# Interaction of two pulsating spherical bubbles in external pressure field near the contact

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**Abstract**. The problem of coalescence of two pulsating spherical bubbles is studied using Lagrangian formalism in assumptions that the acoustic field is weak, the frequency of the external impact is appreciably less than natural oscillation frequency and influence of viscosity on phase of radii pulsation is small. The obtained necessary condition for coalescence of the bubbles is determined by the dimensionless parameter, whose boundary value is nonlinearly depending on the quotient of the bubbles' radii.

## 1. Introduction

The studies on the interaction of two oscillating spheres with periodically varying radii were first reported by V.F.K. Bjerknes [1].

When the distance $x$ (figure 1, $x = x_2 - x_1$, $u_1 = \dot{x}_1$, $u_2 = -\dot{x}_2$) between the centers of the spheres is large, the force averaged over the period (secondary Bjerknes force) is inversely proportional to the squared distance and equals

$$F_B = -\frac{4\pi\rho}{x^2} < a_1^2 a_2^2 \dot{a}_1 \dot{a}_2 >, \tag{1}$$

where $a_1$, $a_2$ are the radii of bubbles, and we consider that $a_1 \leq a_2$, $\rho$ is the fluid density and the dot signifies the time derivation.

In this paper we use the model of two interacting gas bubbles in the presence of a weak acoustic field

$$p = p_\infty + \Delta p \cos(\omega t), \ \ \Delta p << p_\infty. \tag{2}$$

Assuming that the frequency of the external impact $\omega$ is considerably smaller than the natural oscillation frequency $\omega_0$ and satisfies the condition $\omega\gamma << \omega_0^2$, where $\gamma = 4\mu / \rho a^2$, we obtain that [2] $a_i = a_{0i}(1 + \alpha_0 \cos\omega t)$, $\alpha_0 = -\Delta P / 3\kappa p_\infty$, where $\kappa$ is the polytropic exponent and $\mu$ is the fluid's viscosity . In this case, as the bubbles are pulsating in phase, the secondary Bjerknes force between them is attractive.

When the distance $x$ is small, the attraction force is [3]

$$F_B \approx -\frac{\pi}{8}\rho a^2 < \dot{h}^2 > \ln\frac{a}{h}, \quad a = \frac{a_1 a_2}{a_1 + a_2}, \quad h = x - (a_1 + a_2). \tag{3}$$

The classical formula for the viscous interaction force for the bubbles with fixed radii which are nearly touching is [4]

$$F_\mu = -\frac{6\pi\mu}{\left(a_1^{-1} + a_2^{-1}\right)^2}\frac{\dot{h}}{h} = -\frac{6\pi\mu}{\left(a_1^{-1} + a_2^{-1}\right)^2}\frac{d\ln(h)}{dt}. \tag{4}$$

The force of the viscous interaction averaged over the period is equal to zero. So the sum

$$\left\langle F_B + F_\mu \right\rangle < 0. \tag{5}$$

It means that bubbles should always coalesce. But in some cases the bubbles coalesce, in others do not. The purpose of this work was finding the necessary conditions for the coalescence of the bubbles.

## 2. Theoretical part

The initial point of our research is the kinetic energy of the fluid $T$ . We consider the surrounding medium to be incompressible and the bubble shape to be spherical. The exact solution of the problem of interaction of two solid spheres moving along the line of their centres was first obtained by Hicks [5]. The general expression for $T$ is [3,6]

$$\frac{T}{2\pi\rho} = A_1 u_1^2 + 2Bu_1 u_2 + A_2 u_2^2 + D_1\dot{a}_1^2 + 2E\dot{a}_1\dot{a}_2 + D_2\dot{a}_2^2 + [C_{12}\dot{a}_2 + C_{11}\dot{a}_1]u_1 + [C_{21}\dot{a}_1 + C_{22}\dot{a}_2]u_2$$

$$2A_i = \frac{a_i^3}{3} + \sum_{n=1}^{\infty}\left(\frac{a_i}{A_n^i}\right)^3, \quad 2B = \sum_{n=1}^{\infty}\left(\frac{a_k}{B_n^i}\right)^3, \quad D_i = a_i^3 + \sum_{n=1}^{\infty}\frac{a_i^3}{A_n^i}\left[1 + ((B_n^k)^2 - 1)\ln\left(1 - \frac{1}{(B_n^k)^2}\right)\right], \tag{6}$$

$$E = \frac{(a_1 a_2)^2}{a_k}\sum_{n=0}^{\infty}\left(\frac{1}{B_{n+1}^k} - B_n^i \ln\left(1 + \frac{1}{B_{n+1}^k B_n^i}\right)\right), C_{ii} = \sum_{n=1}^{\infty}\frac{a_i^3}{(A_n^i)^2 B_n^k}, C_{ki} = \sum_{n=1}^{\infty}\frac{a_k^3}{(B_n^i)^2 A_{n-1}^k}, i \neq k,\ i,k = 1,2$$

The coefficients $A_n^i$ and $B_n^i$ can be calculated from recurrence relations with the initial conditions

$$A_0^i = 1, B_0^i = 0, \ A_n^i = \frac{x}{a_i}B_n^k - \frac{a_k}{a_i}A_{n-1}^i, \quad B_n^i = \frac{x}{a_i}A_{n-1}^k - \frac{a_k}{a_i}B_{n-1}^i. \tag{7}$$

The Lagrange function for this system equals

$$L = L(x_i, a_i, \dot{x}_i, \dot{a}_i, t) = T - \Pi, \quad \Pi = \sum_{i=1}^{2}\left\{\frac{4\pi}{3}a_i^3\left[p(t) + \frac{p_{0i}}{\kappa - 1}\left(\frac{a_{0i}}{a_i}\right)^{3\kappa}\right] + 4\pi\sigma a_i^2\right\} \tag{8}$$

Using the thin-layer approximation, the more accurate formula for the force of the viscous interaction of the bubbles is [3]:

$$F_\mu = -\frac{6\pi\mu}{\left(a_1^{-1} + a_2^{-1}\right)^2}\left[\frac{\dot{h}}{h} + \left(\frac{\dot{a}_1}{a_1^2} + \frac{\dot{a}_2}{a_2^2}\right)\frac{\ln[h/(a_1 + a_2)]}{a_1^{-1} + a_2^{-1}}\right]. \tag{9}$$

The system of Lagrange equations is of 4th order. Assuming that $\omega^2 << \omega_0^2$, $\omega\gamma << \omega_0^2$ the system is reduced to 2nd order. In transition from the coordinates $x_1$ and $x_2$ to $h = x - a_1 - a_2$ and $y = x_1 + x_2$ it can be noticed that $y$ is a cyclic coordinate. In this case, using the Routh transformation, the problem is reduced to a single differential equation

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{\delta}} = \frac{\partial R}{\partial \delta} + F_\mu \frac{\partial h}{\partial \delta}, \tag{10}$$

where $R = \pi\rho\left(\mu_{11}a_{01}^5(1+\alpha)^5\dot{\delta}^2 + \mu_{12}a_{01}^5(1+\alpha)^4\dot{\alpha}\dot{\delta} + \mu_{22}a_{01}^5(1+\alpha)^3\dot{\alpha}^2\right)$, $\alpha = \alpha_0\cos(\omega t)$, $\delta = h/a_1$, $\mu_{11}, \mu_{12}, \mu_{22} -$ are functions of variables $\delta$ and $a_1/a_2$.

Assuming that $\alpha << 1$, the differential equation takes the form

$$\frac{d}{dt'}[\mu_{11}\dot{\delta} + \mu_{12}\dot{\alpha}] = f_B - \dot{\alpha}(\mu_{11}\dot{\delta} + \mu_{12}\dot{\alpha}) + f_\mu, \tag{11}$$

$$f_B = \frac{1}{2}(d_{11}\dot{\delta}^2 + 2d_{12}\dot{\alpha}\dot{\delta} + d_{22}\dot{\alpha}^2),\ d_{ij} = \frac{\partial\mu_{ij}}{\partial\delta}, i, j = 1,2, \tag{12}$$

$$f_\mu = \frac{F_\mu}{2\pi\rho\omega^2 a_{01}^4} = -M\left(\frac{\dot{\delta}}{\delta} + \dot{\alpha}\ln\left(\delta\frac{a_{01}}{a_{01}+a_{02}}\right)\right),\ M = \frac{3\mu a_{02}^2}{\rho\omega a_{01}^2\left(a_{01}+a_{02}\right)^2} \propto \gamma/\omega. \tag{13}$$

Here and hereafter, the dots signify the derivatives with respect to the dimensionless time $t' = \omega t$.

If the bubbles do not coalesce, they will pulsate near a certain distance $\delta_0$. In this case, the average force of the interaction is equal to zero. To find this force we have looked for the steady-state solution of (11) in the form $\delta = \delta_0 + \Delta(t')$. We take into account the linear and quadratic terms with respect to the oscillation amplitude $\alpha$. Transferring the linear terms to the left-hand side and the quadratic terms to the right-hand side, we obtain, in linear approximation, the amplitude and the phase for $\Delta$. The dimensionless average total force $f$ was obtained by averaging the quadratic terms in the right side of (11).

## 3. Results and discussion

We have obtained the dependence of the dimensionless average total force $f$ on the relative equilibrium distance $\delta_0$ for different quotients of bubbles' radii. These results are presented in figures 2-4.

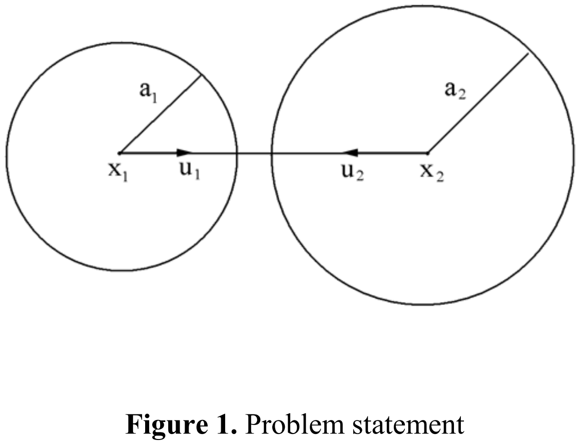


**Figure 1.** Problem statement

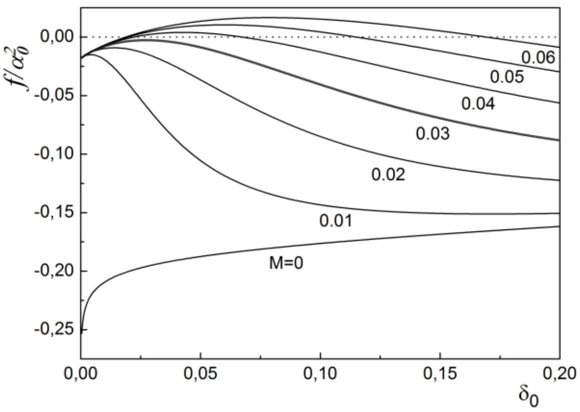


**Figure 2.** The dependences of the dimensionless average total force $f/\alpha_0^2$ on the relative distance $\delta_0$ for $a_2/a_1 = 1$.

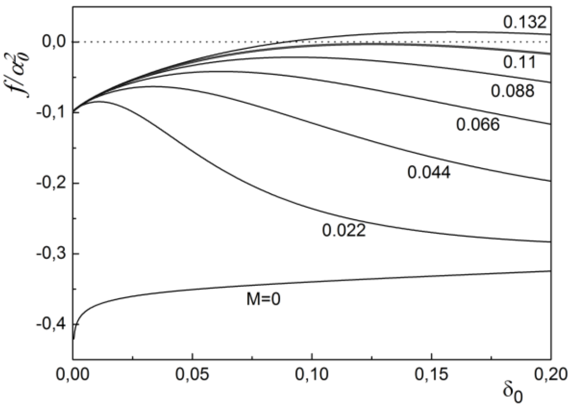


**Figure 3.** The dependences of the dimensionless average total force $f / \alpha_0^2$ on the relative distance $\delta_0$ for $a_2 / a_1 = 1.5$

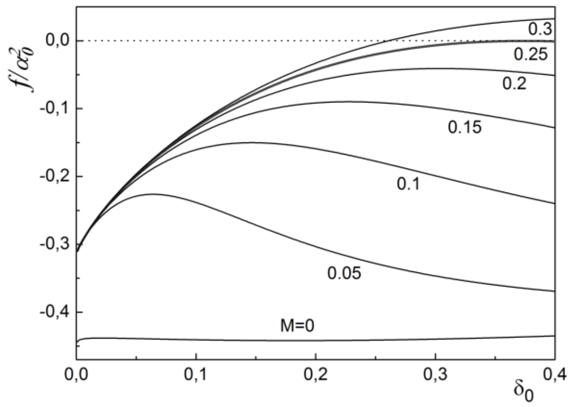


**Figure 4.** The dependences of the dimensionless average total force $f / \alpha_0^2$ on the relative distance $\delta_0$ for $a_2 / a_1 = 2$

The range of numbers $M$, in which the averaged force of the interaction is negative ( $\mu_{11} -$ is a positive function and may be considered as effective mass), determines the necessary condition for epy coalescence of the bubbles. In case of equal radii (figure 2), the coalescence takes place at $M \le 0.03$. When the quotient of the radii is $a_2 / a_1 = 1.5$ (figure 3), the area of the parameter $M$, for which the attraction is observed, expands to $M \le 0.11$. When $a_2 / a_1 = 2$ (figure 4), the attraction occurs at $M \le 0.25$.

The obtained results are consistent with the data of recent experiments with two bubbles ( $a_1 \approx a_2 \approx 25 \mu m$ ) in an ultrasonic standing wave at 22.4 kHz [7] witch coalescence. For this values the parameter $M$ is about $0.015$ and it is less than $0.03$ and therefore it satisfies the necessary condition for coalescence. To be noted, in our model we have neglected the buoyancy force, as it can be assumed to be compensated by the first Bjerknes force. This fact was also investigated in [8].

## 4. Conclusion

In this work we studied the forces of bubbles' interaction that depend on their size, the parameters of the fluid and the periodic external pressure field. The necessary condition for coalescence of the bubbles is determined by the dimensionless parameter *M*. With the growth of the quotient of the bubbles' radii the boundary value of this parameter is rising nonlinearly.

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## References

[1] Bjerknes V F K 1906 *Fields of Force* (Columbia University Press)
[2] Harkin A, Kaper T J and Nadim A 2001 *J. Fluid. Mech.* **445** 377 − 411
[3] Petrov A G 2011 *Fluid Dynamics* **45** (4) 579-595
[4] Batchelor G K 2000 *An Introduction to Fluid Dynamics* (Cambridge : Cambridge Univ. Press)
[5] Hicks W M 1880 *Phil. Trans. R. Soc. Lond.* **171,** 455-492
[6] Voinov O V and Petrov A G 1976, "Motion of Bubbles in a Fluid," in: *Itogi Nauki i Tekhniki. Gidromekhanika.* **10** 86-177
[7] Jiao J, He Y, Yasui K, Kentish S E, Ashokkumar M, Manasseh R and Lee J 2015 *Ultrasonics Sonochemistry* **22** 70-77
[8] Jiao J, He Y, Leong T, Kentish S E , Ashokkumar M, Manasseh R and Lee J 2013 *J. Phys. Chem.* **117** 12549–12555