

Application of a repetitive process setting to design of monotonically convergent iterative learning control

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Abstract. This paper deals with the problem of designing an iterative learning control algorithm for discrete linear systems using repetitive process stability theory. The resulting design produces a stabilizing output feedback controller in the time domain and a feedforward controller that guarantees monotonic convergence in the trial-to-trial domain. The results are also extended to limited frequency range design specification. New design procedure is introduced in terms of linear matrix inequality (LMI) representations, which guarantee the prescribed performances of ILC scheme. A simulation example is given to illustrate the theoretical developments.

1. Introduction

Iterative learning control (ILC) is a method of iteratively updating the control signal to a given system that repeats the same task. Each execution is known as a trial, or pass, and the sequence of operations is that a trial is completed, the system resets to the starting position and then the next trial begins, either immediately after the resetting is complete or a period of time has elapsed since completion of the resetting. The novel feature of this control law design method is the use information from the previous trial to update the control input applied on the next trial and thereby improve performance from trial-to-trial. In particular, the control objective is to find a control input so that the corresponding output precisely tracks a reference signal that is specified over a finite time interval.

Since the original work of Arimoto [2], ILC has remained as a significant area of control systems research with many algorithms experimentally verified in the research laboratory and applied in industrial applications. An overview of developments can be found in, e.g., the survey papers [1, 3, 12], where the last of these has a special focus on run-to-run control as found in the chemical process industries. Applications areas include industrial robotics, see, e.g., [7], where the pick and place operation common in many mass manufacturing processes is an immediate fit to ILC, and wafer stage motion systems, see, for example, [5].

One common approach, see, for example, [1, 3] as starting points for the literature, to ILC design is to first apply a feedback control law to stabilize and/or produce acceptable along the trial dynamics and then apply ILC to force monotonic trial-to-trial error convergence of the resulting system. This is a two step design approach with separate design of the feedback



and learning filters where, for example, the ILC learning update is calculated as the inverse of the dynamics resulting from the feedback controller design.

An alternative approach to ILC design is to use the two-dimensional/repetitive systems setting, i.e., systems that propagate information in two independent directions where for ILC these directions are from trial-to-trial and along each trial respectively. Repetitive processes [10] are a distinct class of two-dimensional systems where information in the temporal domain is limited to a finite duration and hence a more natural match to ILC. These processes make a series of sweeps, termed passes or trials in the ILC setting, through a set of dynamics defined over a finite duration and once each pass is complete the process resets to the starting location. On each pass and output, termed the pass profile, is produced that acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. The result can be oscillations that increase in amplitude from pass-to-pass.

Repetitive processes cannot be controlled by direct application of standard systems theory and algorithms and this has led to the development of a stability theory for them and substantial progress on control system specification and design [10, 14, 15]. The result of repetitive process setting is a one step design for trial-to-trial error convergence and transient response along the trials and hence simultaneous treatment of the trial-to-trial error and transient response along the trials is possible.

This paper deals with simultaneous synthesis of both feedback and learning controllers in an ILC scheme for error convergence and performance, starting with a new result for monotonic trial-to-trial error convergence. This result is achieved by converting the problem to one of stability along the trial for a discrete linear repetitive process, leading to design based on Linear Matrix Inequality (LMI) based computations. Another contribution of the paper is the development of a method deal with the case when the system to be controlled has relative degree greater than unity through the use of an anticipative feedforward control law.

Throughout this paper, the null and identity matrices with the required dimensions are denoted by 0 and I , respectively, and the notation $X \succ Y$ means that the matrix $X - Y$ is positive definite. Also $\text{sym}\{M\}$ is used to denote the symmetric matrix $M + M^T$ and $\rho(\cdot)$ denotes the spectral radius of its matrix argument, i.e., if $\lambda_i, 1 \leq i \leq q$, denotes the eigenvalues of a $q \times q$ matrix, say H , $\rho(H) = \max_{1 \leq i \leq q} |\lambda_i|$. The superscript $*$ denotes the complex conjugate transpose of a matrix and \otimes the matrix Kronecker product.

Use will also be made of the following results, where the first is the generalized KYP lemma and the second the Projection Lemma.

Lemma 1. [6] Consider matrices $\mathbb{A}, \mathbb{B}_0, \Theta$ and

$$\Phi = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \Psi = \begin{bmatrix} 0 & e^{j\omega_c} \\ e^{-j\omega_c} & -2\cos(\omega_d) \end{bmatrix}, \quad (1)$$

with $\omega_c = (\omega_l + \omega_u)/2$, $\omega_d = (\omega_u - \omega_l)/2$ and ω_l, ω_u satisfying $-\pi \leq \omega_l \leq \omega_u \leq \pi$. Suppose also that $\det(e^{j\omega}I - \mathbb{A}) \neq 0$ for all $\omega \in [\omega_l, \omega_u]$. Then the following statements are equivalent.

i)

$$\begin{bmatrix} (e^{j\omega}I - \mathbb{A})^{-1}\mathbb{B}_0 \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (e^{j\omega}I - \mathbb{A})^{-1}\mathbb{B}_0 \\ I \end{bmatrix} \prec 0, \quad \forall \omega \in [\omega_l, \omega_u]. \quad (2)$$

ii) There exist $\mathcal{Q} \succ 0$ and a symmetric \mathcal{P} such that

$$\begin{bmatrix} \mathbb{A} & \mathbb{B}_0 \\ I & 0 \end{bmatrix}^T (\Phi \otimes \mathcal{P} + \Psi \otimes \mathcal{Q}) \begin{bmatrix} \mathbb{A} & \mathbb{B}_0 \\ I & 0 \end{bmatrix} + \Theta \prec 0. \quad (3)$$

Lemma 2. [4] *Given a symmetric matrix $\Gamma \in \mathbb{R}^{p \times p}$ and two matrices Λ, Σ of column dimension p , there exists a matrix \mathcal{W} such that the following inequality holds*

$$\Gamma + \text{sym}\{\Lambda^\top \mathcal{W} \Sigma\} \prec 0, \quad (4)$$

if, and only if the following two projection inequalities with respect to \mathcal{W} are satisfied

$$\Lambda^\perp{}^\top \Gamma \Lambda^\perp \prec 0, \quad \Sigma^\perp{}^\top \Gamma \Sigma^\perp \prec 0, \quad (5)$$

where Λ^\perp and Σ^\perp are arbitrary matrices whose columns form a basis of nullspaces of Λ and Σ , respectively.

2. Background and problem formulation

The notation for variables in this paper is of the form $w_k(p)$, $0 \leq p \leq \alpha < \infty$, $k \geq 0$, where w is the scalar or vector-valued variable under consideration, α is the finite trial length and k is the trial number. The plant dynamics is assumed to be discrete linear time-invariant and described in the ILC setting as

$$\begin{aligned} x_k(p+1) &= Ax_k(p) + Bu_k(p), \\ y_k(p) &= Cx_k(p), \end{aligned} \quad (6)$$

where $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the output vector and $u_k(p) \in \mathbb{R}^l$ is the control input vector. Also we define z as the standard forward shift operator along discrete-time axis, i.e.,

$$zx_k(p) = x_k(p+1),$$

see, e.g., the cited papers [3] for the details of how the z -transform can be applied over the finite trial length without errors arising from the basic definition of this transform over an infinite interval.

This paper considers system models where the input has no direct affect on the output and hence they have relative degree $r \geq 1$, i.e.,

$$CA^{r-1}B \neq 0, \text{ and } CA^iB = 0, \forall i = 1, 2, \dots, r-2. \quad (7)$$

The form of ILC considered in this paper is shown in the block diagram of Figure 1 and consists of a unity negative feedback control loop with controller C applied on the current trial k and the ILC law as shown within the shaded part of this figure. In this paper no loss of generality arising from assuming that the initial state vector on each trial is zero and the memory block in Figure 1 represents the use of previous trial information in the computation of the current trial control input and y_d denotes the supplied reference. In the literature L is often termed the learning filter and Q the robustness filter. All computations within the shaded part of Figure 1 are completed in the time elapsed between the end of one trial and the beginning of the next, i.e., off-line.

The first task is to find guidelines for the choice of the C , L and Q filters, where to improve learning and robustness a basic rule is to choose Q as a low-pass filter whose magnitude is unity for the frequency range where reference tracking is required and zero at all other frequencies. Based on the block diagram of Figure 1, the ILC law is

$$F_{k+1}(z) = Q(z) (F_k(z) + L(z)E_k(z)),$$

and

$$E_k(z) = (I + G(z)C(z))^{-1}Y_d(z) - (I + G(z)C(z))^{-1}G(z)F_k(z).$$

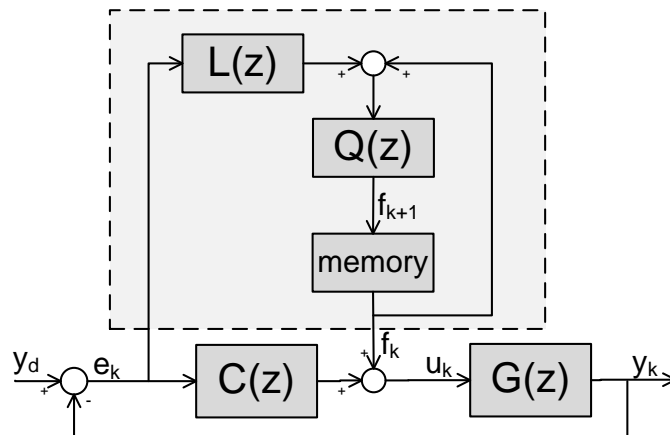


Figure 1. ILC Block Diagram Representation.

Hence, the previous trial error feedforward contribution (assuming $Y_d(z) = 0$) to the current trial error is

$$E_k(z) = -[(I + G(z)C(z))^{-1}G(z)]F(z) = -S_P(z)F_k(z),$$

where $S_P(z) = (I + G(z)C(z))^{-1}G(z)$ denotes the sensitivity function and the propagation of the error from trial-to-trial is

$$E_{k+1}(z) = M(z)E_k(z), \quad (8)$$

where

$$M(z) = Q(z)(I - S_P(z)L(z)). \quad (9)$$

One concludes from (8) is that the tracking error converges as $k \rightarrow \infty$, i.e. the error reduces between successive trials, if and only if all eigenvalues of $M(z)$ are less than one in magnitude, i.e.

$$\rho(M(e^{j\omega})) < 1, \forall \omega \in [-\pi, \pi]. \quad (10)$$

However, practical experience shows that some ILC laws have poor transients during the convergence process even if the above condition is satisfied (e.g. the tracking error may grow over some number of trials). To avoid these problems, a stronger convergence criteria is required for engineering practice. One can ensure that the Euclidean norm of the tracking error decreases monotonically for every trial if $M(e^{j\omega})$ satisfies the sufficient stability condition

$$\bar{\sigma}(M(e^{j\omega})) < 1, \forall \omega \in [-\pi, \pi], \quad (11)$$

where $\bar{\sigma}(\cdot)$ stands for the maximum singular value of its matrix argument. Although (10) is the true stability condition, (11) is more practical and commonly used in practise. From the same reason the condition (11) is used for this paper developments.

2.1. ILC as a repetitive process

This section formulates the ILC design problem considered in the repetitive process setting, starting with the dynamics of these processes. The state-space model of a discrete linear repetitive process [10] over the pass length α is

$$\begin{aligned} x_{k+1}(p+1) &= \mathbb{A}x_{k+1}(p) + \mathbb{B}u_{k+1}(p) + \mathbb{B}_0y_k(p), \\ y_{k+1}(p) &= \mathbb{C}x_{k+1}(p) + \mathbb{D}u_{k+1}(p) + \mathbb{D}_0y_k(p), \end{aligned} \quad (12)$$

where on pass k where $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the output vector and $u_k(p) \in \mathbb{R}^l$ is the control input vector. Furthermore, it is necessary to specify the boundary conditions, i.e., the pass state initial vector sequence $x_k(0)$, $k \geq 1$, and the initial pass profile $y_0(p)$. In this paper, the initial state vector on each pass is taken as the zero vector and the initial pass profile vector entries are assumed to be specified over the pass length as known functions of p .

The terms $\mathbb{B}_0 y_k(p)$ and $\mathbb{D}_0 y_k(p)$ in (12) represent the contributions of the previous pass profile vector to the current pass state and pass profile vectors respectively. This inter-pass interaction is the source of the unique control problem for these processes where the sequence of pass profiles $\{y_k\}_{k \geq 1}$ can contain oscillations that increase in amplitude from pass-to-pass, i.e., with increasing k . The stability theory [10] for these processes requires that a bounded initial pass profile produces a bounded sequence of pass profiles, where in the strongest form this property is required for all possible values of the pass length. In the remainder of this paper trial is replaced by pass to conform with the repetitive process terminology.

To formulate the ILC design problem in the repetitive process terms assume $Q = I$ (at this stage frequency attenuation using the Q -filter is not considered) and rewrite (9) as

$$M(z) = I - G(z) \begin{bmatrix} C(z) & L(z) \end{bmatrix} \left(\begin{bmatrix} (I + G(z)C(z)) & G(z)L(z) \\ 0 & I \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

It is seen from the above formula that $M(z)$ can be represented as a particular case of a general control configuration [11], where the generalized plant P , i.e., the interconnection system of the controlled system, is

$$P(z) = \left[\begin{array}{c|c} P_{11}(z) & P_{12}(z) \\ \hline P_{21}(z) & P_{22}(z) \end{array} \right] = \left[\begin{array}{c|c} I & -G(z) \\ \hline 0 & -G(z) \\ I & 0 \end{array} \right], \quad (13)$$

which is the transfer-function matrix from $[e_k^\top \ u_{k+1}^\top]^\top$ to $[e_{k+1}^\top \ y_{k+1}^\top]^\top$, where $y_{k+1} = [(\bar{y}_{k+1})^\top (\hat{y}_{k+1})^\top]^\top$. The generalized ILC controller to be found is given by $K(z) = [C(z) \ L(z)]$ and hence $M(z)$ can be rewritten as

$$M(z) = P_{11}(z) + P_{12}(z)K(z)(I - P_{22}(z)K(z))^{-1}P_{21}(z).$$

Furthermore, the state-space representation of (13) is

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p), \\ \bar{y}_{k+1}(p) &= -Cx_{k+1}(p), \\ \hat{y}_{k+1}(p) &= e_k(p), \\ e_{k+1}(p) &= -Cx_{k+1}(p) + e_k(p). \end{aligned} \quad (14)$$

Also, suppose that the ILC controller K is static, i.e.

$$u_{k+1}(p) = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} \bar{y}_{k+1}(p) \\ \hat{y}_{k+1}(p) \end{bmatrix}. \quad (15)$$

The resulting ILC dynamics can now be written as

$$\begin{bmatrix} x_{k+1}(p+1) \\ e_{k+1}(p) \end{bmatrix} = \begin{bmatrix} A - BK_1C & BK_2 \\ -C & I \end{bmatrix} \begin{bmatrix} x_{k+1}(p) \\ e_k(p) \end{bmatrix}, \quad (16)$$

which is a *discrete linear repetitive process* state-space model of the form (12). Treated as a repetitive process, the stability along the pass property will force the trial-to-trial error to converge monotonically to zero - see the next section.

If an example described by (6) has relative degree $r > 0$, an r -step delay between its input and output results. To overcome this problem, the previous trial error is shifted by r samples to form an anticipative feedforward control law [13]. Obviously, this is possible in practice since the error signal from trial k is available once this trial is complete. As a consequence this anticipative control law, the transfer-function matrix of the learning controller is taken as $z^r L$ instead of L and the generalized plant P resulting from this modification is

$$P(z) = \left[\begin{array}{c|c} P_{11}(z) & P_{12}(z) \\ \hline P_{21}(z) & P_{22}(z) \end{array} \right] = \left[\begin{array}{c|c} I & -G(z) \\ \hline 0 & -z^r G(z) \\ I & 0 \end{array} \right], \quad (17)$$

where the block P_{22} includes anticipatory operator and the forward time shift is applied to the error signal transmitted through \hat{y}_{k+1} — see the 3rd equation of (14). Then, on introducing the following notation for (16)

$$\bar{A} = A - BK_1C, \quad \bar{B}_0 = BK_2, \quad \bar{C} = -C, \quad \bar{D}_0 = I, \quad (18)$$

and the transfer-function matrix $M(z)$ from e_k to e_{k+1} is

$$M(z) = \bar{C}(zI - \bar{A})^{-1}\bar{B}_0z^r + \bar{D}_0 = \bar{C}z^r(zI - \bar{A})^{-1}\bar{B}_0 + \bar{D}_0. \quad (19)$$

Consider the term $z^r(zI - \bar{A})^{-1}$ and assuming $(zI - \bar{A})^{-1}$ is nonsingular $(zI - \bar{A})(zI - \bar{A})^{-1} = I$, and hence $z(zI - \bar{A})^{-1} = I + \bar{A}(zI - \bar{A})^{-1}$. Also

$$z^2(zI - \bar{A})^{-1} = z(I + \bar{A}(zI - \bar{A})^{-1}) = zI + \bar{A} + \bar{A}^2(zI - \bar{A})^{-1},$$

and hence

$$z^r(zI - \bar{A})^{-1} = \sum_{j=0}^{r-1} z^{r-1-j}\bar{A}^j + \bar{A}^r(zI - \bar{A})^{-1}. \quad (20)$$

In view of (7) and (18)

$$\bar{C}\bar{A}^{r-1} = \bar{C}A^{r-1} = -CA^{r-1}, \quad \bar{C}\bar{A}^r = -CA^{r-1}\bar{A} = -CA^{r-1}(A - BK_1C),$$

and then $M(z)$ in (19) can be rewritten as

$$\begin{aligned} M(z) &= \bar{C} \left(\sum_{j=0}^{r-1} z^{r-1-j}\bar{A}^j + \bar{A}^r(zI - \bar{A})^{-1} \right) \bar{B}_0 + \bar{D}_0 \\ &= -CA^{r-1}\bar{B}_0 - CA^{r-1}(A - BK_1C)(zI - (A - BK_1C)^{-1})\bar{B}_0 + \bar{D}_0. \end{aligned}$$

or

$$M(z) = \mathbb{C}(zI - \mathbb{A})^{-1}\mathbb{B}_0 + \mathbb{D}_0, \quad (21)$$

where the matrices in the state-space quadruple $\{\mathbb{A}, \mathbb{B}_0, \mathbb{C}, \mathbb{D}_0\}$ are

$$\mathbb{A} = A - BK_1C, \quad \mathbb{B}_0 = BK_2, \quad \mathbb{D}_0 = I - CA^{r-1}BK_2, \quad \mathbb{C} = -CA^r + CA^{r-1}BK_1C. \quad (22)$$

These matrices define a *discrete linear repetitive process* state-space model of the form (12) and will be used to formulate the ILC design problem in terms of repetitive process stability theory.

3. ILC design

In this section, the stability theory for discrete linear repetitive processes is used to develop a new ILC design algorithm. It has to be emphasized that several sets and concepts for stability of discrete linear repetitive processes are known [10]. From the practical point of view, the most important are those which guarantee asymptotic stability or stability along the pass (trial). Asymptotic stability guarantees a bounded sequence of pass profiles (i.e. output signals) for a bounded initial pass profile over the finite and fixed pass length α , whereas stability along the pass is stronger since it requires this property uniformly, that is, for all possible values of the pass length and hence, it is not surprising that asymptotic stability is a necessary condition for stability along the pass.

The following result characterizes stability along the pass of examples described by (12).

Lemma 3. [10] *A discrete linear repetitive process described by (12) and (22) is stable along the pass if and only if*

- i) $\rho(\mathbb{D}_0) < 1$,
- ii) $\rho(\mathbb{A}) < 1$,
- iii) *all eigenvalues of $M(e^{j\omega}) = \mathbb{C}(e^{j\omega}I - \mathbb{A})^{-1}\mathbb{B}_0 + \mathbb{D}_0$, $\forall \omega \in [-\pi, \pi]$ have modulus strictly less than unity.*

Obviously, in terms of checking the conditions of the above result, only the third condition is a problem because it requires to check if all eigenvalues of $M(e^{j\omega})$, that is the characteristic loci, lie inside of the unit circle. Simply, the condition *iii*) of Lemma 3 requires to check if (10) holds. However, as discussed previously, we focus our attention on sufficient condition for (10), that is given by (11). Again, it is important to stress that (11) must hold for entire frequency range. In such the case only, i.e. when $\omega \in [-\pi, \pi]$, one can take $\Psi = 0$ in (1) and then making the similar steps to these shown in [9, 8] it is possible to obtain the following result.

Theorem 1. *An ILC scheme described as a discrete linear repetitive process of the form (12) and (22) is stable along the pass and hence monotonic trial-to-trial error convergence occurs if there exist matrices $S \succ 0$, \mathcal{W} and a symmetric matrix \mathcal{P} such that the following matrix inequalities are feasible*

$$\begin{bmatrix} S - \mathcal{W} - \mathcal{W}^\top & \mathcal{W}^\top \mathbb{A} \\ \mathbb{A}^\top \mathcal{W} & -S \end{bmatrix} \prec 0, \quad (23)$$

$$\begin{bmatrix} -\mathcal{P} & -\mathcal{W} & 0 & 0 \\ -\mathcal{W}^\top & \mathcal{P} + \mathbb{A}^\top \mathcal{W} + \mathcal{W}^\top \mathbb{A} & \mathcal{W}^\top \mathbb{B}_0 & \mathbb{C}^\top \\ 0 & \mathbb{B}_0^\top \mathcal{W} & -I & \mathbb{D}_0^\top \\ 0 & \mathbb{C} & \mathbb{D}_0 & -I \end{bmatrix} \prec 0. \quad (24)$$

Proof. Assume that (23) and (24) hold. Then $\mathcal{W} + \mathcal{W}^\top \succ S \succ 0$ which implies that \mathcal{W} is non-singular and hence invertible. Next, rewrite (23) as

$$\begin{bmatrix} S & 0 \\ 0 & -S \end{bmatrix} + \text{sym} \left\{ \begin{bmatrix} -I \\ \mathbb{A}^\top \end{bmatrix} \mathcal{W} \begin{bmatrix} I & 0 \end{bmatrix} \right\} \prec 0,$$

and by virtue of Lemma 2 yields $\mathbb{A}^\top S \mathbb{A} - S \prec 0$. Hence, from the Lyapunov stability theory for standard discrete linear systems, condition *ii*) of Lemma 3 holds. Next, application of the Schur's complement formula enables (24) to be rewritten as

$$\begin{bmatrix} -\mathcal{P} & 0 & 0 \\ 0 & \mathcal{P} + \mathbb{C}^\top \mathbb{C} & \mathbb{C}^\top \mathbb{D}_0 \\ 0 & \mathbb{D}_0^\top \mathbb{C} & \mathbb{D}_0^\top \mathbb{D}_0 - I \end{bmatrix} + \text{sym} \left\{ \begin{bmatrix} -I \\ \mathbb{A}^\top \\ \mathbb{B}_0^\top \end{bmatrix} \mathcal{W} \begin{bmatrix} 0 & I & 0 \end{bmatrix} \right\} \prec 0, \quad (25)$$

where \mathcal{W} is a slack matrix variable. Hence by the result of Lemma 2 one can find that (25) holds if and only if

$$\begin{bmatrix} \mathbb{A}^\top & I & 0 \\ \mathbb{B}_0^\top & 0 & I \end{bmatrix} \begin{bmatrix} -\mathcal{P} & 0 & 0 \\ 0 & \mathcal{P} + \mathbb{C}^\top \mathbb{C} & \mathbb{C}^\top \mathbb{D}_0 \\ 0 & \mathbb{D}_0^\top \mathbb{C} & \mathbb{D}_0^\top \mathbb{D}_0 - I \end{bmatrix} \begin{bmatrix} \mathbb{A} & \mathbb{B}_0 \\ I & 0 \\ 0 & I \end{bmatrix} \prec 0. \quad (26)$$

For (26) to hold requires $\mathbb{D}_0^\top \mathbb{D}_0 - I \prec 0$ and hence that condition *i*) of Lemma 3 holds. Moreover, (26) can be written in the form (3) where

$$\Theta = \begin{bmatrix} \mathbb{C} & \mathbb{D}_0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathbb{C} & \mathbb{D}_0 \\ 0 & I \end{bmatrix}, \quad (27)$$

and hence (2) for this case with Θ defined above is equivalent to

$$M(e^{j\omega})^* M(e^{j\omega}) \prec I, \quad \forall \omega \in [-\pi, \pi]. \quad (28)$$

Hence condition *iii*) of Lemma 3 holds and the proof is complete. \square

Remark 1. The inequality (28) can be rewritten as

$$M(e^{j\omega})^* M(e^{j\omega}) \prec \mu^2 I, \quad \forall \omega \in [-\pi, \pi], \quad (29)$$

where $0 < \mu \leq 1$. Then $\bar{\sigma}(M(e^{j\omega})) < \mu$ and by minimizing μ a higher speed of monotonic trial-to-trial error convergence is obtained.

The inequalities in Theorem 1 are not LMIs as they are bilinear in \mathcal{W} and the matrices defining the controller K . This means no computationally effective method exists for checking (23) and (24). However, by using a particular set of transformations, the inequality (24) can be converted into an LMI. To overcome problems with conversions of (24) into an LMI it is sufficient to find a matrix X such that

$$C\mathcal{W} = XC. \quad (30)$$

It is known that the above equality is equivalent to imposing a block diagonal structural constraint on the matrix variable \mathcal{W} . To proceed, observe that for a full rank matrix C , the singular value decomposition (SVD) of C can be represented as $C = U[R \ 0]V^T$ where U and V are unitary matrices and R is a diagonal matrix with positive diagonal elements in decreasing order. Application of SVD allows us to find the equivalent representation of the equality $C\mathcal{W} = XC$. Simply, take

$$\mathcal{W} = V \begin{bmatrix} \mathcal{W}_{11} & 0 \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix} V^T, \quad (31)$$

and suppose X is chosen as $X = UR\mathcal{W}_{11}R^{-1}U^T$ and then

$$XC = \overbrace{UR\mathcal{W}_{11}R^{-1}U^T}^X \overbrace{U[R \ 0]V^T}^C = UR\mathcal{W}_{11}[I \ 0]V^T = UR[I \ 0]V^T V \begin{bmatrix} \mathcal{W}_{11} & 0 \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix} V^T = C\mathcal{W}.$$

Hence, the matrix structure as in (31) allows us to reformulate the result of Theorem 1 to an LMI-based characterization for the existence of an ILC controller K of (15).

Theorem 2. *An ILC scheme described as a discrete linear repetitive process of the form (12) and (22) is stable along the pass and hence monotonic trial-to-trial error convergence occurs if there exist matrices $\widehat{\mathcal{W}}$ of (31), K_2 , N_1 , $\widehat{S} \succ 0$ and $\widehat{\mathcal{P}} \succ 0$ such that the following LMIs are feasible*

$$\begin{bmatrix} \widehat{S} - \widehat{\mathcal{W}} - \widehat{\mathcal{W}}^\top & A\widehat{\mathcal{W}} - BN_1C \\ (A\widehat{\mathcal{W}} - BN_1C)^\top & -\widehat{S} \end{bmatrix} \prec 0, \quad (32)$$

$$\begin{bmatrix} -\widehat{\mathcal{P}} & -\widehat{\mathcal{W}}^\top & 0 & 0 \\ -\widehat{\mathcal{W}} & \widehat{\mathcal{P}} + \text{sym} \left\{ A\widehat{\mathcal{W}} - BN_1C \right\} & BK_2 & (-CA^r + CA^{r-1}BN_1C)^\top \\ 0 & K_2^\top B^\top & -I & (I - CA^{r-1}BK_2)^\top \\ 0 & -CA^r + CA^{r-1}BN_1C & I - CA^{r-1}BK_2 & -I \end{bmatrix} \prec 0. \quad (33)$$

Moreover, if the above LMIs are feasible, K_1 is obtained from

$$K_1 = N_1 X^{-1}. \quad (34)$$

Proof. Suppose that LMIs (32) and (33) are feasible. Then $\widehat{\mathcal{W}} + \widehat{\mathcal{W}}^\top \succ \widehat{S} \succ 0$ which implies that $\widehat{\mathcal{W}}$ with the structure defined by (31) is non-singular and hence invertible. Next post- and pre-multiply (32) by $\text{diag} \{ \widehat{\mathcal{W}}^{-1}, \widehat{\mathcal{W}}^{-1} \}$ to obtain (23) by setting $\mathcal{W} = \widehat{\mathcal{W}}^{-1}$. Moreover, the LMI (33) results from left and right multiplying (24) by $\text{diag} \{ \mathcal{W}, \mathcal{W}, I, I \}$ and its transpose, respectively. Setting $\widehat{\mathcal{P}} = \mathcal{W}^\top \mathcal{P} \mathcal{W}$, $\widehat{S} = \mathcal{W}^\top S \mathcal{W}$, $N_1 = K_1 X$ and applying (30) to the resulting inequality completes the proof. \square

3.1. ILC with design specifications over finite frequency ranges

The objective now considered is to develop an approach for shaping the maximum singular value of $M(e^{j\omega})$ to meet the control specifications in different frequency ranges without introducing weighting filters, where instead of considering the entire frequency range $[-\pi, \pi]$ attention can be limited to $\omega \in [0, \pi]$ only. Moreover, the results that follow are based on the fact that if this frequency range is partitioned into H intervals (not necessarily with the same length) such that

$$[0, \pi] = \bigcup_{h=1}^H [\omega_{h-1}, \omega_h], \quad (35)$$

where $\omega_0 = 0$ and $\omega_H = \pi$ then Lemma 1 can be applied at any frequency in any of these intervals. The possibility to specify different performance specifications has considerable practical significance since common performance issues occur over different frequency ranges. For example, trial-to-trial error convergence rate is in the ‘low’ frequency range whereas low sensitivity to disturbances and sensor noise are in ‘high’ frequency range. These objectives are included in the requirement that

$$\bar{\sigma}(M(e^{j\omega})) < \mu_h, \quad \forall \omega \in [\omega_{h-1}, \omega_h], \quad h = 1, \dots, H, \quad (36)$$

where entire frequency range is partitioned according to (35) and

$$0 < \mu_h \leq 1, \quad h = 1, \dots, H. \quad (37)$$

The choice of the frequency intervals and the μ_h are determined by knowledge of the application under consideration. Making use of Lemma 1 gives the following result for design to meet the specifications (36).

Theorem 3. Consider an ILC scheme described as a discrete linear repetitive process of the form (12) and (22). Furthermore, suppose that the entire frequency range is arbitrarily divided into H possible different frequency intervals as in (35). Then i) the resulting repetitive process is stable along the pass, ii) monotonic trial-to-trial error convergence occurs and iii) the finite frequency performance specifications (36) are satisfied if there exist matrices \widehat{W} of the structure defined in (31), K_2 , N_1 , $\widehat{S} \succ 0$, $\widehat{Q}_h \succ 0$, $\widehat{P}_h \succ 0$ and arbitrary chosen scalars μ_h satisfying (37) such that the following LMIs are feasible

$$\begin{bmatrix} \widehat{S} - \widehat{W} - \widehat{W}^\top & A\widehat{W} - BN_1C \\ (A\widehat{W} - BN_1C)^\top & -\widehat{S} \end{bmatrix} \prec 0, \quad (38)$$

$$\begin{bmatrix} -\widehat{P}_h & e^{j\omega_{ch}} \widehat{Q}_h - \widehat{W}^\top & 0 & 0 \\ e^{-j\omega_{ch}} \widehat{Q}_h - \widehat{W} & \Lambda_{22} & BK_2 & (-CA^r + CA^{r-1}BN_1C)^\top \\ 0 & K_2^\top B^\top & -\mu_h^2 I & (I - CA^{r-1}BK_2)^\top \\ 0 & -CA^r + CA^{r-1}BN_1C & I - CA^{r-1}BK_2 & -I \end{bmatrix} \prec 0 \quad (39)$$

for all $h = 1, \dots, H$, where

$$\Lambda_{22} = \widehat{P}_h - 2 \cos(\omega_{dh}) \widehat{Q}_h + \text{sym} \left\{ A\widehat{W} - BN_1C \right\}, \quad \omega_{ch} = \frac{\omega_{h-1} + \omega_h}{2}, \quad \omega_{dh} = \frac{\omega_h - \omega_{h-1}}{2}. \quad (40)$$

Moreover, if these LMIs are feasible, the controller gain K_1 can be obtained from (34).

Proof. This follows that for Theorem 2 where Ψ is chosen as in (1) for each of H frequency ranges satisfying (35) with frequency specifications given in (36). Since $\bar{\sigma}(M(e^{j\omega})) < 1$ in each frequency interval, it follows from (35) that $\bar{\sigma}(M(e^{j\omega})) < 1$ holds over entire frequency range the proof is complete. \square

3.2. Q -filter design procedure

Theorem 3 allows us to determine the frequency range where $\bar{\sigma}(M(e^{j\omega})) < 1$ and perform learning over this frequency range only. The remaining frequencies should be cut-off by the Q -filter because the inverse of the plant G is not sufficiently matched at higher frequencies. Therefore, the Q -filter can be chosen as a low-pass filter (which can be implemented as the zero-phase filter, e.g., by using the `filtfilt` routine in MATLAB) with cut-off frequency equal to ω_H , i.e., the highest frequency for which the result of Theorem 3 is valid.

4. Numerical example

In this section a numerical example is considered that illustrate the effectiveness and application and of the new result in this paper. The example given in this section uses a model of laboratory servomechanism system. The system consists of DC motor and the inertial mass which are connected through the rigid shaft - see Figure 2 where a configuration scheme of the modular servo used in this example is given. Rotational motion of the mass is exited by the DC motor. As

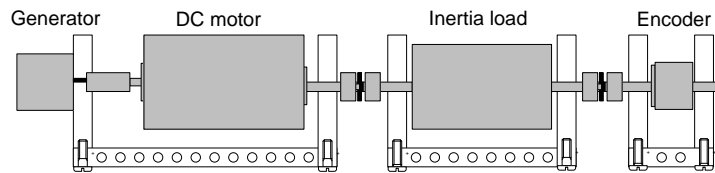


Figure 2. Diagram configuration of the modular servo.

it is in common, the rotational speed of the mass is considered as the output and the armature voltage is considered as the input and hence the following transfer-function represents model of the controlled plant

$$G(s) = \frac{\dot{\Theta}(s)}{V(s)} = \frac{K}{(Js + b)(Ls + R) + K^2}, \quad (41)$$

where K represents both the motor torque constant (K_t) and the back emf constant (K_e), J is the total moment of inertia of the rotor and the mass, b is the motor viscous friction constant, L is the electric inductance and R stands for the electric resistance. By choosing the following values $J = 0.001118$, $b = 3.5077 \cdot 10^{-6}$, $K = 0.056$, $R = 2$ and $L = 0.001$ the model of (41) results in

$$G(s) = \frac{0.056}{1.118 \cdot 10^{-6}s^2 + 0.002236s + 0.003143}. \quad (42)$$

The above transfer-function has been discretized with a sampling time of $T_s = 0.01$ seconds to give a discrete linear state-space model of the form (6) with

$$A = \begin{bmatrix} 0.986 & -3.3769 \cdot 10^{-5} \\ 0.0001 & 0 \end{bmatrix}, B = \begin{bmatrix} 16 \\ 0 \end{bmatrix}, C = [0.0148 \ 12.6705]. \quad (43)$$

Executing the design procedure of Theorem 3 for frequency range from 0 to 2[Hz] gives

$$K_2 = 0.0013, N_1 = 3.2071 \cdot 10^{-4}, W_{11} = 25.8652, X = 25.8652,$$

and hence by (34) gives

$$K_1 = N_1 X^{-1} = 1.2399 \cdot 10^{-5}.$$

The resulting singular value plot of $M(e^{j\omega})$ is given in Figure 3 and confirms that the design specifications are met, i.e $M(e^{j\omega}) < 1$ in chosen frequency range.

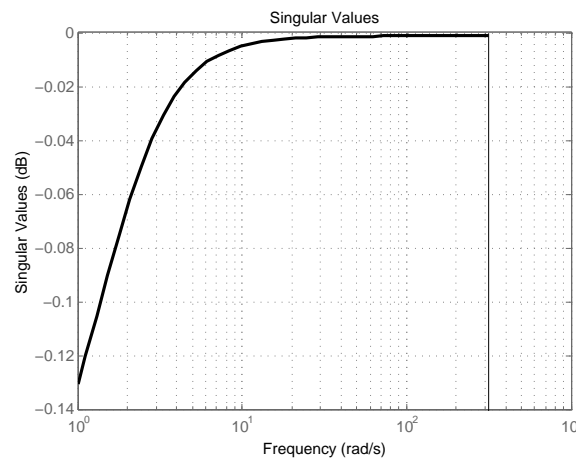


Figure 3. Singular value plot of $M(e^{j\omega})$

5. Conclusions

In this paper a systematic procedure for design of ILC schemes using the repetitive process setting has been developed. Using this design it is possible to avoid adding weighting filters to obtain the required frequency response matrix to meet differing performance specifications.

The results have been established by using the generalized KYP lemma to transform frequency domain inequalities over finite and/or semi-finite frequency ranges for a transfer-function to LMIs. A simulation study using a model for a servomechanism system illustrates the effectiveness of the design.

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