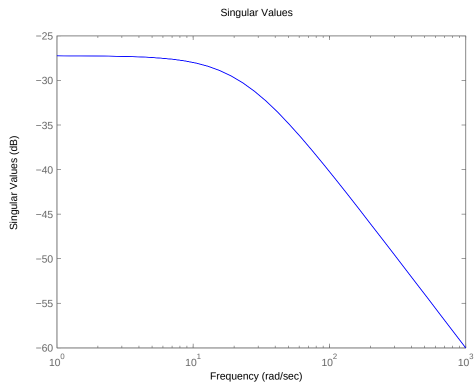

Figure 2. Singular values of the unitary approximation of the system.

and the steady-state value of the residual transfer function matrices are

$$\boldsymbol{G}_f(0) = \boldsymbol{V}^{-1}\boldsymbol{C}\boldsymbol{A}_e^{-1}\boldsymbol{F} = -\begin{bmatrix} 0.0435 & 0.0000 & 0.0000 \\ 0.0000 & 0.0435 & 0.0000 \\ 0.0000 & 0.0000 & 0.0435 \end{bmatrix},$$

$$\boldsymbol{G}_d(0) = \boldsymbol{V}^{-1}\boldsymbol{C}\boldsymbol{A}_e^{-1}\boldsymbol{E} = -\begin{bmatrix} 0.0027 \\ 0.0027 \\ -0.0517 \end{bmatrix}.$$

For completeness it was verified that in sense of Lyapunov stability there exist the positive definite matrix

$$\boldsymbol{P} = \begin{bmatrix} 3.5370 & 0.2172 & -1.1139 & 0.1226 \\ 0.2172 & 0.5332 & -0.1550 & -0.0333 \\ -1.1139 & -0.1550 & 0.8587 & -0.1443 \\ 0.1226 & -0.0333 & -0.1443 & 0.6646 \end{bmatrix}.$$

such that (69)-(71) are affirmative.

Note, using the coordinate transformation defined by the transform matrix (8), the block matrix $\boldsymbol{A}_{o22}$ in the matrix structure (18) is uncontrollable in the closed-loop structure (52)-(54) which describe the equivalent unitary systems. That means, the eigenvalues of $\boldsymbol{A}_{o22}$ determine the unprescribed subset of singular eigenvalues of $\rho(\boldsymbol{A}_e)$ (compare the value $\boldsymbol{A}_{o22} = -23.3200$ and the first singular value of $\rho(\boldsymbol{A}_e)$ in this example).

The results are closer to the theoretical expectations, and are simpler than those that have been published in [22].

7. Concluding Remarks

Based on the singular value approach a simpler method, but of the same precision, for unitary approximation of unknown fault transfer function matrix for continuous-time linear systems is introduced in the paper. Presented version is derived in terms numerical procedures to manipulate dynamics of the residual fault transfer function matrix. Formulated in sense of the second Lyapunov method and expressed through LMIs, stability conditions guaranteing the asymptotic convergence of unitary system observer state are derived over and above. The numerical simulation results show very good approximation performances.

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