Design of FTC structures with PI virtual actuators

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Abstract. The paper presents new conditions, adequate in design of proportional-integral virtual actuators and utilizable in fault-tolerant control structures which are stabilizable by dynamic output controllers. Taking into account disturbance conditions and changes of variables after the virtual actuator activation, the design conditions are outlined in terms of the linear matrix inequalities within the bounded real lemma forms. Using tuning parameters in design, and with suitable choice of the order of dynamic output controller, the approach provides a way to obtain acceptable dynamics of the closed loop system after activation of the virtual actuator.

1. Introduction

To increase the reliability of systems, fault-tolerant control structures (FTC) usually fix a system with faults so that it can continue its mission with certain limitations of functionality and quality. Considering this, the different approaches were studied in FTC design (see, e.g., [1], [15] and the references therein). The standard way of control reconfiguration discards the nominal controller from the control loop and replace it with a new one so that its parameters are re-tuned in occurred fault conditions and, in dependency on the remaining set of sensors and actuators, to recover in a certain extent the performance given on the fault-free control system [16], [25], [27].

By contrast, instead of adapting the controller to the faulty plant, the virtual approach keeps the nominal controller in the reconfigured closed-loop system and virtually adapt the faulty plant to the nominal controller in such a way that the activated virtual reconfiguration block, together with the faulty plant, imitates the fault-free plant. Since in healthy conditions the virtual blocks are not active, and control is realized by the nominal controller, the design of the virtual reconfiguration blocks may seem to be independent of the controller. Designated to sensor faults the reconfiguration block is termed virtual sensor (VS), while in the case of actuator faults is named virtual actuator (VA). Especially, an FTC strategy based on virtual actuator approach for linear piecewise affine systems with actuator faults is presented in [21], for non-linear systems that can be approximated by linear parameter-varying (LPV) models in discrete-time or continuous-time description, this policy is proposed in [17], [22], [24], respectively, and applying to continuous-time Lipschitz nonlinear systems, this practise is introduced in [6].

Until the first ideas of control reconfiguration by using of VA [2], [13] were summarized in the books [19], [23], several aspects have been used in VA design. Introducing the generalized VA it was shown [14] that reconfiguration after an actuator fault can be related to disturbance decoupling. Then, subsequently, H_∞-based VA synthesis was presented and the dual principle was conveyed in [20]. In general, these conditions for VA design are formulated in terms of a finite set of linear matrix inequalities (LMI) for proportional VAs.

The vast majority of applications is realized in conjunction with static output controllers, but this structure causes that the response of fault isolation has to be fast and the peaks of the system variables, immediately after the activation of a VA, must be restrained [12]. Considering characteristic changes of variables in FTC after VA activation, the synthesis of dynamic virtual actuators (DOC) was stated in [8], [9], [10], [11]. The approach in the paper respects specifically the generalized disturbance transfer matrix in proportional-integral (PI) VA design.

The paper is organized as follows. In Sec. 2, the H_∞ approach is presented with results on bounded real lemmas (BRL) for DOC design. Formulating the separation principle in Sec. 3, the BRL based design methods are outlined for PI VA. Finally, the illustrating example is given in Sec. 4 and some concluding remarks are stated in Sec. 5.

Throughout the paper, the following notations are used: $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and the matrix $\boldsymbol{X}$, respectively, $rank(\,\cdot\,)$ remits the rank of a matrix, for a square matrix $\boldsymbol{X} < 0$ means that $\boldsymbol{X}$ is a symmetric negative definite matrix, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, $I\!R$ denotes the set of real numbers, $I\!R^n$, $I\!R^{n\times r}$ refer to the set of all n-dimensional real vectors and $n\times r$ real matrices, respectively.

2. Dynamic Output Controllers

In the paper, there are taken into account systems described in fault-free conditions as

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}_c(t) + \boldsymbol{V}\boldsymbol{v}(t)\,, \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t)\,, \tag{2}$$

where $\boldsymbol{q}(t) \in I\!R^n$ stands up for the system state, $\boldsymbol{u}_c(t) \in I\!R^r$ denotes the control input, $\boldsymbol{y}(t) \in I\!R^m$ is the measurable output, $\boldsymbol{v}(t) \in I\!R^{r_v}$ is the vector of unknown disturbance, the matrices $\boldsymbol{A} \in I\!R^{n\times n}$, $\boldsymbol{B} \in I\!R^{n\times r}$, $\boldsymbol{C} \in I\!R^{m\times n}$ and $\boldsymbol{V} \in I\!R^{n\times r_v}$ are finite valued.

It is assumed that the system is controlled by bi-proper DOC of the form

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{J}\boldsymbol{p}(t) + \boldsymbol{L}\boldsymbol{y}(t)\,, \tag{3}$$

$$\boldsymbol{u}_c(t) = \boldsymbol{M}\boldsymbol{p}(t) + \boldsymbol{N}\boldsymbol{y}(t) \tag{4}$$

and of an order p, where it can be accepted $1 \le p < n$ (reduced order), $p = n$ (full order) and $n < p \le p_m$ (upgraded order), while $\boldsymbol{p}(t) \in I\!R^p$ is the vector of the controller state variables and $\boldsymbol{J} \in I\!R^{p\times p}$, $\boldsymbol{L} \in I\!R^{p\times m}$, $\boldsymbol{M} \in I\!R^{r\times p}$, $\boldsymbol{N} \in I\!R^{r\times m}$.

To analyze the stability of the closed-loop system structure it can be formulated

$$\begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \dot{\boldsymbol{p}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}+\boldsymbol{B}\boldsymbol{N}\boldsymbol{C} & \boldsymbol{B}\boldsymbol{M} \\ \boldsymbol{L}\boldsymbol{C} & \boldsymbol{J} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{p}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{V} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{v}(t)\,, \tag{5}$$

$$\boldsymbol{y}(t) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_p \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{p}(t) \end{bmatrix}. \tag{6}$$

Introducing the notations

$$\boldsymbol{q}^{\bullet T}(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{p}^T(t) \end{bmatrix},\ \boldsymbol{V}^{\bullet T} = \begin{bmatrix} \boldsymbol{V}^T & \boldsymbol{0} \end{bmatrix},\ \boldsymbol{I}^{\bullet} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix}, \tag{7}$$

$$\boldsymbol{A}^{\bullet} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},\ \boldsymbol{B}^{\bullet} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{B} \\ \boldsymbol{I}_p & \boldsymbol{0} \end{bmatrix},\ \boldsymbol{C}^{\bullet} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_p \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix},\boldsymbol{K}^{\bullet} = \begin{bmatrix} \boldsymbol{J} & \boldsymbol{L} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix}, \tag{8}$$

where $\boldsymbol{A}^{\bullet} \in I\!R^{(n+p)\times(n+p)}$, $\boldsymbol{B}^{\bullet} \in I\!R^{(n+p)\times(p+r)}$, $\boldsymbol{C}^{\bullet} \in I\!R^{(p+m)\times(n+p)}$, $\boldsymbol{V}^{\bullet} \in I\!R^{(n+p)\times r_v}$, $\boldsymbol{I}^{\bullet} \in I\!R^{m\times(p+m)}$, it yields for stabilizable $(\boldsymbol{A}^{\bullet},\boldsymbol{B}^{\bullet})$ and detectable $(\boldsymbol{A}^{\bullet},\boldsymbol{C}^{\bullet})$ [3], [11],

$$\dot{\boldsymbol{q}}^{\bullet}(t) = \boldsymbol{A}^{\bullet}_c\boldsymbol{q}^{\bullet}(t) + \boldsymbol{V}^{\bullet}\boldsymbol{v}(t)\,, \quad \boldsymbol{A}^{\bullet}_c = \boldsymbol{A}^{\bullet} + \boldsymbol{B}^{\bullet}\boldsymbol{K}^{\bullet}\boldsymbol{C}^{\bullet}. \tag{9}$$

$$\boldsymbol{y}^{\bullet}(t) = \boldsymbol{I}^{\bullet}\boldsymbol{C}^{\bullet}\boldsymbol{q}^{\bullet}(t) \tag{10}$$

Proposition 1 *[8] (BRL) The closed-loop system, consisting of the plant (1), (2) and DOC (3), (4), is stable with the quadratic performance $\gamma^\bullet$ if there exist a symmetric positive definite matrix $\boldsymbol{Q}^\bullet \in I\!R^{(n+p)\times(n+p)}$, a regular matrix $\boldsymbol{H}^\bullet \in I\!R^{(p+m)\times(p+m)}$, a matrix $\boldsymbol{Y}^\bullet \in I\!R^{(p+r)\times(p+m)}$ and a positive scalar $\gamma^\bullet \in I\!R$ such that*

$$\boldsymbol{Q}^\bullet = \boldsymbol{Q}^{\bullet T} > 0, \quad \gamma^\bullet > 0\,, \tag{11}$$

$$\begin{bmatrix} \boldsymbol{A}^\bullet\boldsymbol{Q}^\bullet + \boldsymbol{Q}^\bullet\boldsymbol{A}^{\bullet T} + \boldsymbol{B}^\bullet\boldsymbol{Y}^\bullet\boldsymbol{C}^\bullet + \boldsymbol{C}^{\bullet T}\boldsymbol{Y}^{\bullet T}\boldsymbol{B}^{\bullet T} & * & * \\ \boldsymbol{V}^{\bullet T} & -\gamma^\bullet\boldsymbol{I}_{r_v} & * \\ \boldsymbol{I}^\bullet\boldsymbol{C}^\bullet\boldsymbol{Q}^\bullet & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0\,, \tag{12}$$

$$\boldsymbol{C}^\bullet\boldsymbol{Q}^\bullet = \boldsymbol{H}^\bullet\boldsymbol{C}^\bullet, \tag{13}$$

where the generalized system matrices are defined in (7), (8).

When the above conditions hold

$$\boldsymbol{K}^\bullet = \boldsymbol{Y}^\bullet(\boldsymbol{H}^\bullet)^{-1}. \tag{14}$$

Here and hereafter $*$ denotes the symmetric item in a symmetric matrix.

In order to adjust fault detection and isolation time to the dynamics of the closed-loop system, the selection of the order p of the DOC be provided with a free tuning parameter in control design. One serviceable method is based on incorporation a slack matrix into LMI design conditions. This augmentation is proposed in the following theorem.

Proposition 2 *[11] (enhanced BRL) The closed-loop system, consisting of the plant (1), (2) and the DOC (3), (4), is stable with the quadratic performance $\gamma^\bullet$ if for the given positive scalar $\delta^\bullet \in I\!R$ there exist symmetric positive definite matrices $\boldsymbol{R}^\bullet$, $\boldsymbol{U}^\bullet \in I\!R^{(n+p)\times(n+p)}$, a regular matrix $\boldsymbol{H}^\bullet \in I\!R^{(p+m)\times(p+m)}$, a matrix $\boldsymbol{Y}^\bullet \in I\!R^{(p+r)\times(p+m)}$ and a positive scalar $\gamma^\bullet \in I\!R$ such that*

$$\boldsymbol{R}^\bullet = \boldsymbol{R}^{\bullet T} > 0, \quad \boldsymbol{U}^\bullet = \boldsymbol{U}^{\bullet T} > 0, \quad \gamma^\bullet > 0\,, \tag{15}$$

$$\begin{bmatrix} \boldsymbol{A}^\bullet\boldsymbol{R}^\bullet + \boldsymbol{R}^\bullet\boldsymbol{A}^{\bullet T} + \boldsymbol{B}^\bullet\boldsymbol{Y}^\bullet\boldsymbol{C}^\bullet + \boldsymbol{C}^{\bullet T}\boldsymbol{Y}^{\bullet T}\boldsymbol{B}^{\bullet T} & * & * & * \\ \boldsymbol{U}^\bullet - \boldsymbol{R}^\bullet + \delta\boldsymbol{A}^\bullet\boldsymbol{R}^\bullet + \delta\boldsymbol{B}^\bullet\boldsymbol{Y}^\bullet\boldsymbol{C}^\bullet & -2\delta^\bullet\boldsymbol{R}^\bullet & * & * \\ \boldsymbol{V}^{\bullet T} & \delta^\bullet\boldsymbol{V}^{\bullet T} & -\gamma^\bullet\boldsymbol{I}_{r_v} & * \\ \boldsymbol{I}^\bullet\boldsymbol{C}^\bullet\boldsymbol{R}^\bullet & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0\,, \tag{16}$$

$$\boldsymbol{C}^\bullet\boldsymbol{R}^\bullet = \boldsymbol{H}^\bullet\boldsymbol{C}^\bullet, \tag{17}$$

where the generalized system matrices are defined in (7), (8) and the positive $\delta^\bullet \in I\!R$ is the tuning parameter.

When the above conditions hold then 14 yields.

Consider the case $r = m$ (square plants), where with each output signal is associated a reference signal. Such regime is called the forced regime and for DOC is defined as follows:

Definition 1 *The forced regime for (1), (2) with DOC (3), (4) is given by the control policy*

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{J}\boldsymbol{p}(t) + \boldsymbol{L}\boldsymbol{y}(t)\,, \tag{18}$$

$$\boldsymbol{u}(t) = \boldsymbol{M}\boldsymbol{p}(t) + \boldsymbol{N}\boldsymbol{y}(t) + \boldsymbol{W}\boldsymbol{w}(t)\,, \tag{19}$$

where $\boldsymbol{w}(t) \in I\!R^m$ is desired output signal vector, and $\boldsymbol{W} \in I\!R^{m\times m}$ is the signal gain matrix.

Theorem 1 *[10] If a square system (1), (2) is stabilizable by the control policy (18), (19), and*

$$\mathtt{rank}\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = n + m\,, \tag{20}$$

then the matrix $\boldsymbol{W}$ takes the form

$$\boldsymbol{W} = -\Big(\boldsymbol{C}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{M}\boldsymbol{J}^{-1}\boldsymbol{L}\boldsymbol{C} + \boldsymbol{B}\boldsymbol{N}\boldsymbol{C})^{-1}\boldsymbol{B}\Big)^{-1}. \tag{21}$$

3. PI Virtual Actuators

The state-space description of the system with a single actuator fault is considered as follows

$$\dot{\boldsymbol{q}}_{fa}(t) = \boldsymbol{A}\boldsymbol{q}_{fa}(t) + \boldsymbol{B}_f\boldsymbol{u}_{fa}(t) + \boldsymbol{V}\boldsymbol{v}(t)\,, \tag{22}$$

$$\boldsymbol{y}_{fa}(t) = \boldsymbol{C}\boldsymbol{q}_{fa}(t)\,, \tag{23}$$

where $\boldsymbol{q}_{fa}(t) \in I\!R^n$, $\boldsymbol{u}_{fa}(t) \in I\!R^r$, $\boldsymbol{y}_{fa}(t) \in I\!R^m$ denote the faulty system state variables vector, the vector of the acting control input variables and the vector of faulty output variables, respectively, and the matrix $\boldsymbol{B}_f \in I\!R^{n\times r}$ is finite valued, while $rank(\boldsymbol{B}_f) < rank(\boldsymbol{B})$. Moreover, it is supposed that $(\boldsymbol{A}, \boldsymbol{B}_f)$ is controllable and $\boldsymbol{u}_{fa}(t)$ is available for reconfiguration.

Analogously, using the same system variable notations, the state-space description of DOC, acting on the system with a single actuator fault, but without DVA, is of the form

$$\dot{\boldsymbol{p}}_{fa}(t) = \boldsymbol{J}\boldsymbol{p}_{fa}(t) + \boldsymbol{L}\boldsymbol{y}_{fa}(t)\,, \tag{24}$$

$$\boldsymbol{u}_c(t) = \boldsymbol{M}\boldsymbol{p}_{fa}(t) + \boldsymbol{N}\boldsymbol{y}_{fa}(t)\,, \tag{25}$$

where $\boldsymbol{p}_{fa}(t) \in I\!R^p$ denotes the controller state variables vector in the faulty system control.

To obtain the DVA state-space description, the following theorem is proven at first.

Theorem 2 *(separation principle) The dynamic virtual actuator for the system with a single actuator fault (22), (23) takes the form*

$$\dot{\boldsymbol{e}}_{fa}(t) = (\boldsymbol{A} - \boldsymbol{B}_f\boldsymbol{G}_P)\boldsymbol{e}_{fa}(t) - \boldsymbol{B}_f\boldsymbol{G}_I\boldsymbol{e}_{pfa}(t) - \boldsymbol{B}\boldsymbol{u}_c(t)\,, \tag{26}$$

$$\dot{\boldsymbol{z}}_{fa}(t) = \boldsymbol{C}\boldsymbol{e}_{fa}(t)\,. \tag{27}$$

$$\boldsymbol{u}_{fa}(t) = -\boldsymbol{G}_P\boldsymbol{e}_{fa}(t) - \boldsymbol{G}_I\boldsymbol{z}_{fa}(t)\,, \tag{28}$$

where

$$\boldsymbol{e}_{fa}(t) = \boldsymbol{q}_{fa}(t) - \boldsymbol{q}(t)\,, \quad \boldsymbol{e}_{pfa}(t) = \boldsymbol{p}_{fa}(t) - \boldsymbol{p}(t)\,, \tag{29}$$

$\boldsymbol{e}_{pfa}(t) \in I\!R^p$, $\boldsymbol{e}_{fa}(t) \in I\!R^n$ *and* $\boldsymbol{G}_P \in I\!R^{r\times n}$, $\boldsymbol{G}_I \in I\!R^{r\times m}$ *are real matrices.*

Proof: Using (1), (2), (3) and (22), (23), (24) and proposing integral part of VA as follows

$$\boldsymbol{z}_{fa}(t) = \int_0^t \boldsymbol{C}(\boldsymbol{q}_{fa}(r) - \boldsymbol{q}(r))\mathrm{d}r, \quad \dot{\boldsymbol{z}}_{fa}(t) = \boldsymbol{C}(\boldsymbol{q}_{fa}(t) - \boldsymbol{q}(t))\,, \tag{30}$$

then the expression of the common system variable model is

$$\begin{bmatrix} \dot{\boldsymbol{q}}_{fa}(t) \\ \dot{\boldsymbol{q}}(t) \\ \dot{\boldsymbol{p}}_{fa}(t) \\ \dot{\boldsymbol{p}}(t) \\ \dot{\boldsymbol{z}}_{fa}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{LC} & \boldsymbol{0} & \boldsymbol{J} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{LC} & \boldsymbol{0} & \boldsymbol{J} & \boldsymbol{0} \\ \boldsymbol{C} & -\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{fa}(t) \\ \boldsymbol{q}(t) \\ \boldsymbol{p}_{fa}(t) \\ \boldsymbol{p}(t) \\ \boldsymbol{z}_{fa}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_f & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{fa}(t) \\ \boldsymbol{u}_c(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{V} \\ \boldsymbol{V} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{v}(t)\,. \tag{31}$$

Using the transform matrix $\boldsymbol{T}$ of the form

$$\boldsymbol{T} = \boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{I}_n & -\boldsymbol{I}_n & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_p & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_p & -\boldsymbol{I}_p & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix}, \quad \boldsymbol{T} \begin{bmatrix} \boldsymbol{q}_{fa}(t) \\ \boldsymbol{q}(t) \\ \boldsymbol{p}_{fa}(t) \\ \boldsymbol{p}(t) \\ \boldsymbol{z}_{fa}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_{fa}(t) \\ \boldsymbol{e}_{fa}(t) \\ \boldsymbol{p}_{fa}(t) \\ \boldsymbol{e}_{pfa}(t) \\ \boldsymbol{z}_{fa}(t) \end{bmatrix}, \tag{32}$$

where $\boldsymbol{e}_{fa}(t)$, $\boldsymbol{e}_{fa}(t)$ are defined in (29), then (31) can be rewritten as

$$\begin{bmatrix} \dot{\boldsymbol{q}}_{fa}(t) \\ \dot{\boldsymbol{e}}_{fa}(t) \\ \dot{\boldsymbol{p}}_{fa}(t) \\ \dot{\boldsymbol{e}}_{pfa}(t) \\ \dot{\boldsymbol{z}}_{fa}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{LC} & \boldsymbol{0} & \boldsymbol{J} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{LC} & \boldsymbol{0} & \boldsymbol{J} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{fa}(t) \\ \boldsymbol{e}_{fa}(t) \\ \boldsymbol{p}_{fa}(t) \\ \boldsymbol{e}_{pfa}(t) \\ \boldsymbol{z}_{fa}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_f & \boldsymbol{0} \\ \boldsymbol{B}_f & -\boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{fa}(t) \\ \boldsymbol{u}_c(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{V} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{v}(t)\,. \tag{33}$$

Defining the covering of the faulty control input as (28), the substitution of (28) in (33) leads to

$$\begin{aligned} \begin{bmatrix} \dot{\boldsymbol{q}}_{fa}(t) \\ \dot{\boldsymbol{e}}_{fa}(t) \\ \dot{\boldsymbol{p}}_{fa}(t) \\ \dot{\boldsymbol{e}}_{pfa}(t) \\ \dot{\boldsymbol{z}}_{fa}(t) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}_f\boldsymbol{G}_P & \boldsymbol{0} & -\boldsymbol{B}_f\boldsymbol{G}_I & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}-\boldsymbol{B}_f\boldsymbol{G}_P & \boldsymbol{0} & -\boldsymbol{B}_f\boldsymbol{G}_I & \boldsymbol{0} \\ \boldsymbol{LC} & \boldsymbol{0} & \boldsymbol{J} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{LC} & \boldsymbol{0} & \boldsymbol{J} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{fa}(t) \\ \boldsymbol{e}_{fa}(t) \\ \boldsymbol{p}_{fa}(t) \\ \boldsymbol{e}_{pfa}(t) \\ \boldsymbol{z}_{fa}(t) \end{bmatrix} \\ &- \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{B} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{u}_c(t) + \begin{bmatrix} \boldsymbol{V} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{v}(t)\,. \end{aligned} \tag{34}$$

Thus, the second and the fifth row of the equation (34) imply (26), (27). Obviously, in view of the block structure of (34), the separation principle yields. This concludes the proof. ■

Corollary 1 *The equations of the closed-loop faulty system with activated PI VA are as follows*

$$\dot{\boldsymbol{q}}^{\circ}_{fa}(t) = \boldsymbol{A}^{\circ}_c\boldsymbol{q}^{\circ}_{fa}(t) + \boldsymbol{V}^{\circ}_{fa}\boldsymbol{d}^{\circ}_{fa}(t)\,, \quad \boldsymbol{A}^{\circ}_c = \boldsymbol{A}^{\circ} - \boldsymbol{B}^{\circ}_f\boldsymbol{G}^{\circ}\,, \tag{35}$$

$$\boldsymbol{y}_{fa}(t) = \boldsymbol{C}^{\circ}\boldsymbol{q}^{\circ}_{fa}(t)\,, \tag{36}$$

$$\boldsymbol{q}^{\circ}_{fa}(t) = \begin{bmatrix} \boldsymbol{q}_{fa}(t) \\ \boldsymbol{z}_{fa}(t) \end{bmatrix}, \ \boldsymbol{d}^{\circ}_{fa}(t) = \begin{bmatrix} \boldsymbol{G}_P\boldsymbol{q}(t) \\ \boldsymbol{v}(t) \\ \boldsymbol{q}(t) \end{bmatrix}, \ \boldsymbol{V}^{\circ}_{fa} = \begin{bmatrix} \boldsymbol{B}_f & \boldsymbol{V} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{C} \end{bmatrix}, \tag{37}$$

$$\boldsymbol{A}^{\circ} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \ \boldsymbol{B}^{\circ}_f = \begin{bmatrix} \boldsymbol{B}_f & \boldsymbol{B}_f \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \ \boldsymbol{G}^{\circ} = \begin{bmatrix} \boldsymbol{G}_P \\ \boldsymbol{G}_I \end{bmatrix}, \ \boldsymbol{C}^{\circ} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}, \tag{38}$$

and $\boldsymbol{q}^{\circ}_{fa}(t) \in I\!R^{(n+m)}$, $\boldsymbol{d}^{\circ}_{fa}(t) \in I\!R^{(r+r_v+n)}$, $\boldsymbol{A}^{\circ} \in I\!R^{(n+m)\times(n+m)}$, $\boldsymbol{B}^{\circ} \in I\!R^{(n+m)\times 2r}$, $\boldsymbol{C}^{\circ} \in I\!R^{m\times(n+m)}$, $\boldsymbol{G}^{\circ} \in I\!R^{2r\times(n+m)}$, $\boldsymbol{V}^{\circ}_{fa} \in I\!R^{(n+m)\times(r+r_v+n)}$. *Obviously, the stability of the closed-loop system in the reconfiguration regime is determined by the system matrix* $\boldsymbol{A}^{\circ}_c$.

Proof: Using (29), the first and the fifth row of the equation (34) as well as (23) give

$$\begin{bmatrix} \dot{\boldsymbol{q}}_{fa}(t) \\ \dot{\boldsymbol{z}}_{fa}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}-\boldsymbol{B}_f\boldsymbol{G}_P & -\boldsymbol{B}_f\boldsymbol{G}_I \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{fa}(t) \\ \boldsymbol{z}_{fa}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_f & \boldsymbol{V} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{G}_P\boldsymbol{q}(t) \\ \boldsymbol{v}(t) \\ \boldsymbol{q}(t) \end{bmatrix}, \tag{39}$$

$$\boldsymbol{y}_{fa}(t) = \boldsymbol{C}\boldsymbol{q}_{fa}(t) = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{fa}(t) \\ \boldsymbol{z}_{fa}(t) \end{bmatrix} \tag{40}$$

and using the relation

$$\begin{bmatrix} \boldsymbol{A}-\boldsymbol{B}_f\boldsymbol{G}_P & -\boldsymbol{B}_f\boldsymbol{G}_I \\ \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{B}_f & \boldsymbol{B}_f \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{G}_P \\ \boldsymbol{G}_I \end{bmatrix}, \tag{41}$$

with the notations (37), (38) then (39), (40) imply (35), (36), respectively. This concludes the proof. ■

Corollary 2 *The state-space description of the PI VA (26), (27) with the covering of faulty control input (28) is as follows*

$$\dot{\boldsymbol{e}}^{\circ}_{fa}(t) = (\boldsymbol{A}^{\circ} - \boldsymbol{B}^{\circ}_{f}\boldsymbol{G}^{\circ})\boldsymbol{e}^{\circ}_{fa}(t) - \boldsymbol{B}^{\circ}\boldsymbol{u}_c(t)\,, \tag{42}$$

$$\boldsymbol{u}_{fa}(t) = -\boldsymbol{G}^{\circ}\boldsymbol{e}^{\circ}_{fa}(t)\,, \tag{43}$$

where

$$\boldsymbol{e}^{\circ}_{fa}(t) = \begin{bmatrix} \boldsymbol{e}_{fa}(t) \\ \boldsymbol{z}_{fa}(t) \end{bmatrix}, \ \boldsymbol{B}^{\circ} = \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix}. \tag{44}$$

In the autonomous regime the stability of the PI VA is determined by the same system matrix $\boldsymbol{A}^{\circ}_c$ *as stability of the closed-loop system in the reconfiguration regime.*

Proof: Writing the equations (26)-(28) in the following form

$$\begin{bmatrix} \dot{\boldsymbol{e}}_{fa}(t) \\ \dot{\boldsymbol{z}}_{fa}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{B}_f\boldsymbol{G}_P & -\boldsymbol{B}_f\boldsymbol{G}_I \\ \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{fa}(t) \\ \boldsymbol{z}_{fa}(t) \end{bmatrix} - \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{u}_c(t)\,, \tag{45}$$

$$\boldsymbol{u}_{fa}(t) = -\begin{bmatrix} \boldsymbol{G}_P & \boldsymbol{G}_I \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{fa}(t) \\ \boldsymbol{z}_{fa}(t) \end{bmatrix}, \tag{46}$$

respectively, and using the notations (38), (44) as well as the relation (41), then (45), (46) imply (42), (43). This concludes the proof. ■

Corollary 3 *The state-space description of DOC masked in inputs by PI VA and acting on the system with a single actuator fault is of the form*

$$\begin{bmatrix} \dot{\boldsymbol{p}}_{fa}(t) \\ \boldsymbol{u}_c(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{J} & \boldsymbol{L} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_{fa}(t) \\ \boldsymbol{y}_{fa}(t) \end{bmatrix} - \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{pfa}(t) \\ \boldsymbol{C}\boldsymbol{e}_{fa}(t) \end{bmatrix}, \tag{47}$$

where $\boldsymbol{y}_{fa}(t)$ *is the measurable output of the closed-loop faulty system.*

Proof: Using (23) and (29), then (2) can be rewritten as

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) = \boldsymbol{C}(\boldsymbol{q}_{fa}(t) - (\boldsymbol{q}_{fa}(t) - \boldsymbol{q}(t))) = \boldsymbol{y}_{fa}(t) - \boldsymbol{C}\boldsymbol{e}_{fa}(t)\,. \tag{48}$$

Considering (48) as the input to the nominal DOC, by using (29) and (48) then (3), (4) imply

$$\begin{bmatrix} \dot{\boldsymbol{p}}(t) \\ \boldsymbol{u}_c(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{J} & \boldsymbol{L} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}(t) \\ \boldsymbol{y}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{J} & \boldsymbol{L} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_{fa}(t) - \boldsymbol{e}_{pfa}(t) \\ \boldsymbol{y}_{fa}(t) - \boldsymbol{C}\boldsymbol{e}_{fa}(t) \end{bmatrix}, \tag{49}$$

$$\begin{bmatrix} \dot{\boldsymbol{p}}(t) \\ \boldsymbol{u}_c(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{J} & \boldsymbol{L} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_{fa}(t) \\ \boldsymbol{y}_{fa}(t) \end{bmatrix} - \begin{bmatrix} \boldsymbol{J} & \boldsymbol{L} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{pfa}(t) \\ \boldsymbol{C}\boldsymbol{e}_{fa}(t) \end{bmatrix}, \tag{50}$$

respectively. Separating the equation given by the forth row of (34) as follows

$$\dot{\boldsymbol{e}}_{pfa}(t) = \boldsymbol{J}\boldsymbol{e}_{pfa}(t) + \boldsymbol{L}\boldsymbol{C}\boldsymbol{e}_{fa}(t)\,, \tag{51}$$

(50) can be rewritten as

$$\begin{bmatrix} \dot{\boldsymbol{p}}(t) \\ \boldsymbol{u}_c(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{J} & \boldsymbol{L} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_{fa}(t) \\ \boldsymbol{y}_{fa}(t) \end{bmatrix} - \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix} \dot{\boldsymbol{e}}_{pfa}(t) - \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{pfa}(t) \\ \boldsymbol{C}\boldsymbol{e}_{fa}(t) \end{bmatrix}. \tag{52}$$

Since the time derivative of (29) takes the form

$$\dot{\boldsymbol{e}}_{pfa}(t) = \dot{\boldsymbol{p}}_{fa}(t) - \dot{\boldsymbol{p}}(t)\,, \tag{53}$$

using (53) then (52) implies the equations of DOC covered by PI VA (47). This concludes the proof. ■

4. Design of PI VA

To offer a tuning parameter in PI VA design the following theorem is proven.

Theorem 3 *(enhanced BRL) The closed-loop system, consisting of the plant with a single actuator fault (22), (23), DOC (3), (4) and DVA (26), (27), is stable with the quadratic performance* γ° *if for given positive* $\delta^\circ \in I\!R$ *there exist symmetric positive definite matrices* $\boldsymbol{R}^\circ, \boldsymbol{U}^\circ \in I\!R^{(n+m)\times(n+m)}$*, a matrix* $\boldsymbol{Y}^\circ \in I\!R^{2r\times(n+m)}$ *and a positive scalar* $\gamma^\circ \in I\!R$ *such that*

$$\boldsymbol{R}^\circ = \boldsymbol{R}^{\circ T} > 0, \quad \boldsymbol{U}^\circ = \boldsymbol{U}^{\circ T} > 0, \quad \gamma^\circ > 0\,, \tag{54}$$

$$\begin{bmatrix} \boldsymbol{A}^\circ\boldsymbol{R}^\circ + \boldsymbol{R}^\circ\boldsymbol{A}^{\circ T} + \boldsymbol{B}_f^\circ\boldsymbol{Y}^\circ + \boldsymbol{Y}^{\circ T}\boldsymbol{B}_f^{\circ T} & * & * & * \\ \boldsymbol{U}^\circ - \boldsymbol{R}^\circ + \delta^\circ\boldsymbol{A}^\circ\boldsymbol{R}^\circ + \delta^\circ\boldsymbol{B}_f^\circ\boldsymbol{Y}^\circ & -2\delta^\circ\boldsymbol{R}^\circ & * & * \\ \boldsymbol{V}_{fa}^{\circ T} & \delta^\circ\boldsymbol{V}_{fa}^{\circ T} & -\gamma^\circ\boldsymbol{I}_{r+r_v+n} & * \\ \boldsymbol{C}^\circ\boldsymbol{R}^\circ & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0\,, \tag{55}$$

where the system matrices are defined in (37), (38) and positive δ° *is the tuning parameter.*

When the above conditions hold

$$\boldsymbol{G}^\circ = \boldsymbol{Y}^\circ(\boldsymbol{R}^\circ)^{-1}. \tag{56}$$

Proof: Since (35) implies

$$\boldsymbol{A}_c^\circ\boldsymbol{q}_{fa}^\circ(t) + \boldsymbol{V}_{fa}^\circ\boldsymbol{d}_{fa}^\circ(t) - \dot{\boldsymbol{q}}_{fa}^\circ(t) = \boldsymbol{0}\,, \tag{57}$$

$$(\boldsymbol{q}_{fa}^{\circ T}(t)\boldsymbol{S}^\circ + \delta^\circ\dot{\boldsymbol{q}}_{fa}^{\circ T}(t)\boldsymbol{S}^\circ)(\boldsymbol{A}_c^\circ\boldsymbol{q}_{fa}^\circ(t) + \boldsymbol{V}_{fa}^\circ\boldsymbol{d}_{fa}^\circ(t) - \dot{\boldsymbol{q}}_{fa}^\circ(t)) = 0\,, \tag{58}$$

where $\boldsymbol{S}^\circ \in I\!R^{(n+m)\times(n+m)}$ is symmetric positive definite and $\delta^\circ \in I\!R$ is a positive scalar and by considering the Lyapunov function candidate as follows

$$v(\boldsymbol{q}_{fa}^\circ(t)) = \boldsymbol{q}_{fa}^{\circ T}(t)\boldsymbol{P}^\circ\boldsymbol{q}_{fa}^\circ(t) + \int_0^t (\boldsymbol{y}_{fa}^T(\tau)\boldsymbol{y}_{fa}(\tau) - \gamma^\circ\boldsymbol{d}_{fa}^{\circ T}(\tau)\boldsymbol{d}_{fa}^\circ(\tau))\mathrm{d}\tau > 0\,, \tag{59}$$

where $\boldsymbol{P}^\circ \in I\!R^{(n+m)\times(n+m)}$ is symmetric positive definite matrix and $\gamma^\circ \in I\!R$ is square of the H_∞ norm of the transfer function matrix of the disturbance $\boldsymbol{d}_{fa}^\circ$, then

$$\dot{v}(\boldsymbol{q}_{fa}^\circ(t)) = \dot{\boldsymbol{q}}_{fa}^{\circ T}(t)\boldsymbol{P}^\circ\boldsymbol{q}_{fa}^\circ(t) + \boldsymbol{q}_{fa}^{\circ T}(t)\boldsymbol{P}^\circ\dot{\boldsymbol{q}}_{fa}^\circ(t) + \boldsymbol{y}_{fa}^T(t)\boldsymbol{y}_{fa}(t) - \gamma^\circ\boldsymbol{d}_{fa}^{\circ T}(t)\boldsymbol{d}_{fa}^\circ(t) < 0\,. \tag{60}$$

and by adding (58) as well as its transpose to (60) and then inserting (36), it can see that

$$\begin{aligned} \dot{v}(\boldsymbol{q}_{fa}^\circ(t)) &= \dot{\boldsymbol{q}}_{fa}^{\circ T}(t)\boldsymbol{P}^\circ\boldsymbol{q}_{fa}^\circ(t) + \boldsymbol{q}_{fa}^{\circ T}(t)\boldsymbol{P}^\circ\dot{\boldsymbol{q}}_{fa}^\circ(t) \\ &+ (\boldsymbol{q}_{fa}^{\circ T}(t)\boldsymbol{S}^\circ + \delta^\circ\dot{\boldsymbol{q}}_{fa}^{\circ T}(t)\boldsymbol{S}^\circ)(\boldsymbol{A}_c^\circ\boldsymbol{q}_{fa}^\circ(t) + \boldsymbol{V}_{fa}^\circ\boldsymbol{d}_{fa}^\circ(t) - \dot{\boldsymbol{q}}_{fa}^\circ(t)) \\ &+ (\boldsymbol{A}_c^\circ\boldsymbol{q}_{fa}^\circ(t) + \boldsymbol{V}_{fa}^\circ\boldsymbol{d}_{fa}^\circ(t) - \dot{\boldsymbol{q}}_{fa}^\circ(t))^T(\boldsymbol{S}^\circ\boldsymbol{q}_{fa}^\circ(t) + \delta^\circ\boldsymbol{S}^\circ\dot{\boldsymbol{q}}_{fa}^\circ(t)) \\ &+ \boldsymbol{q}_{fa}^{\circ T}(t)\boldsymbol{C}^{\circ T}\boldsymbol{C}^\circ\boldsymbol{q}_{fa}^\circ(t) - \gamma^\circ\boldsymbol{d}_{fa}^{\circ T}(t)\boldsymbol{d}_{fa}^\circ < 0\,. \end{aligned} \tag{61}$$

The inequality (59) can be written as

$$\dot{v}(\boldsymbol{q}_c^\circ(t)) = \boldsymbol{q}_{ce}^{\circ T}(t)\boldsymbol{P}_{ce}^\circ\boldsymbol{q}_{ce}^\circ(t) < 0\,, \tag{62}$$

where

$$\boldsymbol{P}_{ce}^\circ = \begin{bmatrix} \boldsymbol{S}^\circ\boldsymbol{A}_c^\circ + \boldsymbol{A}_c^{\circ T}\boldsymbol{S}^\circ + \boldsymbol{C}^{\circ T}\boldsymbol{C}^\circ & * & * \\ \boldsymbol{P}^\circ - \boldsymbol{S}^\circ + \delta^\circ\boldsymbol{S}^\circ\boldsymbol{A}_c^\circ & -2\delta^\circ\boldsymbol{S}^\circ & * \\ \boldsymbol{V}_{fa}^{\circ T}\boldsymbol{S}^\circ & \delta^\circ\boldsymbol{V}_{fa}^{\circ T}\boldsymbol{S}^\circ & -\gamma^\circ\boldsymbol{I}_{r+r_v+n} \end{bmatrix}, \ \boldsymbol{q}_{ce}^\circ(t) = \begin{bmatrix} \boldsymbol{q}_{fa}^\circ(t) \\ \dot{\boldsymbol{q}}_{fa}^\circ(t) \\ \boldsymbol{d}_{fa}^\circ \end{bmatrix}. \tag{63}$$

Since, using the Schur complement property, (63) can be rewritten as

$$\begin{bmatrix} \boldsymbol{S}^\circ \boldsymbol{A}_c^\circ + \boldsymbol{A}^{\circ T} \boldsymbol{S}^\circ & * & * & * \\ \boldsymbol{P}^\circ - \boldsymbol{S}^\circ + \delta^\circ \boldsymbol{S}^\circ \boldsymbol{A}_c^\circ & -2\delta^\circ \boldsymbol{S}^\circ & * & * \\ \boldsymbol{V}_{fa}^{\circ T} \boldsymbol{S}^\circ & \delta^\circ \boldsymbol{V}_{fa}^{\circ T} \boldsymbol{S}^\circ & -\gamma^\circ \boldsymbol{I}_{r+r_v+n} & * \\ \boldsymbol{C}^\circ & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0\,, \tag{64}$$

inserting (35) in (64) and then pre-multiplying and post-multiplying the result by the matrix

$$\boldsymbol{T}_{fe}^\circ = \operatorname{diag}\left[\ \boldsymbol{R}^\circ \quad \boldsymbol{R}^\circ \quad \boldsymbol{I}_{r+r_v+n} \quad \boldsymbol{I}_m\ \right], \quad \boldsymbol{R}^\circ = (\boldsymbol{S}^\circ)^{-1}, \tag{65}$$

it can be obtained

$$\begin{bmatrix} (\boldsymbol{A}^\circ + \boldsymbol{B}_f^\circ \boldsymbol{G}^\circ)\boldsymbol{R}^\circ + \boldsymbol{R}^\circ(\boldsymbol{A}^\circ + \boldsymbol{B}_f^\circ \boldsymbol{G}^\circ)^T & * & * & * \\ \boldsymbol{R}^\circ \boldsymbol{P}^\circ \boldsymbol{R}^\circ - \boldsymbol{R}^\circ + \delta^\circ(\boldsymbol{A}^\circ + \boldsymbol{B}_f^\circ \boldsymbol{G}^\circ)\boldsymbol{R}^\circ & -2\delta^\circ \boldsymbol{R}^\circ & * & * \\ \boldsymbol{V}_{fa}^{\circ T} & \delta^\circ \boldsymbol{V}_{fa}^{\circ T} & -\gamma^\circ \boldsymbol{I}_{r+r_v+n} & * \\ \boldsymbol{C}^\circ \boldsymbol{R}^\circ & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0\,. \tag{66}$$

Introducing the notations

$$\boldsymbol{U}^\circ = \boldsymbol{R}^\circ \boldsymbol{P}^\circ \boldsymbol{R}^\circ, \quad \boldsymbol{Y}^\circ = \boldsymbol{G}^\circ \boldsymbol{R}^\circ \tag{67}$$

then (66), (67) implies (55), (56), respectively. This concludes the proof. ■

Theorem 4 *(BRL) The closed-loop system, consisting of the plant with a single actuator fault (22), (23), DOC (3), (4) and PI VA (26), (27), is stable with the quadratic performance γ° if there exist a symmetric positive definite matrix $\boldsymbol{X}^\circ \in I\!R^{(n+m)\times(n+m)}$, a matrix $\boldsymbol{Y}^\circ \in I\!R^{2r\times(n+m)}$ and a positive scalar $\gamma^\circ \in I\!R$ such that*

$$\boldsymbol{X}^\circ = \boldsymbol{X}^{\circ T} > 0, \quad \gamma^\circ > 0\,, \tag{68}$$

$$\begin{bmatrix} \boldsymbol{A}^\circ \boldsymbol{X}^\circ + \boldsymbol{X}^\circ \boldsymbol{A}^{\circ T} + \boldsymbol{B}_f^\circ \boldsymbol{Y}^\circ + \boldsymbol{Y}^{\circ T} \boldsymbol{B}_f^{\circ T} & * & * \\ \boldsymbol{V}_{fa}^{\circ T} & -\gamma^\circ \boldsymbol{I}_{r+r_v+n} & * \\ \boldsymbol{C}^\circ \boldsymbol{X}^\circ & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0\,, \tag{69}$$

where the generalized system matrices are defined in (37), (38).

When the above conditions hold

$$\boldsymbol{G}^\circ = \boldsymbol{Y}^\circ (\boldsymbol{X}^\circ)^{-1}. \tag{70}$$

Proof: (compare [12]) Setting in (55) that $\boldsymbol{U}^\circ = \boldsymbol{R}^\circ = \boldsymbol{X}^\circ$, $\delta^\circ = 0$ and by eliminating the resulting zero row and zero column, then (55) implies (69). This concludes the proof. ■

5. Illustrative Example

The considered system is represented by the model (1), (2) with the matrix parameters [26]

$$\boldsymbol{A} = \begin{bmatrix} 0.5432 & 0.0137 & 0 & 0.9778 & 0 \\ 0 & -0.1178 & 0.2215 & 0 & -0.9661 \\ 0 & -10.5130 & -0.9967 & 0 & 0.6176 \\ 2.6221 & -0.0030 & 0 & -0.5057 & 0 \\ 0 & 0.7075 & -0.0939 & 0 & -0.2120 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} -0.0318 & -0.0548 & -0.0548 & -0.0318 & 0.0004 \\ 0.0024 & 0.0095 & -0.0095 & 0.0024 & 0.0287 \\ -2.2849 & -1.9574 & 1.9574 & 2.2849 & 1.4871 \\ -0.4628 & -0.8107 & 0.8107 & -0.4628 & 0.0024 \\ 0.0944 & -0.1861 & -0.1861 & 0.0944 & -0.8823 \end{bmatrix}, \boldsymbol{V} = \begin{bmatrix} 0.7593 \\ 0.4116 \\ 0.8793 \\ 0.0272 \\ 0.0389 \end{bmatrix}, \sigma_v^2 = 7.1\times 10^{-3}.$$

The system is controlled by DOC (5), (6), whose parameters were determined using (15)-(17) for order $p = 1$ and the tuning parameter $\delta^{\bullet} = 10$. Using the SeDuMi package [18] then with $\gamma^{\bullet} = 13.4697$ the DOC matrix parameters take the values

$$\boldsymbol{J} = -1.3767\,, \qquad \boldsymbol{L} = 10^{-8}\begin{bmatrix} -0.0673 & -0.0049 & -0.1764 & -0.0886 \end{bmatrix},$$

$$\boldsymbol{M} = 10^{-8}\begin{bmatrix} -0.2501 \\ 0.3877 \\ 0.0318 \\ 0.0714 \\ -0.0043 \end{bmatrix}, \quad \boldsymbol{N} = \begin{bmatrix} 9.9037 & -18.8161 & 5.2467 & 16.0287 \\ 8.8524 & 11.2393 & 4.6291 & -10.3231 \\ 14.8863 & 0.4394 & 7.7691 & -0.7511 \\ 6.5785 & -0.7130 & 3.4584 & 1.3946 \\ -2.8555 & -6.3480 & -1.4051 & 9.3270 \end{bmatrix}$$

and the signal gain matrix $\boldsymbol{W}$ is calculated by (21) as

$$\boldsymbol{W} = \begin{bmatrix} -5.1176 & -0.1008 & 0.0521 & -0.2320 \\ -7.0937 & 0.5788 & -0.4934 & -1.4523 \\ -11.5871 & -0.2253 & 1.4234 & 0.6632 \\ -3.0013 & -0.7243 & -1.0022 & 3.3029 \\ 2.6838 & 2.3520 & -0.6184 & -5.3107 \end{bmatrix}.$$

The closed-loop system is stable with the closed-loop system matrix eigenvalue spectrum

$$\rho(\boldsymbol{A}^{\bullet} + \boldsymbol{B}^{\bullet}\boldsymbol{G}^{\bullet}\boldsymbol{C}^{\bullet}) = \{\, -4.3447 \quad -1.9657 \quad -0.5713 \quad -1.2897 \quad -1.0802 \quad -1.3767 \,\}\,.$$

The control reconstruction by PI VA is illustrated for single fault of the second actuator, where the extended system matrices are constructed as follows:

$$\boldsymbol{A}^{\circ} = \begin{bmatrix} 0.5432 & 0.0137 & 0 & 0.9778 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1178 & 0.2215 & 0 & -0.9661 & 0 & 0 & 0 & 0 \\ 0 & -10.5130 & -0.9967 & 0 & 0.6176 & 0 & 0 & 0 & 0 \\ 2.6221 & -0.0030 & 0 & -0.5057 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7075 & -0.0939 & 0 & -0.2120 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{B}_f^{\circ} = \begin{bmatrix} -0.0318 & 0 & -0.0548 & -0.0318 & 0.0004 \\ 0.0024 & 0 & -0.0095 & 0.0024 & 0.0287 \\ -2.2849 & 0 & 1.9574 & 2.2849 & 1.4871 \\ -0.4628 & 0 & 0.8107 & -0.4628 & 0.0024 \\ 0.0944 & 0 & -0.1861 & 0.0944 & -0.8823 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{C}^{\circ} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{V}_{fa}^{\circ} = \begin{bmatrix} -0.0318 & 0 & -0.0548 & -0.0318 & 0.0004 & 0.7593 & 0 & 0 & 0 & 0 & 0 \\ 0.0024 & 0 & -0.0095 & 0.0024 & 0.0287 & 0.4116 & 0 & 0 & 0 & 0 & 0 \\ -2.2849 & 0 & 1.9574 & 2.2849 & 1.4871 & 0.8793 & 0 & 0 & 0 & 0 & 0 \\ -0.4628 & 0 & 0.8107 & -0.4628 & 0.0024 & 0.0272 & 0 & 0 & 0 & 0 & 0 \\ 0.0944 & 0 & -0.1861 & 0.0944 & -0.8823 & 0.0389 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

For this fault scenario the closed-loop system with above designed parameters is instable and PI VA has to be activated to stabilize the faulty system.

The parameters of PI VA were determined using (54), (55), conditioned by the tuning parameter setting $\delta^\circ = 0.217$. Exploiting the SeDuMi package, with $\gamma_\circ = 4.3700$ the PI VA gain matrices are

$$\boldsymbol{G}_P = \begin{bmatrix} -18.3055 & 0.3354 & -0.1140 & -4.0384 & -0.5380 \\ 0 & 0 & 0 & 0 & 0 \\ -10.1172 & -1.6384 & -0.0834 & -1.1921 & 0.2553 \\ -9.0562 & -3.1597 & -0.1456 & -2.6616 & 1.0467 \\ -0.7671 & 2.5586 & 0.3847 & -0.4279 & -3.0123 \end{bmatrix},$$

$$\boldsymbol{G}_I = \begin{bmatrix} -2.3650 & -0.2736 & 1.1260 & -0.3141 \\ 0 & 0 & 0 & 0 \\ -1.6237 & -0.2461 & 1.2897 & -0.0650 \\ -0.9579 & -0.1727 & 0.1167 & 0.1702 \\ -0.0191 & 0.5438 & -0.1140 & -0.3134 \end{bmatrix},$$

The eigenvalue spectrum of the matrix $\boldsymbol{A}_c^\circ = \boldsymbol{A}^\circ - \boldsymbol{B}_f^\circ \boldsymbol{G}^\circ$ is

$$\rho(\boldsymbol{A}_c^\circ) = \left\{ \begin{array}{ccc} -1.1166 & -0.1636 & -0.1433 \\ -1.4956 \pm 1.0372i & -1.5616 \pm 0.7682i & -0.1928 \pm 0.0352i \end{array} \right\}$$

and this spectrum determines the dynamics of the closed-loop system after activation of PI VA.

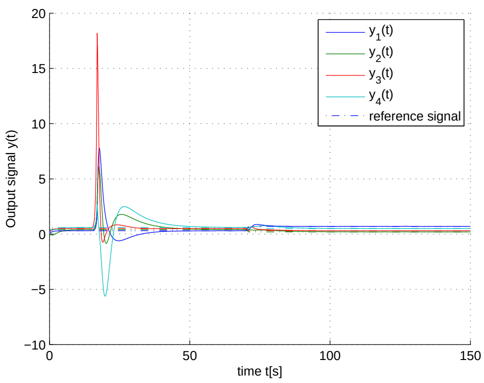

Figure 1. System output response

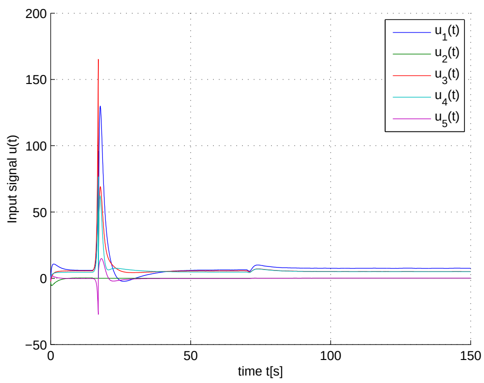

Figure 2. Control variables

In Fig. 1 and Fig. 2 are shown the time responses of the system output and control variables for the control realized by DOC of order $p = 1$ in the controller forced mode and PI VA acting on the faulty system. The single second actuator fault occurred at time instant $t = 15\,s$ and PI VA was activated at time instant $t = 17\,s$. In simulation, the initial condition was set $\boldsymbol{q}_0 = 0$ and the desired output values $\boldsymbol{w}(t) = [\,0.3\ 0.4\ 0.5\ 0.6\,]$ were changed step-wise at the time instant $t = 70\,s$ to $\boldsymbol{w}(t) = [\,0.7\ 0.2\ 0.3\ 0.5\,]$. Although the values of output variables were achieved, the output and control variables peaks after the activation of PI VA are excessively high.

Within the same simulation conditions, Fig. 3 and Fig. 4 show the time responses of the system output and control variables for control realized by DOC of order $p = 4$ and Fig. 5 and Fig. 6 show the time response of the same variables for control with DOC of order $p = 7$ and PI VA. It is obvious that an appropriate conjunction of orders of DOC gives the possibility to significantly reduce the variables peaks after activation of VAs. Note, there are no rules for the optimal order selection of DOC [10].

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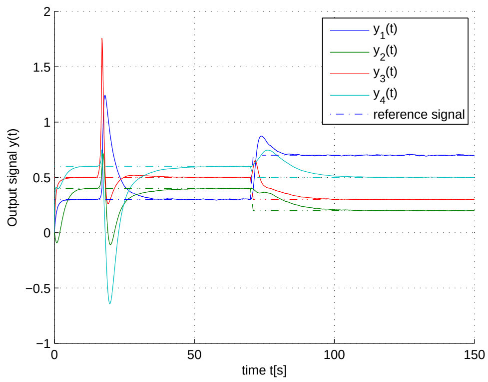

Figure 3. System output response

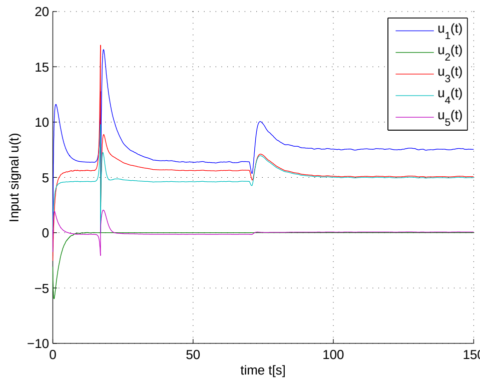

Figure 4. Control variables

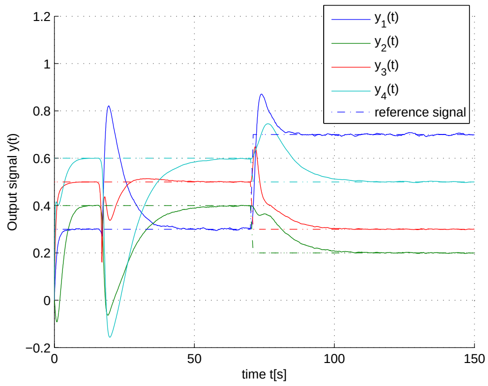

Figure 5. System output response

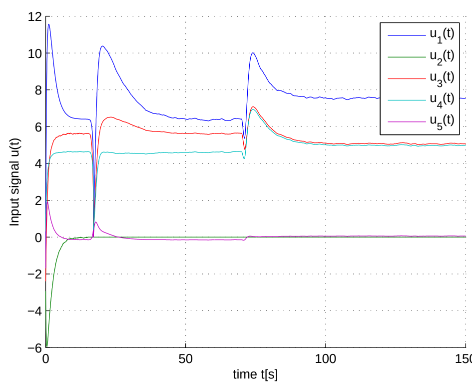

Figure 6. Control variables

6. Concluding Remarks

A key contribution of the proposed approach is the blending of the virtual actuator technique and the output control principle in a unique dynamic scheme, able to provide fault tolerance against actuator faults with such acceptable responses of the system variables, primarily after activation of DVA, which cannot by reached by applying a static output controller on the exactly same plant.

The proposed model of the dynamic effect of a virtual actuator in the FTC structure relies on newly introduced generalized disturbance patterns, reflecting fading of the nominal system state variables after DVA activation. This allows to include in the DVA design conditions the disturbance input/system output model property by H_∞ norm approach.

The application of the proposed approach requires that a fault detection and isolation subsystem is available. However, it becomes clear that the desired performances depend on the fault isolation time, but a suitable order conjunction of the both dynamic components allows significantly extend the time limit of fault detection.

Acknowledgments

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