Robust state estimation and control for nonlinear system with uncertain parameters

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Abstract. The paper focuses on robust state estimation and control for nonlinear system with uncertain parameters. In particular, the problem is oriented towards a practical application for a laboratory three-tank system. The proposed approach starts with a general description of the system and assumptions regarding uncertain parameters and a nonlinear function. The subsequent part of the paper is concerned with the design of the robust observer and controller for nonlinear systems. To confirm the performance of the proposed approaches, the final part presents results obtained for the laboratory three-tank system.

1. Introduction

The problem of designing a robust observer and a state feedback control to guarantee the global stability of a nonlinear system with uncertain parameters has received considerable attention during the last three decades. There are meany technique for state estimation and control for nonlinear system, for example: extended Kalman filter, adaptive technique, sliding mode, LPV technique, neural-networks, fuzzy rules base on Takagi–Sugeno etc. [10, 7, 11, 3, 16, 22, 20, 13, 9, 2]. Irrespective of the large spectrum of available methods non of them is universal to be suitable for arbitrary nonlinear systems. Thus, the complexity of the nonlinear systems, in particular multi–input multioutput system forces the development of new methods. It can be perceived as a generalization of the approaches proposed for Lipschitz systems as well as its recent one-sided counterpart. The proposed approach provides a novel output feedback control strategy that is divide into two traditional steps, i.e., observer design and controller design. Firstly, a robust observer design strategy is described that can tackle both parametric uncertainty and additive external disturbances. Subsequently, the robust control strategy is portrayed that follows general ideas employed for the observer design strategy.

Note that the proposed strategy is a preliminary step to designing robust fault-tolerant control scheme[22]. However, this is beyond the scope of this paper.

The paper is organised as follows. Section 2 introduces a description of the system along with assumptions regarding the nonlinearities. Whilst section 3 describes the concept of designing the robust observer. The subsequent section 4 presents the robust control framework. The final part of the paper is concerned with an illustrative example and conclusions.

2. A general description of system

Let us consider a non-linear system:

$$\boldsymbol{x}_{k+1} = (\boldsymbol{A} + \Delta\boldsymbol{A})\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{W}_1\boldsymbol{w}_k, \tag{1}$$

$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{W}_2\boldsymbol{w}_k \tag{2}$$

where matrices $\boldsymbol{A}$, $\boldsymbol{B}$ define the nominal model and $\Delta\boldsymbol{A}$ represent admissible uncertainties which are assumed to be of the form

$$\left[\Delta\boldsymbol{A}\right] = \boldsymbol{H}\boldsymbol{\mathcal{F}}\left[\boldsymbol{E}_A\right] \tag{3}$$

where $\boldsymbol{H}$, $\boldsymbol{E}_A$ are now constant matrices of compatible dimensions, $\boldsymbol{\mathcal{F}}$ is an unknown matrix satisfies

$$\boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}} \preceq \boldsymbol{I}, \tag{4}$$

where $\boldsymbol{x}_k \in \mathbb{X} \subset \mathbb{R}^n$ is the state, $\boldsymbol{u}_k \in \mathbb{U} \in \mathbb{R}^r$ stands for the input, $\boldsymbol{y}_k \in \mathbb{R}^m$ denotes the output, $\boldsymbol{g}\left(\boldsymbol{x}_k\right)$ is non-linear function with respect to $\boldsymbol{x}_k$ and $\boldsymbol{u}_k$ $\boldsymbol{w}_k \in l_2$ is a an exogenous disturbance vector and $\boldsymbol{W}_1 \in \mathbb{R}^{n\times n}$, $\boldsymbol{W}_2 \in \mathbb{R}^{m\times n}$ stand for its distribution matrices while

$$l_2 = \left\{\mathbf{w} \in \mathbb{R}^n \middle|\ \|\mathbf{w}\|_{l_2} < +\infty\right\}, \|\mathbf{w}\|_{l_2} = \left(\sum_{k=0}^{\infty}\|\boldsymbol{w}_k\|^2\right)^{\frac{1}{2}}. \tag{5}$$

Moreover, the following set of assumptions are imposed:

Assumption 1:

$$\boldsymbol{g}\left(\boldsymbol{x}\right)^T\boldsymbol{x} \le \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}, \quad \boldsymbol{M} \in \mathbb{M}. \tag{6}$$

Assumption 2:

$$\boldsymbol{g}\left(\boldsymbol{x}\right)^T\boldsymbol{g}\left(\boldsymbol{x}\right) \le \boldsymbol{x}^T\boldsymbol{M}^T\boldsymbol{M}\boldsymbol{x}, \quad \boldsymbol{M} \in \mathbb{M}. \tag{7}$$

Assumption 3: There exists a matrix $\boldsymbol{M}$ such that

$$\left(\boldsymbol{g}\left(\boldsymbol{a}\right) - \boldsymbol{g}\left(\boldsymbol{b}\right)\right)^T\left(\boldsymbol{a} - \boldsymbol{b}\right) \le \left(\boldsymbol{a} - \boldsymbol{b}\right)^T\boldsymbol{M}(\boldsymbol{a} - \boldsymbol{b}), \quad \forall\boldsymbol{a}, \boldsymbol{b} \in \mathbb{X}. \tag{8}$$

Assumption 4: There exists a matrix $\boldsymbol{M}$ such that

$$\left(\boldsymbol{g}\left(\boldsymbol{a}\right) - \boldsymbol{g}\left(\boldsymbol{b}\right)\right)^T\left(\boldsymbol{g}\left(\boldsymbol{a}, \boldsymbol{u}\right) - \boldsymbol{g}\left(\boldsymbol{b}, \boldsymbol{u}\right)\right) \le \left(\boldsymbol{a} - \boldsymbol{b}\right)^T\boldsymbol{M}^T\boldsymbol{M}(\boldsymbol{a} - \boldsymbol{b}), \quad \forall\boldsymbol{a}, \boldsymbol{b} \in \mathbb{X}. \tag{9}$$

The paper extends the idea of one-sided Lipschitz condition [19, 25] by imposing *Assumption 3*. *Assumption 4* extends the usual Lipschitz condition. Using the Differential Mean Value Theorem (DMVT) [24], it is possible to describe Lipschitz constant matrices by supremum of the norm of Jacobian of the function.

$$\boldsymbol{g}\left(\boldsymbol{a}\right) - \boldsymbol{g}\left(\boldsymbol{b}\right) = \boldsymbol{M}_x(\boldsymbol{a} - \boldsymbol{b}), \tag{10}$$

with

$$\boldsymbol{M}_{x,u} = \left[\left(\frac{\partial g_1}{\partial x}(\boldsymbol{c_1}, \boldsymbol{u})\right)^T \quad \ldots \quad \left(\frac{\partial g_n}{\partial x}(\boldsymbol{c_n}, \boldsymbol{u})\right)^T\right]^T, \tag{11}$$

where $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n \in \mathrm{Co}(\boldsymbol{a}, \boldsymbol{b})$, $\boldsymbol{c}_i \neq \boldsymbol{a}$, $\boldsymbol{c}_i \neq \boldsymbol{b}$, $i = 1, \ldots, n$. Assuming that

$$\bar{a}_{i,j} \ge \frac{\partial g_i}{\partial x_j} \ge \underline{a}_{i,j}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n, \tag{12}$$

it is clear that:

$$\mathbb{M}_x = \left\{\boldsymbol{M} \in \mathbb{R}^{n\times n} | \bar{a}_{i,j} \ge m_{x,i,j} \ge \underline{a}_{i,j},\ i, j = 1, \ldots, n,\right\} \tag{13}$$

It is worthwhile that, if $\boldsymbol{M}^T\boldsymbol{M} = \gamma^2\boldsymbol{I}$, then *Assumption 3* becomes a usual Lipschitz condition [1, 15, 18, 17], with γ being a Lipschitz constant. This property makes the proposed employed strategy more general than those presented in the literature [1, 15, 18, 17].

3. Design of the state observer

The main objective of this section is to present the design procedure of the robust observer that will be able to estimate all state vector for proposed nonlinear system. For that purpose, the following structure is proposed

$$\hat{\boldsymbol{x}}_{k+1} = \boldsymbol{A}\hat{\boldsymbol{x}}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{g}\left(\hat{\boldsymbol{x}}_k\right) + \boldsymbol{K}_o(\boldsymbol{y}_k - \boldsymbol{C}\hat{\boldsymbol{x}}_k), \tag{14}$$

while the state estimation error is given by

$$\boldsymbol{e}_{k+1} = \boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1} = \tilde{\boldsymbol{A}}\boldsymbol{e}_k + \tilde{\boldsymbol{G}}\boldsymbol{x}_k + \boldsymbol{s}_k + \tilde{\boldsymbol{W}}\boldsymbol{w}_k, \tag{15}$$

where

$$\begin{aligned} &\tilde{\boldsymbol{A}} = (\boldsymbol{A} - \boldsymbol{K}_o\boldsymbol{C}), \quad \tilde{\boldsymbol{G}} = (\boldsymbol{H}\boldsymbol{\mathcal{F}}\boldsymbol{E}_A), \quad \boldsymbol{s}_k = \boldsymbol{g}\left(\boldsymbol{x}_k\right) - \boldsymbol{g}\left(\hat{\boldsymbol{x}}_k\right) \\ &\tilde{\boldsymbol{W}} = (\boldsymbol{W}_1 - \boldsymbol{K}_o\boldsymbol{W}_2). \end{aligned} \tag{16}$$

Theorem 1. *For a prescribed disturbance attenuation level* $\mu > 0$*, the observer design for the system* (15) *is solvable if there exist* $\boldsymbol{N}$*,* $\boldsymbol{U}$*,* $\boldsymbol{P} \succ \boldsymbol{0}$*,* $\alpha > 0$*,* $\beta > 0$*,* $\epsilon > 0$ *such that the following condition is satisfied:*

$$\begin{bmatrix}
\boldsymbol{I}-\boldsymbol{P}+\alpha(\boldsymbol{M}+\boldsymbol{M}^T) & \boldsymbol{0} & -\alpha\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{A}^T\boldsymbol{U}^T-\boldsymbol{C}^T\boldsymbol{N}^T & \boldsymbol{M}^T\boldsymbol{U}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & -\boldsymbol{P}+\epsilon\boldsymbol{E}_A^T\boldsymbol{E}_A^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}^T\boldsymbol{U}^T & \boldsymbol{0} \\
-\alpha\boldsymbol{I} & \boldsymbol{0} & -\beta\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{U}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\mu^2\boldsymbol{I} & \boldsymbol{W}_1^T\boldsymbol{U}^T-\boldsymbol{W}_2^T\boldsymbol{N}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{U}\boldsymbol{A}-\boldsymbol{N}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{U} & \boldsymbol{U}\boldsymbol{W}_1-\boldsymbol{N}\boldsymbol{W}_2 & \boldsymbol{P}-\boldsymbol{U}-\boldsymbol{U}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{U}\boldsymbol{M} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \beta\boldsymbol{I}-\boldsymbol{U}-\boldsymbol{U}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\epsilon\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\epsilon\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\epsilon\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\epsilon\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{U}\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\epsilon\boldsymbol{I} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\epsilon\boldsymbol{I}
\end{bmatrix} \prec 0. \tag{17}$$

Proof. The problem of $\mathcal{H}_\infty$ observer design is to determine the matrix $\boldsymbol{K}_o$ such that[12, 14, 23]

$$\lim_{k\to\infty} \boldsymbol{e}_k = \boldsymbol{0} \quad \text{for } \boldsymbol{w}_k = \boldsymbol{0} \tag{18}$$

$$\|\boldsymbol{e}_k\|_{l_2} \leq \mu\|\boldsymbol{w}_k\|_{l_2} \quad \text{for } \boldsymbol{w}_k \neq \boldsymbol{0},\, \boldsymbol{e}_0 = \boldsymbol{0}. \tag{19}$$

In order to settle the above problem it is sufficient to find a Lyapunov function V_k such that:

$$\Delta V_k + \boldsymbol{e}_k^T \boldsymbol{e}_k - \mu^2 \boldsymbol{w}_k^T \boldsymbol{w}_k < 0,\ k = 0, \ldots \infty, \tag{20}$$

where

$$\Delta V_k = V_{k+1} - V_k, \tag{21}$$
$$V_k = \boldsymbol{e}_k^T \boldsymbol{P} \boldsymbol{e}_k + \boldsymbol{x}_k^T \boldsymbol{P} \boldsymbol{x}_k, \tag{22}$$
$$\Delta V_k = \boldsymbol{e}_{k+1}^T \boldsymbol{P} \boldsymbol{e}_{k+1} - \boldsymbol{e}_k^T \boldsymbol{P} \boldsymbol{e}_k - \boldsymbol{x}_k^T \boldsymbol{P} \boldsymbol{x}_k. \tag{23}$$

Consequently, using (15)

$$\begin{aligned}
&\Delta V_k + \boldsymbol{e}_k^T \boldsymbol{e}_k - \mu^2 \boldsymbol{w}_k^T \boldsymbol{w}_k = \\
&\boldsymbol{e}_k^T \left(\tilde{\boldsymbol{A}}^T \boldsymbol{P} \tilde{\boldsymbol{A}} + \boldsymbol{I} - \boldsymbol{P}\right) \boldsymbol{e}_k + \boldsymbol{e}_k^T \left(\tilde{\boldsymbol{A}}^T \boldsymbol{P} \tilde{\boldsymbol{G}}\right) \boldsymbol{x}_k + \boldsymbol{e}_k^T \left(\tilde{\boldsymbol{A}}^T \boldsymbol{P}\right) \boldsymbol{s}_k + \boldsymbol{e}_k^T \left(\tilde{\boldsymbol{A}}^T \boldsymbol{P} \tilde{\boldsymbol{W}}\right) \boldsymbol{w}_k + \\
&\boldsymbol{x}_k^T \left(\tilde{\boldsymbol{G}}^T \boldsymbol{P} \tilde{\boldsymbol{A}}\right) \boldsymbol{e}_k + \boldsymbol{x}_k^T \left(\tilde{\boldsymbol{G}}^T \boldsymbol{P} \tilde{\boldsymbol{G}} - \boldsymbol{P}\right) \boldsymbol{x}_k + \boldsymbol{x}_k^T \left(\tilde{\boldsymbol{G}}^T \boldsymbol{P}\right) \boldsymbol{s}_k + \boldsymbol{x}_k^T \left(\tilde{\boldsymbol{G}}^T \boldsymbol{P} \tilde{\boldsymbol{W}}\right) \boldsymbol{w}_k + \\
&\boldsymbol{s}_k^T \left(\boldsymbol{P} \tilde{\boldsymbol{A}}\right) \boldsymbol{e}_k + \boldsymbol{s}_k^T \left(\boldsymbol{P} \tilde{\boldsymbol{G}}\right) \boldsymbol{x}_k + \boldsymbol{s}_k^T \left(\boldsymbol{P}\right) \boldsymbol{s}_k + \boldsymbol{s}_k^T \left(\boldsymbol{P} \tilde{\boldsymbol{W}}\right) \boldsymbol{w}_k + \\
&\boldsymbol{w}_k^T \left(\tilde{\boldsymbol{W}} \boldsymbol{P} \tilde{\boldsymbol{A}}\right) \boldsymbol{e}_k + \boldsymbol{w}_k^T \left(\tilde{\boldsymbol{W}} \boldsymbol{P} \tilde{\boldsymbol{G}}\right) \boldsymbol{x}_k + \boldsymbol{w}_k^T \left(\tilde{\boldsymbol{W}} \boldsymbol{P}\right) \boldsymbol{s}_k + \boldsymbol{w}_k^T \left(\tilde{\boldsymbol{W}}^T \boldsymbol{P} \tilde{\boldsymbol{W}} - \mu^2 \boldsymbol{I}\right) \boldsymbol{w}_k < \boldsymbol{0}.
\end{aligned} \tag{24}$$

By defining

$$\tilde{\boldsymbol{v}}_k = [\boldsymbol{e}_k^T, \boldsymbol{x}_k^T, \boldsymbol{s}_k^T, \boldsymbol{w}_k^T]^T, \tag{25}$$

it can be shown that (25) is equivalent to

$$\tilde{\boldsymbol{v}}_k^T \begin{bmatrix} \tilde{\boldsymbol{A}}^T \boldsymbol{P} \tilde{\boldsymbol{A}} + \boldsymbol{I} - \boldsymbol{P} & \tilde{\boldsymbol{A}}^T \tilde{\boldsymbol{G}} & \tilde{\boldsymbol{A}}^T \boldsymbol{P} & \tilde{\boldsymbol{A}}^T \boldsymbol{P} \tilde{\boldsymbol{W}} \\ \tilde{\boldsymbol{G}}^T \boldsymbol{P} \tilde{\boldsymbol{A}} & \tilde{\boldsymbol{G}}^T \boldsymbol{P} \tilde{\boldsymbol{G}} - \boldsymbol{P} & \tilde{\boldsymbol{G}}^T \boldsymbol{P} & \tilde{\boldsymbol{G}}^T \boldsymbol{P} \tilde{\boldsymbol{W}} \\ \boldsymbol{P} \tilde{\boldsymbol{A}} & \boldsymbol{P} \tilde{\boldsymbol{G}} & \boldsymbol{P} & \boldsymbol{P} \tilde{\boldsymbol{W}} \\ \tilde{\boldsymbol{W}}^T \boldsymbol{P} \tilde{\boldsymbol{A}} & \tilde{\boldsymbol{W}} \boldsymbol{P} \tilde{\boldsymbol{G}} & \tilde{\boldsymbol{W}}^T \boldsymbol{P} & \tilde{\boldsymbol{W}}^T \boldsymbol{P} \tilde{\boldsymbol{W}} - \mu^2 \boldsymbol{I} \end{bmatrix} \tilde{\boldsymbol{v}}_k \prec \boldsymbol{0}. \tag{26}$$

Analyzing assumption of non-linear function, it can be show that (8) is equivalent to

$$\boldsymbol{s}_k^T \boldsymbol{e}_k \leq \boldsymbol{e}_k^T \boldsymbol{M} \boldsymbol{e}_k = \frac{1}{2} \boldsymbol{e}_k^T (\boldsymbol{M} + \boldsymbol{M}^T) \boldsymbol{e}_k. \tag{27}$$

Inequality (27) can be written as

$$\frac{1}{2} \boldsymbol{e}_k^T (\boldsymbol{M} + \boldsymbol{M}^T) \boldsymbol{e}_k - \boldsymbol{s}_k^T \boldsymbol{e}_k \leq 0,$$

which is equivalent to

$$\frac{1}{2} \boldsymbol{e}_k^T (\boldsymbol{M} + \boldsymbol{M}^T) \boldsymbol{e}_k - \frac{1}{2} \boldsymbol{s}_k^T \boldsymbol{e}_k - \frac{1}{2} \boldsymbol{e}_k^T \boldsymbol{s}_k \geq \boldsymbol{0}. \tag{28}$$

Thus, for any $\alpha > 0$

$$\alpha \tilde{\boldsymbol{v}}_k^T \begin{bmatrix} (\boldsymbol{M} + \boldsymbol{M}^T) & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \tilde{\boldsymbol{v}}_k \geq \boldsymbol{0}. \tag{29}$$

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Similarly, for some β inequality (9) can be written as

$$\beta \boldsymbol{e}_k^T \boldsymbol{M}^T \boldsymbol{M} \boldsymbol{e}_k - \beta \boldsymbol{s}_k^T \boldsymbol{s}_k \geq \boldsymbol{0}. \tag{30}$$

Thus, for any $\beta > 0$

$$\beta \bar{\boldsymbol{v}}_k^T \begin{bmatrix} \boldsymbol{M}^T\boldsymbol{M} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \bar{\boldsymbol{v}}_k \geq \boldsymbol{0}. \tag{31}$$

Combining together (26), (29) and (31) gives

$$\begin{bmatrix} \tilde{\boldsymbol{A}}^T\boldsymbol{P}\tilde{\boldsymbol{A}} + \boldsymbol{I} - \boldsymbol{P} + \alpha(\boldsymbol{M}+\boldsymbol{M}^T) + \beta\boldsymbol{M}^T\boldsymbol{M} & \tilde{\boldsymbol{A}}^T\tilde{\boldsymbol{G}} & \tilde{\boldsymbol{A}}^T\boldsymbol{P} - \alpha\boldsymbol{I} & \tilde{\boldsymbol{A}}^T\boldsymbol{P}\tilde{\boldsymbol{W}} \\ \tilde{\boldsymbol{G}}^T\boldsymbol{P}\tilde{\boldsymbol{A}} & \tilde{\boldsymbol{G}}^T\boldsymbol{P}\tilde{\boldsymbol{G}} - \boldsymbol{P} & \tilde{\boldsymbol{G}}^T\boldsymbol{P} & \tilde{\boldsymbol{G}}^T\boldsymbol{P}\tilde{\boldsymbol{W}} \\ \boldsymbol{P}\tilde{\boldsymbol{A}} - \alpha\boldsymbol{I} & \boldsymbol{P}\tilde{\boldsymbol{G}} & \boldsymbol{P} - \beta\boldsymbol{I} & \boldsymbol{P}\tilde{\boldsymbol{W}} \\ \tilde{\boldsymbol{W}}^T\boldsymbol{P}\tilde{\boldsymbol{A}} & \tilde{\boldsymbol{W}}\boldsymbol{P}\tilde{\boldsymbol{G}} & \tilde{\boldsymbol{W}}^T\boldsymbol{P} & \tilde{\boldsymbol{W}}^T\boldsymbol{P}\tilde{\boldsymbol{W}} - \mu^2\boldsymbol{I} \end{bmatrix} \prec \boldsymbol{0} \tag{32}$$

Lemma 1. *The following statements are equivalent[6]:*

(i) There exists $\boldsymbol{X} \succ 0$ *such that*

$$\boldsymbol{V}^T\boldsymbol{X}\boldsymbol{V} - \boldsymbol{W} \prec 0 \tag{33}$$

(ii) There exists $\boldsymbol{X} \succ 0$ *such that*

$$\begin{bmatrix} -\boldsymbol{W} & \boldsymbol{V}^T\boldsymbol{U}^T \\ \boldsymbol{U}\boldsymbol{V} & \boldsymbol{X} - \boldsymbol{U} - \boldsymbol{U}^T \end{bmatrix} \prec 0. \tag{34}$$

Rewriting (32) as

$$\begin{bmatrix} \tilde{\boldsymbol{A}}^T \\ \tilde{\boldsymbol{G}}^T \\ \boldsymbol{I}^T \\ \tilde{\boldsymbol{W}}^T \end{bmatrix} \boldsymbol{P} \begin{bmatrix} \tilde{\boldsymbol{A}} & \tilde{\boldsymbol{G}} & \boldsymbol{I} & \tilde{\boldsymbol{W}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{I} - \boldsymbol{P} + \alpha(\boldsymbol{M}+\boldsymbol{M}^T) + \beta\boldsymbol{M}^T\boldsymbol{M} & \boldsymbol{0} & -\alpha\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} \\ -\alpha\boldsymbol{I} & \boldsymbol{0} & -\beta\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\mu^2\boldsymbol{I} \end{bmatrix} \prec \boldsymbol{0}, \tag{35}$$

and using *Lemma 1* to (35) yields

$$\begin{bmatrix} \boldsymbol{I} - \boldsymbol{P} + \alpha(\boldsymbol{M}+\boldsymbol{M}^T) & \boldsymbol{0} & -\alpha\boldsymbol{I} & \boldsymbol{0} & \tilde{\boldsymbol{A}}^T\boldsymbol{U}^T & \boldsymbol{M}^T\boldsymbol{U}^T \\ \boldsymbol{0} & -\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} & \tilde{\boldsymbol{G}}^T\boldsymbol{U}^T & \boldsymbol{0} \\ -\alpha\boldsymbol{I} & \boldsymbol{0} & -\beta\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\mu^2\boldsymbol{I} & \tilde{\boldsymbol{W}}^T\boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{U}\tilde{\boldsymbol{A}} & \boldsymbol{U}\tilde{\boldsymbol{G}} & \boldsymbol{U} & \boldsymbol{U}\tilde{\boldsymbol{W}} & \boldsymbol{P} - \boldsymbol{U} - \boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{U}\boldsymbol{M} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \beta\boldsymbol{I} - \boldsymbol{U} - \boldsymbol{U}^T \end{bmatrix} \prec \boldsymbol{0}, \tag{36}$$

with $\boldsymbol{U}\tilde{\boldsymbol{A}} = \boldsymbol{U}\boldsymbol{A} - \boldsymbol{U}\boldsymbol{K}_o\boldsymbol{C} = \boldsymbol{U}\boldsymbol{A} - \boldsymbol{N}\boldsymbol{C}$, $\boldsymbol{U}\tilde{\boldsymbol{W}} = \boldsymbol{U}\boldsymbol{W}_1 - \boldsymbol{U}\boldsymbol{K}_o\boldsymbol{W}_2 = \boldsymbol{U}\boldsymbol{W}_1 - \boldsymbol{N}\boldsymbol{W}_2$ and $\boldsymbol{U}\tilde{\boldsymbol{G}} = \boldsymbol{U}\boldsymbol{H}\boldsymbol{\mathcal{F}}\boldsymbol{E}_A$. Inequality (36) can be written in equivalent form

$$\begin{bmatrix} \boldsymbol{I}-\boldsymbol{P}+\alpha(\boldsymbol{M}+\boldsymbol{M}^T) & \boldsymbol{0} & -\alpha\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{A}^T\boldsymbol{U}^T-\boldsymbol{C}^T\boldsymbol{N}^T & \boldsymbol{M}^T\boldsymbol{U}^T \\ \boldsymbol{0} & -\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ -\alpha\boldsymbol{I} & \boldsymbol{0} & -\beta\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\mu^2\boldsymbol{I} & \boldsymbol{W}_1^T\boldsymbol{U}^T-\boldsymbol{W}_2^T\boldsymbol{N}^T & \boldsymbol{0} \\ \boldsymbol{U}\boldsymbol{A}-\boldsymbol{N}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{U} & \boldsymbol{U}\boldsymbol{W}_1-\boldsymbol{N}\boldsymbol{W}_2 & \boldsymbol{P}-\boldsymbol{U}-\boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{U}\boldsymbol{M} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \beta\boldsymbol{I}-\boldsymbol{U}-\boldsymbol{U}^T \end{bmatrix} +$$

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E}_A^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{F}}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\mathcal{F}}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{F}}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{F}}^T & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{F}}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{F}}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}^T\boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} + \quad (37)$$

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{F}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\mathcal{F}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{F}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{F}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{F}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E}_A & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \prec \boldsymbol{0},$$

Lemma 2. *Let $\boldsymbol{H}$, $\boldsymbol{E}$ be given real matrices of appropriate dimensions and $\boldsymbol{\mathcal{F}}$ satisfy $\boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}} \preceq \boldsymbol{I}$. Then for any $\epsilon > 0$ the following holds*

$$\boldsymbol{H}\boldsymbol{\mathcal{F}}\boldsymbol{E} + \boldsymbol{E}^T\boldsymbol{\mathcal{F}}^T\boldsymbol{H} \preceq \epsilon\boldsymbol{H}\boldsymbol{H}^T + \frac{1}{\epsilon}\boldsymbol{E}^T\boldsymbol{E}. \tag{38}$$

Applying *Lemma 2* to (37), we obtain

$$\begin{bmatrix} \boldsymbol{I}-\boldsymbol{P}+\alpha(\boldsymbol{M}+\boldsymbol{M}^T) & \boldsymbol{0} & -\alpha\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{A}^T\boldsymbol{U}^T-\boldsymbol{C}^T\boldsymbol{N}^T & \boldsymbol{M}^T\boldsymbol{U}^T \\ \boldsymbol{0} & -\boldsymbol{P}+\epsilon\boldsymbol{E}_A^T\boldsymbol{E}_A^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ -\alpha\boldsymbol{I} & \boldsymbol{0} & -\beta\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\mu^2\boldsymbol{I} & \boldsymbol{W}_1^T\boldsymbol{U}^T-\boldsymbol{W}_2^T\boldsymbol{N}^T & \boldsymbol{0} \\ \boldsymbol{U}\boldsymbol{A}-\boldsymbol{N}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{U} & \boldsymbol{U}\boldsymbol{W}_1-\boldsymbol{N}\boldsymbol{W}_2 & \boldsymbol{P}-\boldsymbol{U}-\boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{U}\boldsymbol{M} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \beta\boldsymbol{I}-\boldsymbol{U}-\boldsymbol{U}^T \end{bmatrix} + \tag{39}$$

$$\epsilon^{-1}\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}^T\boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \prec \boldsymbol{0}.$$

Applying the Schur complements leads to (17), which completes the proof. □

4. Design of the state feedback controller

The main objective of this section is to present the design procedure of the robust controller for the proposed system. The controller will be designed in such a way that a predefined disturbance attenuation level with respect to the state of the system is achieved. To solve the above problem the following control scheme is proposed

$$\boldsymbol{u}_k = -\boldsymbol{K}_c\boldsymbol{x}_k. \tag{40}$$

Substituting (40) in to (1) gives

$$\boldsymbol{x}_{k+1} = \bar{\boldsymbol{A}}\boldsymbol{x}_k + \boldsymbol{g}\left(\boldsymbol{x}_k\right) + \bar{\boldsymbol{W}}\boldsymbol{w}_k \tag{41}$$

where

$$\bar{\boldsymbol{A}} = \left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c + \boldsymbol{H}\boldsymbol{\mathcal{F}}\boldsymbol{E}_A\right), \quad \bar{\boldsymbol{W}} = \boldsymbol{W}_1. \tag{42}$$

Theorem 2. *For a prescribed disturbance attenuation level $\mu > 0$, the controller design for the system (41) is solvable if there exist $\boldsymbol{N}$, $\boldsymbol{U}$, $\boldsymbol{P} \succ \boldsymbol{0}$, $\alpha > 0$, $\beta > 0$, $\epsilon > 0$ such that the following condition is satisfied:*

$$\left[\begin{array}{ccccc} \boldsymbol{I} - \boldsymbol{P} + \alpha(\boldsymbol{M} + \boldsymbol{M}^T) + \boldsymbol{H}\boldsymbol{H}^T & -\alpha\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{A}\boldsymbol{U} - \boldsymbol{B}\boldsymbol{N} & \boldsymbol{M}\boldsymbol{U} \\ -\alpha\boldsymbol{I} & -\beta\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{U} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\mu^2\boldsymbol{I} & \boldsymbol{W}_1\boldsymbol{U} & \boldsymbol{0} \\ \boldsymbol{U}^T\boldsymbol{A}^T - \boldsymbol{N}^T\boldsymbol{B}^T & \boldsymbol{U}^T & \boldsymbol{U}^T\boldsymbol{W}_1^T & \boldsymbol{P} - \boldsymbol{U} - \boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{U}^T\boldsymbol{M}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \beta\boldsymbol{I} - \boldsymbol{U} - \boldsymbol{U}^T \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{E}_A\boldsymbol{U} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{array}\right.$$

$$\left.\begin{array}{ccccc} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{U}^T\boldsymbol{E}_A^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ -\epsilon\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\epsilon\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\epsilon\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\epsilon\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\epsilon\boldsymbol{I} \end{array}\right] \prec 0. \tag{43}$$

Proof. The problem of $\mathcal{H}_\infty$ controller design is to determine the matrix $\boldsymbol{K}_c$ such that[21, 4]

$$\lim_{k\to\infty} \boldsymbol{x}_k = \boldsymbol{0} \quad \text{for } \boldsymbol{w}_k = \boldsymbol{0} \tag{44}$$

$$\|\boldsymbol{x}_k\|_{l_2} \leq \mu\|\boldsymbol{v}_k\|_{l_2} \quad \text{for } \boldsymbol{w}_k \neq \boldsymbol{0},\, \boldsymbol{x}_0 = \boldsymbol{0}. \tag{45}$$

In order to settle the above problem it is sufficient to find a Lyapunov function V_k such that:

$$\Delta V_k + \boldsymbol{x}_k^T\boldsymbol{x}_k - \mu^2\boldsymbol{w}_k^T\boldsymbol{w}_k < 0,\, k = 0, \ldots \infty, \tag{46}$$

where

$$\Delta V_k = V_{k+1} - V_k, \tag{47}$$

$$V_k = \boldsymbol{x}_k^T\boldsymbol{P}\boldsymbol{x}_k, \tag{48}$$

$$\Delta V_k = \boldsymbol{x}_{k+1}^T\boldsymbol{P}\boldsymbol{x}_{k+1} - \boldsymbol{x}_k^T\boldsymbol{P}\boldsymbol{x}_k, \tag{49}$$

Consequently, using (41)

$$\begin{aligned}
&\Delta V_k + \boldsymbol{x}_k^T \boldsymbol{x}_k - \mu^2 \boldsymbol{w}_k^T \boldsymbol{w}_k = \\
&\boldsymbol{x}_k^T \left(\bar{\boldsymbol{A}}^T \boldsymbol{P} \bar{\boldsymbol{A}} - \boldsymbol{P} + \boldsymbol{I}\right) \boldsymbol{x}_k + \boldsymbol{x}_k^T \left(\bar{\boldsymbol{A}}^T \boldsymbol{P}\right) \boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{x}_k^T \left(\bar{\boldsymbol{A}}^T \boldsymbol{P} \bar{\boldsymbol{W}}\right) \boldsymbol{w}_k + \\
&\boldsymbol{g}\left(\boldsymbol{x}_k\right)^T \left(\boldsymbol{P}\bar{\boldsymbol{A}}\right) \boldsymbol{x}_k + \boldsymbol{g}\left(\boldsymbol{x}_k\right)^T \left(\boldsymbol{P}\right) \boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{g}\left(\boldsymbol{x}_k\right)^T \left(\boldsymbol{P}\bar{\boldsymbol{W}}\right) \boldsymbol{w}_k + \\
&\boldsymbol{w}_k^T \left(\bar{\boldsymbol{W}}^T \boldsymbol{P} \bar{\boldsymbol{A}}\right) \boldsymbol{x}_k + \boldsymbol{w}_k^T \left(\bar{\boldsymbol{W}}^T \boldsymbol{P}\right) \boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{w}_k^T \left(\bar{\boldsymbol{W}}^T \boldsymbol{P} \bar{\boldsymbol{W}} - \mu^2 \boldsymbol{I}\right) \boldsymbol{w}_k < \boldsymbol{0}.
\end{aligned}$$

By defining

$$\boldsymbol{v}_k = [\boldsymbol{x}_k^T, \boldsymbol{g}\left(\boldsymbol{x}_k\right)^T, \boldsymbol{w}_k^T]^T, \tag{50}$$

it can be shown that (50) is equivalent to

$$\boldsymbol{v}_k^T \begin{bmatrix} \bar{\boldsymbol{A}}^T \boldsymbol{P} \bar{\boldsymbol{A}} + \boldsymbol{I} - \boldsymbol{P} & \bar{\boldsymbol{A}}^T \boldsymbol{P} & \bar{\boldsymbol{A}}^T \boldsymbol{P} \bar{\boldsymbol{W}} \\ \boldsymbol{P}\bar{\boldsymbol{A}} & \boldsymbol{P} & \boldsymbol{P}\boldsymbol{W} \\ \bar{\boldsymbol{W}}^T \boldsymbol{P} \bar{\boldsymbol{A}} & \bar{\boldsymbol{W}}^T \boldsymbol{P} & \bar{\boldsymbol{W}}^T \boldsymbol{P} \bar{\boldsymbol{W}} - \mu^2 \boldsymbol{I} \end{bmatrix} \boldsymbol{v}_k \prec \boldsymbol{0}. \tag{51}$$

Analyzing assumption of non-linear function, it can be show that (6) is equivalent to

$$\boldsymbol{g}\left(\boldsymbol{x}\right)^T \boldsymbol{x} \leq \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} = \frac{1}{2} \boldsymbol{x}^T (\boldsymbol{M} + \boldsymbol{M}^T) \boldsymbol{x}, \quad \boldsymbol{M} \in \mathbb{M}. \tag{52}$$

Inequality (52) can be written as

$$\frac{1}{2} \boldsymbol{x}^T (\boldsymbol{M} + \boldsymbol{M}^T) \boldsymbol{x} - \boldsymbol{g}\left(\boldsymbol{x}_k\right)^T \boldsymbol{x}_k \succeq 0, \quad \boldsymbol{M} \in \mathbb{M}, \tag{53}$$

which is equivalent to

$$\frac{1}{2} \boldsymbol{x}_k^T (\boldsymbol{M} + \boldsymbol{M}^T) \boldsymbol{x}_k - \frac{1}{2} \boldsymbol{g}\left(\boldsymbol{x}_k\right)^T \boldsymbol{x}_k - \frac{1}{2} \boldsymbol{x}_k^T \boldsymbol{g}\left(\boldsymbol{x}_k\right) \geq \boldsymbol{0}. \tag{54}$$

Thus, for any $\alpha > 0$

$$\alpha \boldsymbol{v}_k^T \begin{bmatrix} (\boldsymbol{M} + \boldsymbol{M}^T) & -\boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{v}_k \succeq \boldsymbol{0}, \quad \alpha > 0, \quad \boldsymbol{M} \in \mathbb{M}. \tag{55}$$

Similarly, for some β inequality (7) can be written as

$$\beta \boldsymbol{x}_k^T \boldsymbol{M}^T \boldsymbol{M} \boldsymbol{x}_k - \beta \boldsymbol{g}\left(\boldsymbol{x}_k\right)^T \boldsymbol{g}\left(\boldsymbol{x}_k\right) \succeq 0, \quad \beta > 0. \tag{56}$$

$$\beta \boldsymbol{v}_k^T \begin{bmatrix} \boldsymbol{M}^T \boldsymbol{M} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{v}_k \succeq \boldsymbol{0}. \tag{57}$$

Combining together (51), (55) and (57) gives

$$\begin{bmatrix} \bar{\boldsymbol{A}}^T \boldsymbol{P} \bar{\boldsymbol{A}} + \boldsymbol{I} - \boldsymbol{P} + \alpha(\boldsymbol{M} + \boldsymbol{M}^T) + \beta \boldsymbol{M}^T \boldsymbol{M} & \bar{\boldsymbol{A}}^T \boldsymbol{P} - \alpha \boldsymbol{I} & \bar{\boldsymbol{A}}^T \boldsymbol{P} \bar{\boldsymbol{W}} \\ \boldsymbol{P}\bar{\boldsymbol{A}} - \alpha \boldsymbol{I} & \boldsymbol{P} - \beta \boldsymbol{I} & \boldsymbol{P}\boldsymbol{W} \\ \bar{\boldsymbol{W}}^T \boldsymbol{P} \bar{\boldsymbol{A}} & \bar{\boldsymbol{W}}^T \boldsymbol{P} & \bar{\boldsymbol{W}}^T \boldsymbol{P} \bar{\boldsymbol{W}} - \mu^2 \boldsymbol{I} \end{bmatrix} \prec \boldsymbol{0} \tag{58}$$

Rewriting (58) as

$$\begin{bmatrix}\bar{\boldsymbol{A}}^T\\ \boldsymbol{I}^T\\ \bar{\boldsymbol{W}}^T\end{bmatrix}\boldsymbol{P}\begin{bmatrix}\bar{\boldsymbol{A}} & \boldsymbol{I} & \bar{\boldsymbol{W}}\end{bmatrix}\begin{bmatrix}\boldsymbol{I}-\boldsymbol{P}+\alpha(\boldsymbol{M}+\boldsymbol{M}^T)+\beta\boldsymbol{M}^T\boldsymbol{M} & -\alpha\boldsymbol{I} & \boldsymbol{0}\\ -\alpha\boldsymbol{I} & -\beta\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & -\mu^2\boldsymbol{I}\end{bmatrix}\prec\boldsymbol{0}, \tag{59}$$

and using a transposed version *Lemma 1* to (59) yields

$$\begin{bmatrix}\boldsymbol{I}-\boldsymbol{P}+\alpha(\boldsymbol{M}+\boldsymbol{M}^T) & -\alpha\boldsymbol{I} & \boldsymbol{0} & \bar{\boldsymbol{A}}\boldsymbol{U} & \boldsymbol{M}\boldsymbol{U}\\ -\alpha\boldsymbol{I} & -\beta\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{U} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & -\mu^2\boldsymbol{I} & \bar{\boldsymbol{W}}\boldsymbol{U} & \boldsymbol{0}\\ \boldsymbol{U}^T\bar{\boldsymbol{A}}^T & \boldsymbol{U}^T & \boldsymbol{U}^T\bar{\boldsymbol{W}}^T & \boldsymbol{P}-\boldsymbol{U}-\boldsymbol{U}^T & \boldsymbol{0}\\ \boldsymbol{U}^T\boldsymbol{M}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \beta\boldsymbol{I}-\boldsymbol{U}-\boldsymbol{U}^T\end{bmatrix}\prec\boldsymbol{0}, \tag{60}$$

with $\bar{\boldsymbol{A}}\boldsymbol{U}=\boldsymbol{A}\boldsymbol{U}-\boldsymbol{B}\boldsymbol{K}_c\boldsymbol{U}+\boldsymbol{H}\boldsymbol{\mathcal{F}}\boldsymbol{E}_A\boldsymbol{U}=\boldsymbol{A}\boldsymbol{U}-\boldsymbol{B}\boldsymbol{N}+\boldsymbol{H}\boldsymbol{\mathcal{F}}\boldsymbol{E}_A\boldsymbol{U}$ and $\bar{\boldsymbol{W}}\boldsymbol{U}=\boldsymbol{W}_1\boldsymbol{U}$. Written (60) similarly as inequality (36) and then applying *Lemma 2* and Schur complements leads to (43),which completes the proof. □

5. Case study

To verify the proposed approach, it was implemented for real three-tank system portrayed in Fig. 1 [5, 14, 8]. The system is composed of three separate tanks with variable cross-area

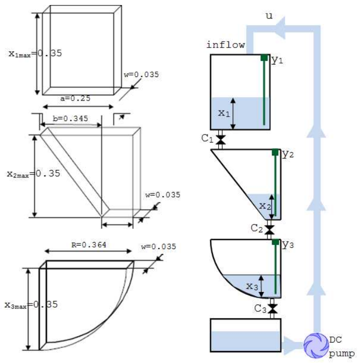

Figure 1. Three tank system

sections, where nonlinearities are imposed from shape of the second and third tank. The model was identified and following system matrices were obtained as well as nonlinear function of the system:

$$\boldsymbol{A}=\begin{bmatrix}0.9982 & 0 & 0\\ 0.0018 & 0.9973 & 0\\ 0 & 0.0025 & 0.9972\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}11.1400\\ 0\\ 0\end{bmatrix},\quad \boldsymbol{W}_1=\begin{bmatrix}0.0100 & 0 & 0\\ 0 & 0.0100 & 0\\ 0 & 0 & 0.0100\end{bmatrix}$$

$$\boldsymbol{H}=\begin{bmatrix}0.1\\ 0.1\\ 0\end{bmatrix},\quad \boldsymbol{E}=\begin{bmatrix}0.2\\ 0\\ 0\end{bmatrix}^T,\quad \boldsymbol{W}_1=0.05\boldsymbol{C},\quad \boldsymbol{g}\left(\boldsymbol{x}_k\right)=\begin{bmatrix}\frac{1}{\beta_1(x_1)}C_1x_{1,k}^{\alpha_1}\\ \frac{1}{\beta_2(x_{2,k})}C_1x_{1,k}^{\alpha_1}-\frac{1}{\beta_2(x_{2,k})}C_2x_{2,k}^{\alpha_2}\\ \frac{1}{\beta_3(x_{3,k})}C_2x_{2,k}^{\alpha_2}-\frac{1}{\beta_3(x_{3,k})}C_3x_{3,k}^{\alpha_3}\end{bmatrix}.$$

where: $\beta_i(x_i)$ is cross-sectional area of i-th tank, α_i denotes the flow coefficient parameter and C_i stands for i-th valve open level, the last two parameters were obtain through identification process. The numerical parameters obtained from identification along with operating ranges are: $0 \leq u \leq 1.5e^{-4}\left[\frac{m^3}{s}\right]$, $0.01 \leq x_{1-3} \leq 0.35[m]$ with $C_1 = 1.6721\mathrm{e}^{-4}$, $C_2 = 1.8804\mathrm{e}^{-4}$, $C_3 = 1.6637\mathrm{e}^{-4}$, $\alpha_1 = 0.4334$, $\alpha_2 = 0.4134$, $\alpha_2 = 0.3673$.

By varying each component of $\boldsymbol{x}_k$ within its possible domain, defined by the physical constraints of the three-tank system, it is possible to obtain the matrices

$$\boldsymbol{M}_{min} = \begin{bmatrix} -1.0094 & 0 & 0 \\ -0.0010 & -1.0457 & 0 \\ 0 & -0.0014 & -1.0593 \end{bmatrix}, \quad \boldsymbol{M}_{max} = \begin{bmatrix} -0.9997 & 0 & 0 \\ 0.0238 & -0.9968 & 0 \\ 0 & 0.0224 & -0.9979 \end{bmatrix}$$

which contain all elements $\bar{a}_{i,j}$ and $\underline{a}_{i,j}$, $i, j = 1, \ldots, n$ defining (13) respectively. Afterwards, taking into account all the possible combinations of elements $\boldsymbol{M}_{min}$ and $\boldsymbol{M}_{max}$, 32 matrices $\boldsymbol{M}_i$ are obtained.

For the purpose of further comparative study, two cases were employed:

Case SC1: It will be assumed that the whole state vector is directly measured

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{K}_o = \begin{bmatrix} 0.7382 & -0.0008 & -0.0000 \\ 0.0012 & 0.7402 & 0.0005 \\ -0.0000 & 0.0030 & 0.7405 \end{bmatrix}, \quad \mu = 0.5623, \quad \boldsymbol{x}_0 = \boldsymbol{0}, \quad \boldsymbol{z}_0 = \boldsymbol{0}$$

Case SC2: It will be assumed that some state variables are not directly measured

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{K}_o = \begin{bmatrix} 0.6977 & -0.0000 \\ 0.0002 & 0.0026 \\ -0.0000 & 0.7056 \end{bmatrix}, \quad \mu = 0.5623, \quad \boldsymbol{x}_0 = \boldsymbol{0}, \quad \boldsymbol{z}_0 = \boldsymbol{0}$$

Figure 2–3 present the results obtained for the proposed observer structure for case SC1. From these results, it is evident that the state estimation is performed with a good quality. Figure 3 also shows the state estimation for second state which is not directly measured. This corresponds to case SC2. From obtained results, it is evident that the state estimation is performed with a good quality. Of course there are differences between the quality of the estimation for the case SC1 and SC2 for state $\boldsymbol{x}_2$. Finally, Fig. 4 shows the system performance for the proposed control scheme. The goal is to control the liquid level in an upper tank. The reference signal (red solid line) is the target that has to be achieved by the controller. The robust controller was designed with (43) and $\mu = 0.5623$. The obtained robust controller is $K_c = [0.05731, 0.00076, 0.000001]$. The state vector is derived by the observer for SC2. From these results it can be seen that the state converges to the required setpoint.

6. Conclusions

The main objective of this paper was to propose problem of robust state estimation and robust control for nonlinear systems. The proposed design procedure for robust observer and robust controller is relatively simple and boils down to solving a set of linear matrix inequalities. The proposed approach can be efficiently implemented to the real-time system. The final part presents comprehensive case study of the three-tank system. The results clearly exhibit the high performance of the proposed scheme.

Acknowledgments

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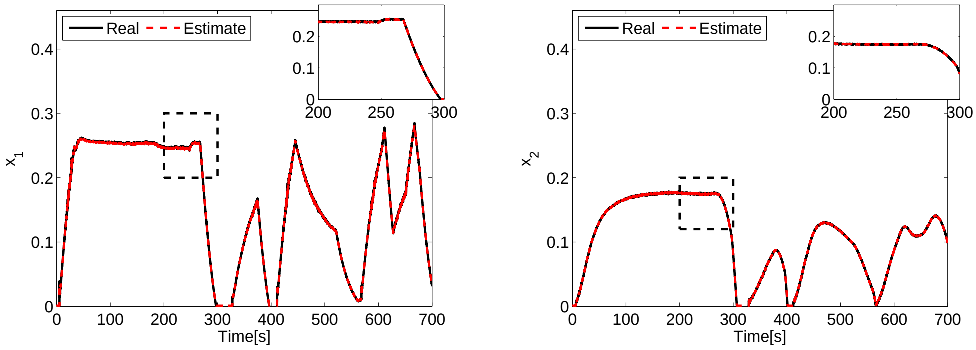

Figure 2. SC1: State variable $\boldsymbol{x}_1$ (left) $\boldsymbol{x}_2$ (right) (black line) and its estimates (red line) for $t = 0, \ldots, 700[s]$

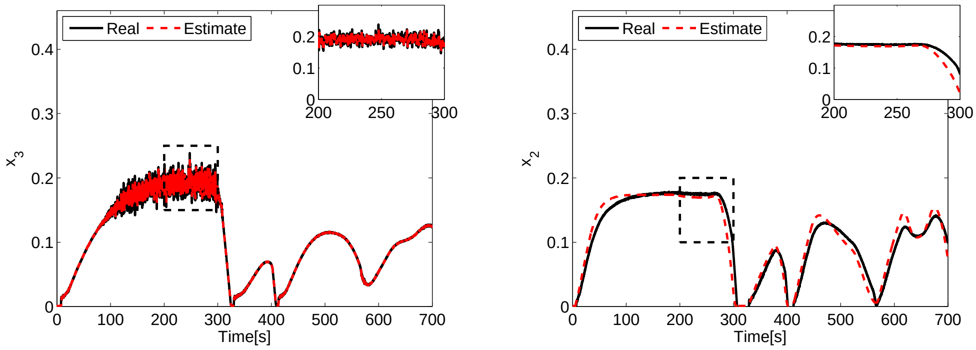

Figure 3. SC1(left): State variable $\boldsymbol{x}_3$ (black line) and its estimates (red line) for $t = 0, \ldots, 700[s]$. SC2(right): State variable $\boldsymbol{x}_2$ (black line) and its estimates (red line) for $t = 0, \ldots, 700[s]$

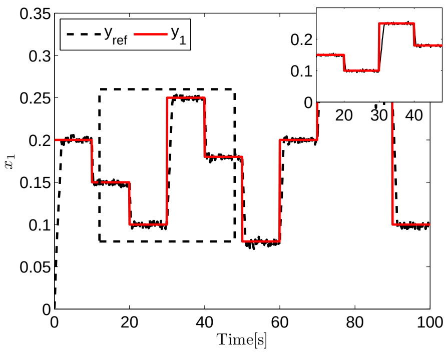

Figure 4. Performance of the system for $t = 0, \ldots, 100[s]$

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