Hydrodynamic representation of the Klein-Gordon-Einstein equations in the weak field limit

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Abstract. Using a generalization of the Madelung transformation, we derive the hydrodynamic representation of the Klein-Gordon-Einstein equations in the weak field limit. We consider a complex self-interacting scalar field with an arbitrary potential of the form $V(|\varphi|^2)$. We compare the results with simplified models in which the gravitational potential is introduced by hand in the Klein-Gordon equation, and assumed to satisfy a (generalized) Poisson equation. Nonrelativistic hydrodynamic equations based on the Schrödinger-Poisson equations or on the Gross-Pitaevskii-Poisson equations are recovered in the limit $c \to +\infty$.

1. Introduction

Scalar fields (SF) play an important role in particle physics, astrophysics, and cosmology [1]. Their evolution is usually described by the Klein-Gordon (KG) equation [2, 3] which can be viewed as a relativistic extension of the Schrödinger equation [4]. The KG equation describes spin-0 particles (bosons) that can be charged (complex SF) or neutral (real SF). The coupling between the KG equation and gravity through the Einstein equations, leading to the Klein-Gordon-Einstein (KGE) equations, was first considered in the context of boson stars [5, 6, 7]. It has also been proposed that dark matter (DM) halos may be made of a SF described by the KGE equations (see, e.g., [8, 9, 10] for recent reviews). Actually, at the galactic scale, the Newtonian limit is valid so that DM halos can be described by the Schrödinger-Poisson (SP) equations or by the Gross-Pitaevskii-Poisson (GPP) equations. In that case, the wave function $\psi(\vec{x}, t)$ describes a Bose-Einstein condensate (BEC) at $T = 0$, and the self-interaction of the bosons is measured by their scattering length a_s. Therefore, DM halos could be gigantic quantum objects made of BECs. The wave properties of bosonic DM may stabilize the system against gravitational collapse, providing halo cores and sharply suppressing small-scale linear power. This may solve the problems of the cold dark matter (CDM) model such as the cusp problem and the missing satellite problem. The scalar field dark matter (SFDM) model and the BEC dark matter (BECDM) model, also called ΨDM models, have received much attention in the last years.

Since DM may be a SF, it is of considerable interest to study the cosmological implications of this scenario. The cosmological evolution of a spatially homogeneous noninteracting real SF described by the KGE equations competing with baryonic matter, radiation and dark energy was considered by Matos *et al.* [11]. They found that real SFs display fast oscillations but that, on the mean, they reproduce the cosmological predictions of the standard ΛCDM model. The study of perturbations was considered by Suárez and Matos [12] for a self-interacting real SF described by the Klein-Gordon-Poisson (KGP) equations and by Magaña *et al.* [13] for a noninteracting real SF described by the KGE equations. These studies show that the perturbations can grow in the linear regime, leading, in the nonlinear regime, to the formation of structures corresponding to DM halos. This is in agreement with the early work of Khlopov *et al.* [14] who studied the Jeans instability of a relativistic SF in a static background. The case of a complex self-interacting SF representing BECDM was considered by Chavanis [15] in the context of Newtonian cosmology. His study is based on the GPP equations. Harko [16] and Chavanis [15] developed an approximate relativistic BEC cosmology and found that the perturbations grow faster in BECDM as compared to ΛCDM. Recently, Li *et al.* [17] developed an exact relativistic cosmology for a complex self-interacting SF/BEC based on the KGE equations. They studied the evolution of the homogeneous background and showed that the Universe undergoes three successive phases: a stiff matter era, followed by a radiation era due to the SF (that exists only for self-interacting SFs), and finally a matter era similar to CDM.

Instead of working directly in terms of a SF, we can adopt a fluid approach and work with hydrodynamic equations. In the case of the Schrödinger equation, this hydrodynamic approach was introduced by Madelung [18]. He showed that the Schrödinger equation is equivalent to the Euler equations for an irrotational fluid with an additional quantum potential arising from the finite value of $\hbar$. This hydrodynamic representation has been used by Böhmer and Harko [19] and by Chavanis [15, 20, 21] among others in the case of BECDM and in the case of BEC stars [22]. This hydrodynamic approach has been generalized by Suárez and Matos [12, 23] in the context of the KG equation. They used it to study the formation of structures in the Universe, assuming that DM is in the form of a fundamental SF with a $\lambda\varphi^4$ potential.

In the works [12, 23], the SF is taken to be real and the gravitational potential is introduced by hand in the KG equation, and assumed to be determined by the classical Poisson equation where the source is the rest-mass density ρ. This leads to the KGP equations. However, this treatment is not self-consistent since it combines relativistic and nonrelativistic equations. In these Proceedings, we derive the hydrodynamic representation of a complex SF coupled to gravity through the Einstein equations in the weak field approximation. In this way, we develop a self-consistent relativistic treatment. Throughout this work, we use the conformal Newtonian gauge which takes into account metric perturbations up to first order. We consider only scalar perturbations. This is sufficient if we are interested in calculating observational consequences of the SF dynamics in the linear regime. We compare our results with those obtained from the heuristic KGP equations. In these Proceedings, we develop the basic formalism and provide the main equations for an arbitrary self-interaction potential of the form $V(|\varphi|^2)$. Details and applications of our formalism can be found in our research papers [24, 25].

2. The conformal Newtonian gauge

We consider the KGE equations in the weak field limit $\Phi/c^2 \ll 1$. The equations that we derive are valid at the order $O(\Phi/c^2)$. We work with the conformal Newtonian gauge which is a perturbed form of the Friedmann-Lemaître-Robertson-Walker (FLRW) line element [26]. We consider the simplest form of the Newtonian gauge, only taking into account scalar perturbations which are the ones that contribute to the formation of structures in cosmology. We neglect anisotropic stresses. We assume that the Universe is flat in agreement with the observations of the cosmic microwave background (CMB). Under these conditions, our line element is given by

$$ds^2 = c^2\left(1 + 2\frac{\Phi}{c^2}\right)dt^2 - a(t)^2\left(1 - 2\frac{\Phi}{c^2}\right)\delta_{ij}dx^i dx^j, \tag{1}$$

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where $\Phi/c^2 \ll 1$. In this metric, $\Phi(\vec{x}, t)$ represents the gravitational potential of classical Newtonian gravity and $a(t)$ is the (dimensionless) scale factor.

3. The Lagrangian of the scalar field

We assume that DM can be described by a complex SF which is a continuous function of space and time defined at each point by $\varphi(x^\mu) = \varphi(x, y, z, t)$. The action of the relativistic SF is

$$S_\varphi = \int d^4x \sqrt{-g} \mathcal{L}_\varphi, \tag{2}$$

where $\mathcal{L}_\varphi = \mathcal{L}_\varphi(\varphi, \varphi^*, \partial_\mu\varphi, \partial_\mu\varphi^*)$ is the Lagrangian density and $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor. We adopt the following generic Lagrangian density

$$\mathcal{L}_\varphi = \frac{1}{2} g^{\mu\nu} \partial_\mu\varphi^* \partial_\nu\varphi - \frac{m^2c^2}{2\hbar^2}|\varphi|^2 - V(|\varphi|^2) \tag{3}$$

which is written for a metric signature $(+, -, -, -)$. The second (quadratic) term is the rest-mass term. The potential $V(|\varphi|^2)$ takes into account the self-interaction of the SF. In certain applications, it is relevant to consider a quartic potential of the form

$$V(|\varphi|^2) = \frac{m^2}{2\hbar^4}\lambda|\varphi|^4, \tag{4}$$

where λ is the self-interaction coupling strength. If the SF describes a BEC at $T = 0$, the quartic potential (4) models the two-particle self-interaction.[1] In that case, m represents the mass of the bosons and λ is a constant related to the s-wave scattering length of the bosons a_s (measured in the center of mass frame) by $\lambda = 4\pi a_s \hbar^2/m$. This corresponds to the first Born approximation. The potential can then be written as

$$V(|\varphi|^2) = \frac{2\pi a_s m}{\hbar^2}|\varphi|^4. \tag{5}$$

4. The energy-momentum tensor

The energy-momentum tensor of the SF is

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}_\varphi}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_\varphi. \tag{6}$$

For the generic Lagrangian (3), it takes the form

$$T_{\mu\nu} = \frac{1}{2}(\partial_\mu\varphi^*\partial_\nu\varphi + \partial_\nu\varphi^*\partial_\mu\varphi) - g_{\mu\nu}\left[\frac{1}{2}g^{\rho\sigma}\partial_\rho\varphi^*\partial_\sigma\varphi - \frac{m^2c^2}{2\hbar^2}|\varphi|^2 - V(|\varphi|^2)\right]. \tag{7}$$

By analogy with the energy-momentum tensor of a perfect fluid, the energy density and the pressure tensor of the SF are defined by $\epsilon = T_0^0$ and $P_i^j = -T_i^j$. The conservation of the energy-momentum tensor, which results from the Noether theorem, writes $D_\nu T^{\mu\nu} = 0$.

[1] It is a good approximation to ignore higher-order interactions when the boson gas is dilute, i.e., when the particle self-interaction range is much smaller than the mean interparticle distance.

5. The Klein-Gordon equation

The equation of motion for the SF can be obtained from the principle of least action. Imposing $\delta S_\varphi = 0$ for arbitrary variations $\delta\varphi$ and $\delta\varphi^*$, we obtain the Euler-Lagrange equation

$$D_\mu \left[\frac{\partial \mathcal{L}_\varphi}{\partial(\partial_\mu \varphi)^*} \right] - \frac{\partial \mathcal{L}_\varphi}{\partial \varphi^*} = 0, \tag{8}$$

where D is the covariant derivative. For the generic Lagrangian (3), this leads to the KG equation

$$\Box\varphi + \frac{m^2c^2}{\hbar^2}\varphi + 2V(|\varphi|^2)_{,\varphi^*} = 0, \tag{9}$$

where $\Box$ is the d'Alembertian operator

$$\Box \equiv D_\mu(g^{\mu\nu}\partial_\nu) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu) \tag{10}$$

and

$$V(|\varphi|^2)_{,\varphi^*} = \frac{dV}{d|\varphi|^2}\varphi. \tag{11}$$

Computing the d'Alembertian (10) with the Newtonian gauge, we obtain the KG equation

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} + \frac{3H}{c^2}\frac{\partial\varphi}{\partial t} - \frac{1}{a^2}\left(1+\frac{4\Phi}{c^2}\right)\Delta\varphi - \frac{4}{c^4}\frac{\partial\Phi}{\partial t}\frac{\partial\varphi}{\partial t} + \left(1+\frac{2\Phi}{c^2}\right)\frac{m^2c^2}{\hbar^2}\varphi + 2\left(1+2\frac{\Phi}{c^2}\right)V_{,\varphi^*} = 0, \tag{12}$$

where $H = \dot{a}/a$ is the Hubble constant. Using the expression (7) of the energy-momentum tensor, the energy density and the pressure are given by

$$\epsilon = T_0^0 = \frac{1}{2c^2}\left(1-\frac{2\Phi}{c^2}\right)\left|\frac{\partial\varphi}{\partial t}\right|^2 + \frac{1}{2a^2}\left(1+\frac{2\Phi}{c^2}\right)|\vec{\nabla}\varphi|^2 + \frac{m^2c^2}{2\hbar^2}|\varphi|^2 + V(|\varphi|^2), \tag{13}$$

$$P = -\frac{1}{3}(T_1^1 + T_2^2 + T_3^3) = \frac{1}{2c^2}\left(1-\frac{2\Phi}{c^2}\right)\left|\frac{\partial\varphi}{\partial t}\right|^2 - \frac{1}{6a^2}\left(1+\frac{2\Phi}{c^2}\right)|\vec{\nabla}\varphi|^2 - \frac{m^2c^2}{2\hbar^2}|\varphi|^2 - V(|\varphi|^2). \tag{14}$$

6. The Einstein equations

The Einstein equations write

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{15}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, and G is Newton's gravitational constant [27]. The conservation of the energy-momentum tensor is automatically included in the Einstein equations. The time-time component of the Einstein equations is

$$R_0^0 - \frac{1}{2}R = \frac{8\pi G}{c^4}T_0^0. \tag{16}$$

With the Newtonian gauge, it can be written as

$$\frac{\Delta\Phi}{4\pi G a^2} = \frac{\epsilon}{c^2} - \frac{3H^2}{8\pi G} + \frac{3H}{4\pi G c^2}\left(\frac{\partial\Phi}{\partial t} + H\Phi\right). \tag{17}$$

Using the expression (13) of the energy density ϵ, which represents the time-time component of the energy-momentum tensor, we get

$$\frac{\Delta\Phi}{4\pi G a^2} = \frac{1}{2c^4}\left(1-\frac{2\Phi}{c^2}\right)\left|\frac{\partial\varphi}{\partial t}\right|^2 + \frac{1}{2a^2c^2}\left(1+\frac{2\Phi}{c^2}\right)|\vec{\nabla}\varphi|^2 + \frac{m^2}{2\hbar^2}|\varphi|^2$$
$$+\frac{1}{c^2}V(|\varphi|^2) - \frac{3H^2}{8\pi G} + \frac{3H}{4\pi G c^2}\left(\frac{\partial\Phi}{\partial t} + H\Phi\right). \quad (18)$$

Eqs. (12) and (18) form the KGE equations.

7. Spatially homogeneous scalar field

For a spatially homogeneous SF with $\varphi_b(\vec{x},t) = \varphi_b(t)$ and $\Phi_b(\vec{x},t) = 0$, the KG equation (12) reduces to

$$\frac{1}{c^2}\frac{d^2\varphi_b}{dt^2} + \frac{3H}{c^2}\frac{d\varphi_b}{dt} + \frac{m^2c^2}{\hbar^2}\varphi_b + 2V_{,\varphi_b^*} = 0. \quad (19)$$

In that case, the energy-momentum tensor is diagonal and isotropic, $T^\mu_\nu = \mathrm{diag}(\epsilon_b, -P_b, -P_b, -P_b)$. The energy density $\epsilon_b(t)$ and the pressure $P_b(t)$ are given by

$$\epsilon_b = \frac{1}{2c^2}\left|\frac{d\varphi_b}{dt}\right|^2 + \frac{m^2c^2}{2\hbar^2}|\varphi|^2 + V(|\varphi_b|^2), \qquad P_b = \frac{1}{2c^2}\left|\frac{d\varphi_b}{dt}\right|^2 - \frac{m^2c^2}{2\hbar^2}|\varphi|^2 - V(|\varphi_b|^2). \quad (20)$$

From these equations, we obtain the continuity equation

$$\frac{d\epsilon_b}{dt} + 3H(\epsilon_b + P_b) = 0 \quad (21)$$

which is one of the Friedmann equations [27]. The other Friedmann equation is obtained from the Einstein equation (18) that reduces to

$$H^2 = \frac{8\pi G}{3c^2}\epsilon_b. \quad (22)$$

This relation shows that the term $-3H^2/8\pi G = -\epsilon_b$ in the Einstein equation (17) plays the role of a neutralizing background.[2] From Eqs. (21) and (22), we easily obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon_b + 3P_b). \quad (23)$$

8. The Gross-Pitaevskii-Einstein equations

The KG equation without self-interaction can be viewed as a relativistic generalization of the Schrödinger equation. Similarly, the KG equation with a self-interaction can be viewed as a relativistic generalization of the GP equation. In order to recover the Schrödinger and GP equations in the nonrelativistic limit $c \to +\infty$, we make the transformation [1]:

$$\varphi(\vec{x},t) = \frac{\hbar}{m}e^{-imc^2t/\hbar}\psi(\vec{x},t). \quad (24)$$

[2] In a static universe, the source of the gravitational potential is ϵ/c^2. In an expanding universe, the source of the gravitational potential is $(\epsilon - \epsilon_b)/c^2$, so that $\Phi = 0$ when $\epsilon = \epsilon_b$. When we work in the comoving frame, the expansion of the universe amounts to subtracting a neutralizing background $-\epsilon_b/c^2$ to the density ϵ/c^2 [28], as in the Jellium model of plasma physics. This is the correct way to solve the Jeans swindle [29].

The prefactor $\hbar/m$ is justified in Appendix A. Mathematically, we can always make this change of variables. However, we emphasize that it is only in the nonrelativistic limit $c \to +\infty$ that ψ has the interpretation of a wave function, and that $|\psi|^2 = \rho$ has the interpretation of a rest-mass density. In the relativistic regime, ψ and $\rho = |\psi|^2$ do not have a clear physical interpretation. We will call them "pseudo wave function" and "pseudo rest-mass density". Nevertheless, it is perfectly legitimate to work with these variables and, as we shall see, the equations written in terms of these quantities take relatively simple forms that generalize naturally the nonrelativistic ones.

Substituting Eq. (24) into Eqs. (12) and (18), we obtain

$$\begin{aligned} & i\hbar\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2mc^2}\frac{\partial^2\psi}{\partial t^2} - \frac{3}{2}H\frac{\hbar^2}{mc^2}\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2ma^2}\left(1+\frac{4\Phi}{c^2}\right)\Delta\psi - m\Phi\psi \\ & -\left(1+\frac{2\Phi}{c^2}\right)m\frac{dV}{d|\psi|^2}\psi + \frac{3}{2}i\hbar H\psi + \frac{2\hbar^2}{mc^4}\frac{\partial\Phi}{\partial t}\left(\frac{\partial\psi}{\partial t} - \frac{imc^2}{\hbar}\psi\right) = 0, \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{\Delta\Phi}{4\pi G a^2} = & \left(1-\frac{\Phi}{c^2}\right)|\psi|^2 + \frac{\hbar^2}{2m^2c^4}\left(1-\frac{2\Phi}{c^2}\right)\left|\frac{\partial\psi}{\partial t}\right|^2 + \frac{\hbar^2}{2a^2m^2c^2}\left(1+\frac{2\Phi}{c^2}\right)|\vec{\nabla}\psi|^2 \\ & + \frac{1}{c^2}V(|\psi|^2) - \frac{\hbar}{mc^2}\left(1-\frac{2\Phi}{c^2}\right)\mathrm{Im}\left(\frac{\partial\psi}{\partial t}\psi^*\right) - \frac{3H^2}{8\pi G} + \frac{3H}{4\pi G c^2}\left(\frac{\partial\Phi}{\partial t} + H\Phi\right). \end{aligned} \tag{26}$$

Eq. (25) can be interpreted as a generalized Schrödinger equation (in the absence of self-interaction) or as a generalized GP equation (in the presence of self-interaction). It is coupled to the Einstein equation (26). The energy density and the pressure can be written as

$$\begin{aligned} \epsilon = & \frac{\hbar^2}{2m^2c^2}\left(1-\frac{2\Phi}{c^2}\right)\left|\frac{\partial\psi}{\partial t}\right|^2 - \frac{\hbar}{m}\left(1-\frac{2\Phi}{c^2}\right)\mathrm{Im}\left(\frac{\partial\psi}{\partial t}\psi^*\right) \\ & + \frac{\hbar^2}{2a^2m^2}\left(1+\frac{2\Phi}{c^2}\right)|\vec{\nabla}\psi|^2 + \frac{1}{2}c^2\left(1-\frac{2\Phi}{c^2}\right)|\psi|^2 + \frac{1}{2}c^2|\psi|^2 + V(|\psi|^2), \end{aligned} \tag{27}$$

$$\begin{aligned} P = & \frac{\hbar^2}{2m^2c^2}\left(1-\frac{2\Phi}{c^2}\right)\left|\frac{\partial\psi}{\partial t}\right|^2 - \frac{\hbar}{m}\left(1-\frac{2\Phi}{c^2}\right)\mathrm{Im}\left(\frac{\partial\psi}{\partial t}\psi^*\right) \\ & - \frac{\hbar^2}{6a^2m^2}\left(1+\frac{2\Phi}{c^2}\right)|\vec{\nabla}\psi|^2 + \frac{1}{2}c^2\left(1-\frac{2\Phi}{c^2}\right)|\psi|^2 - \frac{1}{2}c^2|\psi|^2 - V(|\psi|^2). \end{aligned} \tag{28}$$

Eqs. (25) and (26) form the GPE equations. In the nonrelativistic limit $c \to +\infty$, they reduce to the GPP equations [15]:

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{3}{2}i\hbar H\psi = -\frac{\hbar^2}{2ma^2}\Delta\psi + m\Phi\psi + m\frac{dV}{d|\psi|^2}\psi, \tag{29}$$

$$\frac{\Delta\Phi}{4\pi G a^2} = |\psi|^2 - \frac{3H^2}{8\pi G}. \tag{30}$$

For the quartic potential (5), we have

$$V(|\psi|^2) = \frac{2\pi a_s\hbar^2}{m^3}|\psi|^4. \tag{31}$$

9. The hydrodynamic representation

Important characteristics of the system are revealed by reformulating the KGE equations in the form of hydrodynamic equations. This can be done at the level of the GPE equations (25)-(26) via the Madelung transformation [18]. To that purpose, we write the pseudo wave function ψ as

$$\psi(\vec{x},t) = \sqrt{\rho(\vec{x},t)}e^{iS(\vec{x},t)/\hbar}, \tag{32}$$

where $\rho = |\psi|^2$ plays the role of a pseudo rest-mass density and S plays the role of a pseudo action. Following Madelung, we also define a pseudo velocity field as

$$\vec{v}(\vec{x},t) = \frac{\vec{\nabla} S}{ma}, \tag{33}$$

where the scale factor a has been introduced in order to take into account the expansion of the Universe. We note that this velocity field is irrotational.

Substituting Eqs. (32)-(33) into the GPE equations (25)-(26), and separating real and imaginary parts, we obtain the system of hydrodynamic equations

$$\frac{\partial\rho}{\partial t} + 3H\rho + \frac{1}{a}\vec{\nabla}\cdot(\rho\vec{v}) = \frac{1}{mc^2}\frac{\partial}{\partial t}\left(\rho\frac{\partial S}{\partial t}\right) + \frac{3H\rho}{mc^2}\frac{\partial S}{\partial t} + \frac{4\rho}{mc^4}\frac{\partial\Phi}{\partial t}\left(mc^2 - \frac{\partial S}{\partial t}\right) - \frac{4\Phi}{ac^2}\vec{\nabla}\cdot(\rho\vec{v}), \tag{34}$$

$$\begin{aligned}\frac{\partial S}{\partial t} + \frac{(\vec{\nabla} S)^2}{2ma^2} = {} & -\frac{\hbar^2}{2mc^2}\frac{\frac{\partial^2\sqrt{\rho}}{\partial t^2}}{\sqrt{\rho}} + \left(1+\frac{4\Phi}{c^2}\right)\frac{\hbar^2}{2ma^2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - \frac{2\Phi}{mc^2a^2}(\vec{\nabla} S)^2 \\ & -m\Phi - \left(1+\frac{2\Phi}{c^2}\right)mh(\rho) + \frac{1}{2mc^2}\left(\frac{\partial S}{\partial t}\right)^2 - \left(3H - \frac{4}{c^2}\frac{\partial\Phi}{\partial t}\right)\frac{\hbar^2}{4mc^2\rho}\frac{\partial\rho}{\partial t},\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{\partial\vec{v}}{\partial t} + H\vec{v} + \frac{1}{a}(\vec{v}\cdot\vec{\nabla})\vec{v} = {} & -\frac{\hbar^2}{2am^2c^2}\vec{\nabla}\left(\frac{\frac{\partial^2\sqrt{\rho}}{\partial t^2}}{\sqrt{\rho}}\right) + \frac{\hbar^2}{2m^2a^3}\vec{\nabla}\left[\left(1+\frac{4\Phi}{c^2}\right)\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right] \\ & -\frac{1}{a}\vec{\nabla}\Phi - \frac{1}{\rho a}\vec{\nabla}p - \frac{2}{ac^2}\vec{\nabla}(h\Phi) - \frac{2}{ac^2}\vec{\nabla}(\Phi v^2) + \frac{1}{2am^2c^2}\vec{\nabla}\left[\left(\frac{\partial S}{\partial t}\right)^2\right] \\ & -\frac{3\hbar^2}{4am^2c^2}H\vec{\nabla}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial t}\right) + \frac{\hbar^2}{am^2c^4}\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\frac{1}{\rho}\frac{\partial\rho}{\partial t}\right),\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\Delta\Phi}{4\pi Ga^2} = {} & \left(1-\frac{\Phi}{c^2}\right)\rho + \frac{\hbar^2}{2m^2c^4}\left(1-\frac{2\Phi}{c^2}\right)\left[\frac{1}{4\rho}\left(\frac{\partial\rho}{\partial t}\right)^2 + \frac{\rho}{\hbar^2}\left(\frac{\partial S}{\partial t}\right)^2\right] \\ & +\frac{\hbar^2}{2a^2m^2c^2}\left(1+\frac{2\Phi}{c^2}\right)\left[\frac{1}{4\rho}(\vec{\nabla}\rho)^2 + \frac{\rho}{\hbar^2}(\vec{\nabla} S)^2\right] \\ & +\frac{1}{c^2}V(\rho) - \frac{1}{mc^2}\left(1-\frac{2\Phi}{c^2}\right)\rho\frac{\partial S}{\partial t} - \frac{3H^2}{8\pi G} + \frac{3H}{4\pi Gc^2}\left(\frac{\partial\Phi}{\partial t} + H\Phi\right),\end{aligned} \tag{37}$$

where $h(\rho) = V'(\rho)$ is a pseudo enthalpy and $p(\rho)$ is a pseudo pressure defined by the relation $h'(\rho) = p'(\rho)/\rho$ [20]. It is explicitly given by $p(\rho) = \rho h(\rho) - \int h(\rho)\,d\rho$, i.e.,

$$p(\rho) = \rho V'(\rho) - V(\rho). \tag{38}$$

The pseudo velocity of sound is $c_s^2 = p'(\rho) = \rho V''(\rho)$. For the quartic potential (5), we have

$$V(\rho) = \frac{2\pi a_s \hbar^2}{m^3}\rho^2, \qquad h(\rho) = \frac{4\pi a_s \hbar^2}{m^3}\rho, \qquad p(\rho) = \frac{2\pi a_s \hbar^2}{m^3}\rho^2, \qquad c_s^2 = \frac{4\pi a_s \hbar^2}{m^3}\rho. \tag{39}$$

The pseudo pressure is given by a polytropic equation of state of index $\gamma = 2$ which is quadratic. We note that this equation of state coincides with the equation of state of a nonrelativistic self-interacting BEC [30]. This coincidence is not obvious because Eqs. (34)-(37) are valid in the relativistic regime. The interpretation of this equation of state is, however, not direct because ρ and p are a pseudo density and a pseudo pressure that coincide with the real density and the real pressure of a BEC only in the nonrelativistic limit $c \to +\infty$.

The energy density and the pressure can be written in terms of hydrodynamic variables as

$$\epsilon = \frac{\hbar^2}{2m^2c^2}\left(1-\frac{2\Phi}{c^2}\right)\left[\frac{1}{4\rho}\left(\frac{\partial\rho}{\partial t}\right)^2 + \frac{\rho}{\hbar^2}\left(\frac{\partial S}{\partial t}\right)^2\right] + \frac{\hbar^2}{2a^2m^2}\left(1+\frac{2\Phi}{c^2}\right)\left[\frac{1}{4\rho}(\vec{\nabla}\rho)^2 + \frac{\rho}{\hbar^2}(\vec{\nabla}S)^2\right]$$
$$-\left(1-\frac{2\Phi}{c^2}\right)\frac{\rho}{m}\frac{\partial S}{\partial t} + \frac{1}{2}\left(1-\frac{2\Phi}{c^2}\right)\rho c^2 + \frac{1}{2}\rho c^2 + V(\rho), \tag{40}$$

$$P = \frac{\hbar^2}{2m^2c^2}\left(1-\frac{2\Phi}{c^2}\right)\left[\frac{1}{4\rho}\left(\frac{\partial\rho}{\partial t}\right)^2 + \frac{\rho}{\hbar^2}\left(\frac{\partial S}{\partial t}\right)^2\right] - \frac{\hbar^2}{6a^2m^2}\left(1+\frac{2\Phi}{c^2}\right)\left[\frac{1}{4\rho}(\vec{\nabla}\rho)^2 + \frac{\rho}{\hbar^2}(\vec{\nabla}S)^2\right]$$
$$-\left(1-\frac{2\Phi}{c^2}\right)\frac{\rho}{m}\frac{\partial S}{\partial t} + \frac{1}{2}\left(1-\frac{2\Phi}{c^2}\right)\rho c^2 - \frac{1}{2}\rho c^2 - V(\rho). \tag{41}$$

We note that, in general, the pressure P defined by Eq. (41) is different from the pressure p defined by Eq. (38).[3] However, they coincide for a homogeneous SF in the regime where the SF oscillations are faster than the Hubble expansion [24, 25].

The hydrodynamic equations (34)-(37) have a clear physical interpretation. Eq. (34), corresponding to the imaginary part of the GPE equations, is the continuity equation. We note that $\int \rho\, d^3x$ is not conserved in the relativistic regime. However, Eq. (34) is consistent with the conservation of the charge of a SF (see section 10 and Appendix B). Eq. (35), corresponding to the real part of the GPE equations, is the Bernoulli or Hamilton-Jacobi equation. Eq. (36), obtained by taking the gradient of Eq. (35), is the momentum equation. Eq. (37) is the Einstein equation. We stress that the hydrodynamic equations (34)-(37) are equivalent to the GPE equations (25)-(26) which are themselves equivalent to the KGE equations (12) and (18). In the nonrelativistic limit $c \to +\infty$, we recover the quantum Euler-Poisson equations [15]:

$$\frac{\partial\rho}{\partial t} + 3H\rho + \frac{1}{a}\vec{\nabla}\cdot(\rho\vec{v}) = 0, \tag{42}$$

$$\frac{\partial S}{\partial t} + \frac{(\vec{\nabla}S)^2}{2ma^2} = \frac{\hbar^2}{2ma^2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - m\Phi - mh(\rho), \tag{43}$$

$$\frac{\partial\vec{v}}{\partial t} + H\vec{v} + \frac{1}{a}(\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{\hbar^2}{2m^2a^3}\vec{\nabla}\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \frac{1}{a}\vec{\nabla}\Phi - \frac{1}{\rho a}\vec{\nabla}p, \tag{44}$$

$$\frac{\Delta\Phi}{4\pi G a^2} = \rho - \frac{3H^2}{8\pi G}. \tag{45}$$

These equations can also be written in terms of the density contrast $\delta = (\rho - \rho_b)/\rho_b$ [15].

[3] In our formalism, p represents the pressure arising from the self-interaction of the bosons (scattering) while P is the pressure of the SF defined by analogy with the pressure of an ideal fluid whose energy momentum tensor writes $T_{\mu\nu} = (P+\epsilon)u_\mu u_\nu/c^2 - Pg_{\mu\nu}$.

10. Cosmological evolution of a spatially homogeneous scalar field

We consider the evolution of a universe induced solely by a spatially homogeneous SF. In the comoving frame, we have $\rho(\vec{x},t) = \rho_b(t)$, $\vec{v}_b(\vec{x},t) = \vec{0}$, $\Phi_b(\vec{x},t) = 0$, and $S_b(\vec{x},t) = S_b(t)$. We introduce the notation $E(t) = -dS_b/dt$ which can be considered as the time-dependent energy of the spatially homogeneous SF in the comoving frame. The pseudo wave function of the SF is $\psi_b(\vec{x},t) = \psi_b(t) = \sqrt{\rho_b(t)}e^{-(i/\hbar)\int E(t)\,dt}$. Using Eq. (24), we have $\varphi_b(\vec{x},t) = \varphi_b(t) = \frac{\hbar}{m}\sqrt{\rho_b(t)}e^{-(i/\hbar)[mc^2 t+\int E(t)\,dt]}$ so the total energy of the SF, including its rest mass, is $E_{\rm tot}(t) = E(t) + mc^2$.

For a spatially homogeneous SF, the hydrodynamic equations (34)-(37) reduce to

$$\frac{d\rho_b}{dt} + 3H\rho_b = -\frac{1}{mc^2}\frac{d}{dt}(\rho_b E) - \frac{3H\rho_b}{mc^2}E, \tag{46}$$

$$\left(\frac{E}{2mc^2}+1\right)E = \frac{\hbar^2}{2mc^2}\frac{\frac{d^2\sqrt{\rho_b}}{dt^2}}{\sqrt{\rho_b}} + mh(\rho_b) + \frac{3H\hbar^2}{4mc^2\rho_b}\frac{d\rho_b}{dt}, \tag{47}$$

$$\frac{3H^2}{8\pi G} = \rho_b + \frac{\hbar^2}{2m^2c^4}\left[\frac{1}{4\rho_b}\left(\frac{d\rho_b}{dt}\right)^2 + \frac{\rho_b}{\hbar^2}E^2\right] + \frac{1}{c^2}V(\rho_b) + \frac{E}{mc^2}\rho_b. \tag{48}$$

In terms of the total energy $E_{\rm tot}(t) = E(t) + mc^2$, the equation of continuity (46) becomes

$$\frac{1}{\rho_b}\frac{d\rho_b}{dt} + \frac{3}{a}\frac{da}{dt} + \frac{1}{E_{\rm tot}}\frac{dE_{\rm tot}}{dt} = 0. \tag{49}$$

It can be rewritten as a conservation law:

$$\frac{d}{dt}(E_{\rm tot}\,\rho_b a^3) = 0. \tag{50}$$

Therefore, the total energy is exactly given by

$$\frac{E_{\rm tot}}{mc^2} = \frac{Qm}{\rho_b a^3}, \tag{51}$$

where Q is a constant. This conservation law was found by Gu and Hwang [31] directly from the KG equation. It can be shown that $Q = \int J^0\, d^3x$ represents the conserved charge density of the complex SF (see Appendix B). The energy density and the pressure of a homogeneous SF are

$$\epsilon_b = \frac{\hbar^2}{8m^2c^2}\frac{1}{\rho_b}\left(\frac{d\rho_b}{dt}\right)^2 + \frac{\rho_b}{m}E\left(1+\frac{E}{2mc^2}\right) + \rho_b c^2 + V(\rho_b), \tag{52}$$

$$P_b = \frac{\hbar^2}{8m^2c^2}\frac{1}{\rho_b}\left(\frac{d\rho_b}{dt}\right)^2 + \frac{\rho_b}{m}E\left(1+\frac{E}{2mc^2}\right) - V(\rho_b). \tag{53}$$

Equations (47), (48) and (51) determine the complete evolution of a universe induced by a spatially homogeneous SF. Working directly on the homogeneous KG equation with a quartic self-interaction potential, Li *et al.* [17] have shown that a universe filled with a relativistic complex SF first undergoes an intrinsic stiff matter era, followed by a radiation era due to its self-interaction, before finally entering in the matter era. The stiff matter era occurs when the

SF oscillations are slower than the Hubble expansion while the radiation and matter eras occur when the SF oscillations are faster than the Hubble expansion. These different regimes can be recovered from the hydrodynamic equations (46)-(48) [24].

In the nonrelativistic limit $c \to +\infty$, Eqs. (46)-(48) reduce to

$$\frac{d\rho_b}{dt} + 3H\rho_b = 0, \qquad E = mh(\rho_b), \qquad \frac{3H^2}{8\pi G} = \rho_b. \tag{54}$$

We find that $\rho_b \propto a^{-3}$, $a \propto t^{2/3}$ and $\rho_b = 1/(6\pi G t^2)$ (Einstein-de Sitter solution), so the homogeneous SF/BEC behaves as CDM. For the quartic potential (5), we have $E(t) = 4\pi a_s \hbar^2 \rho_b/m^2 = 2a_s\hbar^2/3Gm^2t^2$ and $S_b(t) = 2a_s\hbar^2/3Gm^2t + C$.

11. Generalized Klein-Gordon-Poisson equations

In this section, we consider a simplified model in which we introduce the gravitational potential $\Phi(\vec{x}, t)$ in the ordinary KG equation by hand, as an external potential, and assume that this potential is produced by the SF itself via a generalized Poisson equation in which the source is the energy density ϵ. This leads to the generalized KGP equations. We then show that these equations can be rigorously justified from the KGE equations in the limit $\Phi/c^2 \to 0$. However, this simplified model is not sufficient to study the evolution of the perturbations in the linear relativistic regime since it precisely neglects terms of order Φ/c^2.

We consider the FLRW metric that describes an isotropic and homogeneous expanding background. The line element in the comoving frame is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = c^2dt^2 - a(t)^2\delta_{ij}dx^i dx^j. \tag{55}$$

For this metric, the d'Alembertian operator (10) writes

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{3H}{c^2}\frac{\partial}{\partial t} - \frac{1}{a^2}\Delta. \tag{56}$$

In order to take the self-gravity of the SF into account, we introduce a Lagrangian of interaction that couples the gravitational potential $\Phi(\vec{x}, t)$ to the scalar field $\varphi(\vec{x}, t)$ according to

$$\mathcal{L}_{\rm int} = -\frac{m^2}{\hbar^2}\Phi|\varphi|^2. \tag{57}$$

The total Lagrangian of the system (SF + gravity) is given by $\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_{\rm int}$. The equation of motion resulting from the stationarity of the total action $S = S_\varphi + S_{\rm int}$, obtained by writing $\delta S = 0$, is the KG equation

$$\Box\varphi + \frac{m^2c^2}{\hbar^2}\varphi + 2V(|\varphi|^2)_{,\varphi^*} + \frac{2m^2}{\hbar^2}\Phi\varphi = 0, \tag{58}$$

where the d'Alembertian operator is given by Eq. (56) and the gravitational potential $\Phi(\vec{x}, t)$ acts here as an external potential. The energy density and the pressure, defined from the diagonal part of the energy-momentum tensor (7), are given by

$$\epsilon = \frac{1}{2c^2}\left|\frac{\partial\varphi}{\partial t}\right|^2 + \frac{1}{2a^2}|\vec{\nabla}\varphi|^2 + \frac{m^2c^2}{2\hbar^2}|\varphi|^2 + V(|\varphi|^2), \tag{59}$$

$$P = \frac{1}{2c^2}\left|\frac{\partial\varphi}{\partial t}\right|^2 - \frac{1}{6a^2}|\vec{\nabla}\varphi|^2 - \frac{m^2c^2}{2\hbar^2}|\varphi|^2 - V(|\varphi|^2). \tag{60}$$

Eq. (58) is the ordinary KG equation for a SF in an external potential $\Phi(\vec{x},t)$ in an expanding background. We now state that $\Phi(\vec{x},t)$ is actually the gravitational potential produced by the SF itself. We phenomenologically assume that the gravitational potential is determined by a generalized Poisson equation of the form

$$\frac{\Delta\Phi}{4\pi G a^2}=\frac{1}{c^2}(\epsilon-\epsilon_b) \tag{61}$$

in which the source of the gravitational potential is the energy density ϵ of the SF (more precisely, its deviation from the homogeneous background density $\epsilon_b(t)$). Using Eq. (59) for the energy density of a SF, and recalling the Friedmann equation (22), the generalized Poisson equation can be written as

$$\frac{\Delta\Phi}{4\pi G a^2}=\frac{1}{2c^4}\left|\frac{\partial\varphi}{\partial t}\right|^2+\frac{1}{2a^2c^2}|\vec{\nabla}\varphi|^2+\frac{m^2}{2\hbar^2}|\varphi|^2+\frac{1}{c^2}V(|\varphi|^2)-\frac{3H^2}{8\pi G}. \tag{62}$$

Eqs. (58) and (62) form the generalized KGP equations. They have been introduced in an *ad hoc* manner but they can be rigorously justified from the KGE equations (12) and (18) in the limit $\Phi/c^2\to 0$ (which, of course, is different from the nonrelativistic limit $c\to+\infty$). We see that the gravitational potential Φ appears in the KG equation (58) due to the cancelation of c^2 in the product $\Phi/c^2\times c^2$ in Eq. (12). Therefore, we do not have to introduce Φ by hand: the generalized KGP equations can be obtained from the KGE equations by simply neglecting terms of order Φ/c^2 in these equations. Similarly, the equations related to the generalized KGP equations can be obtained from the ones related to the KGE equations by neglecting terms of order Φ/c^2. For example, the generalized GPP equations write

$$i\hbar\frac{\partial\psi}{\partial t}-\frac{\hbar^2}{2mc^2}\frac{\partial^2\psi}{\partial t^2}-\frac{3}{2}H\frac{\hbar^2}{mc^2}\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2ma^2}\Delta\psi-m\Phi\psi-m\frac{dV}{d|\psi|^2}\psi+\frac{3}{2}i\hbar H\psi=0, \tag{63}$$

$$\frac{\Delta\Phi}{4\pi G a^2}=|\psi|^2+\frac{\hbar^2}{2m^2c^4}\left|\frac{\partial\psi}{\partial t}\right|^2+\frac{\hbar^2}{2a^2m^2c^2}|\vec{\nabla}\psi|^2+\frac{1}{c^2}V(|\psi|^2)-\frac{\hbar}{mc^2}\mathrm{Im}\left(\frac{\partial\psi}{\partial t}\psi^*\right)-\frac{3H^2}{8\pi G}. \tag{64}$$

The energy density and the pressure are given by

$$\epsilon=\frac{\hbar^2}{2m^2c^2}\left|\frac{\partial\psi}{\partial t}\right|^2-\frac{\hbar}{m}\mathrm{Im}\left(\frac{\partial\psi}{\partial t}\psi^*\right)+\frac{\hbar^2}{2a^2m^2}|\vec{\nabla}\psi|^2+c^2|\psi|^2+V(|\psi|^2), \tag{65}$$

$$P=\frac{\hbar^2}{2m^2c^2}\left|\frac{\partial\psi}{\partial t}\right|^2-\frac{\hbar}{m}\mathrm{Im}\left(\frac{\partial\psi}{\partial t}\psi^*\right)-\frac{\hbar^2}{6a^2m^2}|\vec{\nabla}\psi|^2-V(|\psi|^2). \tag{66}$$

The corresponding hydrodynamic equations write

$$\frac{\partial\rho}{\partial t}+3H\rho+\frac{1}{a}\vec{\nabla}\cdot(\rho\vec{v})=\frac{1}{mc^2}\frac{\partial}{\partial t}\left(\rho\frac{\partial S}{\partial t}\right)+\frac{3H\rho}{mc^2}\frac{\partial S}{\partial t}, \tag{67}$$

$$\frac{\partial S}{\partial t}+\frac{(\vec{\nabla}S)^2}{2ma^2}=-\frac{\hbar^2}{2mc^2}\frac{\frac{\partial^2\sqrt{\rho}}{\partial t^2}}{\sqrt{\rho}}+\frac{\hbar^2}{2ma^2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}-m\Phi-mh(\rho)+\frac{1}{2mc^2}\left(\frac{\partial S}{\partial t}\right)^2-\frac{3H\hbar^2}{4mc^2\rho}\frac{\partial\rho}{\partial t}, \tag{68}$$

$$\begin{aligned}\frac{\partial\vec{v}}{\partial t}+H\vec{v}+\frac{1}{a}(\vec{v}\cdot\vec{\nabla})\vec{v}=&-\frac{\hbar^2}{2am^2c^2}\vec{\nabla}\left(\frac{\frac{\partial^2\sqrt{\rho}}{\partial t^2}}{\sqrt{\rho}}\right)+\frac{\hbar^2}{2m^2a^3}\vec{\nabla}\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)-\frac{1}{a}\vec{\nabla}\Phi-\frac{1}{\rho a}\vec{\nabla}p\\&+\frac{1}{2am^2c^2}\vec{\nabla}\left[\left(\frac{\partial S}{\partial t}\right)^2\right]-\frac{3\hbar^2}{4am^2c^2}H\vec{\nabla}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial t}\right),\end{aligned} \tag{69}$$

$$\frac{\Delta\Phi}{4\pi G a^2} = \rho + \frac{\hbar^2}{2m^2c^4}\left[\frac{1}{4\rho}\left(\frac{\partial\rho}{\partial t}\right)^2 + \frac{\rho}{\hbar^2}\left(\frac{\partial S}{\partial t}\right)^2\right] + \frac{\hbar^2}{2a^2m^2c^2}\left[\frac{1}{4\rho}(\vec{\nabla}\rho)^2 + \frac{\rho}{\hbar^2}(\vec{\nabla}S)^2\right] + \frac{1}{c^2}V(\rho) - \frac{1}{mc^2}\rho\frac{\partial S}{\partial t} - \frac{3H^2}{8\pi G}. \quad (70)$$

The energy density and the pressure can be written in terms of hydrodynamic variables as

$$\epsilon = \frac{\hbar^2}{2m^2c^2}\left[\frac{1}{4\rho}\left(\frac{\partial\rho}{\partial t}\right)^2 + \frac{\rho}{\hbar^2}\left(\frac{\partial S}{\partial t}\right)^2\right] + \frac{\hbar^2}{2a^2m^2}\left[\frac{1}{4\rho}(\vec{\nabla}\rho)^2 + \frac{\rho}{\hbar^2}(\vec{\nabla}S)^2\right] - \frac{\rho}{m}\frac{\partial S}{\partial t} + \rho c^2 + V(\rho), \quad (71)$$

$$P = \frac{\hbar^2}{2m^2c^2}\left[\frac{1}{4\rho}\left(\frac{\partial\rho}{\partial t}\right)^2 + \frac{\rho}{\hbar^2}\left(\frac{\partial S}{\partial t}\right)^2\right] - \frac{\hbar^2}{6a^2m^2}\left[\frac{1}{4\rho}(\vec{\nabla}\rho)^2 + \frac{\rho}{\hbar^2}(\vec{\nabla}S)^2\right] - \frac{\rho}{m}\frac{\partial S}{\partial t} - V(\rho). \quad (72)$$

This model correctly describes the homogeneous background for which $\Phi = 0$ but it is not sufficient to describe the evolution of the perturbations in the linear regime because we must precisely take into account the terms of order Φ/c^2 in this regime (except, of course, in the nonrelativistic limit $c \to +\infty$). Therefore, the use of the KGE equations is mandatory to study the evolution of the perturbations in the relativistic regime.

Remark: We could also assume that the gravitational potential is determined by a Poisson equation of the form

$$\frac{\Delta\Phi}{4\pi G a^2} = \rho \quad (73)$$

in which the source of the gravitational potential is the pseudo rest-mass density $\rho = |\psi|^2$ of the SF. Eqs. (58) and (73) form the KGP equations. This approximation has been considered in [12]. However, there is an inconsistency in coupling the relativistic KG equation (58) to the classical Poisson equation (73).

12. Conclusion

We have developed a formalism based on a relativistic SF described by the KGE equations in the weak field limit. We have transformed these equations into equivalent hydrodynamic equations. These equations are arguably more tractable than the KGE equations themselves. In the nonrelativistic limit, they reduce to the hydrodynamic equations directly obtained from the GPP equations [15]. Therefore, our formalism clarifies the connection between the relativistic and nonrelativistic treatments. We note that, in the relativistic regime, the hydrodynamic variables ψ, ρ, S, $\vec{v}$, p,... that we have introduced do not have a direct physical interpretation. It is only in the nonrelativistic limit $c \to +\infty$ that they coincide with the wave function, rest-mass density, action, velocity, and pressure. However, these variables are perfectly well-defined mathematically from the SF φ in any regime, and they are totally legitimate. Furthermore, in terms of these variables, the relativistic hydrodynamic equations take a relatively simple form that provides a natural generalization of the nonrelativistic hydrodynamic equations.

The complete study of these relativistic hydrodynamic equations is of considerable interest but it is, of course, of great complexity. In our research papers [24, 25], we have started their study in simple cases. We have checked that the hydrodynamic equations of the SFDM model reproduce the evolution of the homogeneous background obtained previously by Li *et al.* [17] directly from the KGE equations: a stiff matter era, followed by a radiation era (for a self-interacting SF), and a matter era. We have also started to study the evolution of the perturbations in the linear regime in a static and in an expanding universe. We have shown analytically and numerically that perturbations whose wavelength is below the Jeans length oscillate in time while perturbations whose wavelength is above the Jeans length grow linearly with the scale factor as in the CDM model. The growth of perturbations in the SF model is substantially faster than in the CDM model. We have also shown that general relativity attenuates or even prevents the growth of perturbations at very large scales, close to the horizon (Hubble length).

For physically relevant wavelengths, the nonrelativistic limit of our formalism is sufficient to describe the evolution of the perturbations in the matter era. However, even if relativistic corrections are weak in the matter era, we may wonder whether their effect could be detected in an era of precision cosmology. In particular, it would be interesting to see if one can observe differences between the KGE equations considered in these Proceedings and the heuristic KGP equations studied in the past, in which gravity is introduced by hand in the KG equations (see section 11).

In future works, it will be important to consider the nonlinear regime where structure formation actually occurs. In general, this problem must be addressed numerically. The hydrodynamic equations derived in these Proceedings may be very helpful because they may be easier to solve than the KGE equations. Therefore, numerical simulations using fluid dynamics should be developed in the future. As a first step, relativistic effects could be neglected and the nonrelativistic equations of Ref. [15] could be used. These equations are similar to the hydrodynamic equations of CDM except that they include a quantum potential (Heisenberg) and a pressure term (scattering) that avoid singularities at small scales [15]. In this respect, it may be recalled that the SP equations were introduced early by Widrow and Kaiser [32] as a procedure of small-scale regularization (a sort of mathematical trick) to prevent singularities in collisionless simulations of *classical* particles. In their approach, $\hbar$ is not the Planck constant, but rather an ajustable parameter that controls the spatial resolution. Their procedure may find a physical justification if DM is made of self-gravitating BECs [33].

Our relativistic formalism may have applications for other self-gravitating systems described by SFs or BECs besides DM. We can mention, for example, the case of boson stars [5, 6, 7] and the case of microscopic quantum black holes made of BECs of gravitons stuck at a quantum critical point [34, 35, 10]. It has also been proposed [22] that, because of their superfluid core, neutron stars could be BEC stars. Indeed, the neutrons (fermions) could form Cooper pairs and behave as bosons of mass $2m_n$. This idea may solve certain issues regarding the maximum mass of neutron stars. Finally, we may mention analog models of gravity in which BECs described by the GP equation or by the KG equation are used to simulate results of classical and quantum field theory in curved spacetime [36].

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Appendix A. The value of A

The GP equation is obtained from the KG equation by means of the transformation

$$\varphi = Ae^{-imc^2t/\hbar}\psi. \tag{A.1}$$

The constant A can be computed as follows. Substituting Eq. (A.1) into Eq. (13), we find that the energy density of the SF is given by

$$\frac{\epsilon}{c^2} = \frac{T_0^0}{c^2} = \frac{1}{2}\left(1-\frac{2\Phi}{c^2}\right)\frac{m^2}{\hbar^2}A^2|\psi|^2 + \frac{m^2}{2\hbar^2}A^2|\psi|^2 + \frac{A^2}{2c^4}\left(1-\frac{2\Phi}{c^2}\right)\left|\frac{\partial\psi}{\partial t}\right|^2$$
$$+\frac{A^2}{2a^2c^2}\left(1+\frac{2\Phi}{c^2}\right)|\vec{\nabla}\psi|^2 + \frac{1}{c^2}V(|\psi|^2) - \frac{mA^2}{\hbar c^2}\left(1-\frac{2\Phi}{c^2}\right)\mathrm{Im}\left(\frac{\partial\psi}{\partial t}\psi^*\right). \tag{A.2}$$

Taking the nonrelativistic limit $c \to +\infty$ of this equation, we obtain

$$\frac{\epsilon}{c^2} \to \frac{m^2A^2}{\hbar^2}|\psi|^2 = \frac{m^2A^2}{\hbar^2}\rho, \tag{A.3}$$

where $\rho = |\psi|^2$ is the rest-mass density. Since $\epsilon \sim \rho c^2$ in the nonrelativistic limit $c \to +\infty$, we find

$$A = \frac{\hbar}{m}. \tag{A.4}$$

Appendix B. Some comments about the Klein-Gordon equation

In this Appendix, we recall the difficulties associated with the interpretation of the KG equation. We also clarify the relation between the hydrodynamic representation of the KG equation and the conservation of the charge.

The fundamental equation of nonrelativistic quantum mechanics is the Schrödinger [4] equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi. \tag{B.1}$$

If we write the wave function under the form $\psi(\vec{x},t) = \sqrt{\rho(\vec{x},t)}e^{iS(\vec{x},t)/\hbar}$ and define the density ρ and the current $\vec{J}$ by

$$\rho = |\psi|^2, \tag{B.2}$$

$$\vec{J} = \rho\frac{\vec{\nabla}S}{m} = \frac{\hbar}{2im}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*), \tag{B.3}$$

where we have used $S = (\hbar/2i)\ln(\psi/\psi^*)$, we obtain the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0. \tag{B.4}$$

This equation shows that the integral of the density $\int \rho d^3x$ is conserved. Furthermore, by definition, the density is positive: $\rho(\vec{x},t) \geq 0$. Therefore, $\rho(\vec{x},t)$ can be interpreted as a probability density.

In the Madelung hydrodynamic representation of the Schrödinger equation, we define the density ρ and the velocity $\vec{v}$ by

$$\rho = |\psi|^2, \qquad \vec{v} = \frac{\vec{\nabla}S}{m}. \tag{B.5}$$

These variables satisfy the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0. \tag{B.6}$$

Since $\vec{J} = \rho\vec{v}$ according to Eqs. (B.3) and (B.5), we immediately see the equivalence between Eqs. (B.4) and (B.6).

We now consider the KG [2, 3] equation

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi + \frac{m^2c^2}{\hbar^2}\varphi = 0 \tag{B.7}$$

which was initially proposed as a relativistic extension of the Schrödinger equation.[4] If we write the SF under the form $\varphi(\vec{x},t) = \sqrt{R(\vec{x},t)}e^{i\sigma(\vec{x},t)/\hbar}$ and introduce the quadricurrent

[4] The KG equation was actually discovered by Schrödinger before he found the equation that now bears his name [1]. The KG equation was also obtained by Fock [37], de Donder and van den Dungen [38], and Kudar [39].

$J^\mu = -R\partial^\mu\sigma/m = -(\hbar/2im)(\varphi^*\partial^\mu\varphi - \varphi\partial^\mu\varphi^*)$, we obtain the continuity equation $\partial_\mu J^\mu = 0$. The quadricurrent can be written as $J^\mu = (J^0, \vec{J})$ with

$$J^0 = -R\frac{\frac{1}{c}\frac{\partial\sigma}{\partial t}}{m} = -\frac{\hbar}{2imc}\left(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}\right), \tag{B.8}$$

$$\vec{J} = R\frac{\vec{\nabla}\sigma}{m} = \frac{\hbar}{2im}(\varphi^*\vec{\nabla}\varphi - \varphi\vec{\nabla}\varphi^*). \tag{B.9}$$

If we introduce the KG density $\rho_{\rm KG} = J^0/c$, we can write the continuity equation as

$$\frac{\partial\rho_{\rm KG}}{\partial t} + \nabla\cdot\vec{J} = 0. \tag{B.10}$$

This equation shows that the integral of the KG density $\int \rho_{\rm KG} d^3x$ is conserved. However, the KG density $\rho_{\rm KG}(\vec{x}, t)$ is *not* definite positive so it cannot be interpreted as a density probability. Another difficulty with the KG equation is that it allows negative kinetic energies as solution. Indeed, decomposing Eq. (B.7) into plane waves, we obtain two solutions of the form $\varphi_\pm(\vec{x}, t) = A_\pm e^{i(\vec{p}\cdot\vec{x} - E_\pm t)/\hbar}$ with $E_\pm = \pm\sqrt{p^2c^2 + m^2c^4}$.

The original difficulties encountered with the KG equation had interesting historical developments. Dirac [40] proposed another relativistic extension of the Schrödinger equation. The Dirac equation describes spin-1/2 massive particles (fermions) such as electrons and quarks. In Dirac's theory, the probability density is positive but negative energies are allowed. Dirac solved the problem of negative energies via the "hole" theory [41]. This leads to the concept of antiparticles that are related to negative energy eigenstates. Antimatter was unsuspected before Dirac's work. The first antiparticle, the positron, was experimentally discovered by Anderson [42] in 1932. On the other hand, Pauli and Weisskopf [43] proposed a new interpretation of the KG equation. They interpreted the KG density $\rho_{\rm KG} = J^0/c$ as a charge density which can be of arbitrary sign. If we define the charge density and the charge current by $\rho_e = meJ^0/c\hbar^2$ and $\vec{J}_e = me\vec{J}/\hbar^2$, where e is an elementary charge, we can rewrite the continuity equation (B.10) as

$$\frac{\partial\rho_e}{\partial t} + \nabla\cdot\vec{J}_e = 0. \tag{B.11}$$

This equation expresses the conservation of the total charge $Q_e = \int \rho_e\, d^3x$. Although the KG equation is not a successful relativistic generalization of the Schrödinger equation (initially introduced to describe the energy spectrum of the electron), this equation was resurrected in the context of quantum field theory where it was shown to describe spin-0 particles (bosons) such as π-mesons, pions, or the Higgs boson. In this interpretation, since $\rho_e^{(\pm)} = \pm(e|E_\pm|/\hbar^2c^2)|\varphi_\pm|^2$, φ_+ specifies particles with charge $+e$ and energy $+E$ while φ_- specifies antiparticles with the same mass but with charge $-e$ and energy $-E$. For a real SF the charge is zero since $\rho_e = 0$ and $\vec{J}_e = \vec{0}$ according to Eqs. (B.8) and (B.9).

In the hydrodynamic representation of the KG equation, we write

$$\varphi(\vec{x}, t) = \frac{\hbar}{m}\sqrt{\rho(\vec{x}, t)}e^{i\left[S(\vec{x},t) - mc^2t\right]/\hbar} \tag{B.12}$$

and define the density ρ and the velocity $\vec{v}$ by

$$\rho = \frac{m^2}{\hbar^2}|\varphi|^2, \qquad \vec{v} = \frac{\vec{\nabla}S}{m}. \tag{B.13}$$

These variables satisfy the continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=\frac{1}{mc^2}\frac{\partial}{\partial t}\left(\rho\frac{\partial S}{\partial t}\right). \tag{B.14}$$

The density $\rho(\vec{x},t)$ is positive by definition but it is not conserved. Therefore, $\rho(\vec{x},t)$ cannot be interpreted as a density probability. However, the continuity equation (B.14) can be rewritten as

$$\frac{\partial}{\partial t}\left(\rho\frac{E_{\rm tot}}{mc^2}\right)+\nabla\cdot(\rho\vec{v})=0, \tag{B.15}$$

where we have defined $E(\vec{x},t)=-\partial S/\partial t$ and $E_{\rm tot}(\vec{x},t)=mc^2+E(\vec{x},t)$. This equation implies the conservation of the integral $\int\rho E_{\rm tot}\,d^3x$. To show that this integral corresponds to the total charge of the SF, we note that $R=(\hbar^2/m^2)\rho$ and $\sigma=S-mc^2t$. Therefore,

$$\rho_e=-\frac{e}{c^2\hbar^2}R\frac{\partial\sigma}{\partial t}=-e\frac{\rho}{m}\frac{\frac{\partial S}{\partial t}-mc^2}{mc^2}=e\frac{\rho}{m}\left(\frac{E}{mc^2}+1\right)=e\frac{\rho}{m}\frac{E_{\rm tot}}{mc^2}, \tag{B.16}$$

$$\vec{J}_e=\frac{e}{\hbar^2}R\vec{\nabla}\sigma=e\frac{\rho}{m}\frac{\vec{\nabla}S}{m}=e\frac{\rho}{m}\vec{v}. \tag{B.17}$$

These relations first establish the equivalence between Eqs. (B.11), (B.14), and (B.15). Furthermore, they show that the total charge of the SF can be written as

$$Q_e=e\int\frac{\rho}{m}\frac{E_{\rm tot}}{mc^2}\,d^3x \tag{B.18}$$

in agreement with Eq. (B.15). Eq. (B.18) is consistent with the expression (51) of the charge of a spatially homogeneous SF in an expanding universe.

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