GEOMETRIC DESCRIPTION OF BLACK HOLE THERMODYNAMICS WITH HOMOGENEOUS FUNDAMENTAL EQUATION

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Abstract. We investigate the Geometrothermodynamics of the 2-dimensional submanifold $\mathcal{E}$ corresponding the space of thermodynamic equilibrium states for a static and spherically symmetric hairy to black hole solution in Lorentz non-invariant massive gravity. We show that it is possible to consider a fundamental thermodynamic function as a homogeneous function of degree 1 in the extensive variables, as classical thermodynamics demand, in order to be consistent with the physical meaning of the intensive variables. The geometry of the space of equilibrium states is computed showing that it contains information about the thermodynamic interaction, critical points, and phase transitions structure.

1. Introduction

The research done by Hawking [1] and Bekenstein[2, 3, 4] in the early 1970s, showed that it is possible to find a relationship between the properties of the black holes (event horizon) and the laws of thermodynamics. For example, the surface gravity is related with the temperature, and the sum of the areas of the black hole horizons cannot decrease, which resembles the classical thermodynamic entropy. Therefore, an entropy which is proportional to the area was attributed to the horizon of the black hole. Thus, they also realized that one can formulate the four laws of black hole dynamics in a manner analogous to the laws of classical thermodynamics [5].

In spite of the fact that the main objection with respect to this analogy was resolved [6] (one that argued that if black holes possess entropy as well as energy, then they must have a non-zero temperature and must radiate which seemed to contradict the view that nothing can escape the black hole horizon) still some others remain, such as the non-homogeneity of the fundamental equation [7] which describe the thermodynamics of the black holes, or the meaning of a phase transition in these exotic thermodynamic systems.

The study of the thermodynamics of the black holes by means of geometry has been a subject of intensive research [8, 9, 10, 11, 12, 13]. This geometric study has been considered in several papers by means of different approaches like Weinhold's theory [14], Ruppeiner's theory [15] or the most recent theory called geometrothermodynamics [16]. Geometrothermodynamics (GTD) is a formalism that relates a contact structure of the phase space $\mathcal{T}$ with the metric structure on a special subspace of $\mathcal{T}$ called the space of equilibrium states $\mathcal{E}$.

On the other hand, the most successful cosmological model that is in agreement with the observational data implies the existence of a vacuum energy related with the cosmological constant Λ whose magnitude is unnatural from the effective field theory point of view [17]. Hence, a dark energy is needed to reconcile general relativity with the observations. The dark energy solves the problem of the accelerated expansion of the Universe. Nevertheless, the interest to explain the acceleration of the universe without resorting to the dark energy has motivated the search for large-distances modified theories of the gravity, the Lorentz- breaking massive gravity is one of these models which is free from pathologies such as ghosts, low strong coupling scales or instabilities at full non-perturbative level [18, 19, 20]. Although these models do not require the existence of Λ, the cosmological constant problem remains open [21]. A generalized Schwarzschild solution for this model has been obtained by D. Comelli et al. which is an exact black hole solution showing a nonanalytic hair [22].

In this work, we will use geometrotermodynamics to formulate an invariant geometric representation of the thermodynamics of a static and spherically symmetric hairy black hole solution in massive gravity, using a thermodynamic fundamental equation of degree 1.

This paper is organized as follows. In Section 2, we review the generalized Schwarzschild solution for massive gravity and its thermodynamics. In Sec. 3, we present a brief review of the GTD formalism. In Sec. 4, we apply the formalism of GTD to the generalized Schwarzschild solution in massive gravity and show that the geometric properties of the equilibrium space are in correspondence with thermodynamic properties of the black hole. Finally, in Sec. 5 we present the conclusions.

2. Thermodynamics of the black hole solution in Lorentz non-invariant massive gravity

The general action corresponding to massive gravity is given by the expression [20, 22, 23, 24, 25, 26]:

$$I = \int_{\mathcal{M}} d^4\sqrt{g}\Big[-\frac{1}{16\pi}R + \Lambda^4\mathcal{F}(X, Vi, W^{ij})\Big] - \int_{\partial\mathcal{M}} d^3\sqrt{\gamma}\frac{K}{8\pi}\,, \tag{1}$$

where $\mathcal{F}$ is a function of four scalar fields ϕ^μ that are minimally coupled to gravity by the covariant derivatives since:

$$X = \Lambda^{-4}g^{\mu\nu}\partial_\mu\phi^0\partial_\nu\phi^0 \;;\; V^i = \Lambda^{-4}\partial^\mu\phi^i\partial_\mu\phi^0 \;;\; W^{ij} = \Lambda^{-4}\partial^\mu\phi^i\partial_\mu\phi^j - \frac{V^iV^j}{X}\,, \tag{2}$$

with latin and greek indices running on the space and spacetime, respectively. The second integral is instead the Gibbons-Hawking-York boundary term [28, 29] where γ is the metric induced on the boundary $\partial\mathcal{M}$ and K is the trace of the extrinsic curvature $K_{ij} = \frac{1}{2}\gamma^k{}_i\nabla_k n_j$ of $\mathcal{M}$ with unit normal n^i. Such a boundary term is required to have a well-defined variational principle in the presence of the border $\partial\mathcal{M}$.

Spherically symmetric black holes in massive gravity have been investigated in [22, 28, 29] and it was found that a set of coordinates always can be found where the solution can be written in the form:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2\sin^2\theta d\varphi^2)\,, \tag{3}$$

and the fields are given by the expressions [28]:

$$\phi^0 = \Lambda^2[t + h(r)] \;\;;\;\; \phi^i = \phi(r)\frac{\Lambda^2 x^i}{r}\,. \tag{4}$$

The metric functions f, h and ϕ are given by the following equations,

$$
\begin{aligned}
f &= 1 - \frac{2M}{r} - \frac{Q}{r^\lambda} && (5)\\
h &= \pm \int \frac{dr}{f}\left[1 - f\Big(\frac{\lambda(\lambda-1)Q}{12m^2 r^{\lambda+2}} + 1\Big)^{-1}\right]^{\frac{1}{2}}, && (6)\\
\phi(r) &= r\,, && (7)
\end{aligned}
$$

where $\lambda \neq 1$ is a positive constant, M and Q are integration constants. In the case $\lambda > 1$ the gravitational potential is asymptotically Newtonian and the parameter M coincides with the ADM mass, while Q is a scalar charge whose presence reflects the modification of the gravitational interaction as compared to General Relativity.

The roots of the lapse function $f(g_{tt} = 0)$ define the horizons $r = r_\pm$ of the spacetime. In particular, the null hypersurface $r = r_+$ can be shown to correspond to an event horizon, which in this case is also a Killing horizon, whereas the inner horizon at r_- is a Cauchy horizon. Therefore from $f(r_+) = 0$ [26] we get,

$$
1 - \frac{2M}{r_+} - \frac{Q}{r_+^\lambda} = 0\,, \tag{8}
$$

From the expression (8) for the horizon radii r_+ we obtain the following relationship,

$$
M(r_+, Q) = \frac{1}{2} r_+ \left[1 - \frac{Q}{r_+^\lambda}\right], \tag{9}
$$

From the area-entropy relationship, $S = \pi r^2{}_+$, the equation (8) can be rewritten as

$$
M(S, Q) = \frac{1}{2}\left(\frac{S}{\pi}\right)^{\frac{1}{2}}\left[1 - \frac{Q}{\left(\frac{S}{\pi}\right)^{\frac{\lambda}{2}}}\right]. \tag{10}
$$

This equation relates all the thermodynamic variables entering the black hole metric in the form of a fundamental thermodynamic equation $M = M(S, Q)$. Also, this can be considered as the thermodynamic potential related to the canonical ensemble of the system so that the corresponding heat capacity contains information about the phase transitions of the system. Equation (10) is an inhomogeneous function in the extensive variables S and Q. Following Davies [30] we homogenisize the fundamental equation by redefining the parameters Q and S as,

$$
Q = q^\lambda\,, \qquad ; \qquad S = s^2\,. \tag{11}
$$

Then, equation (10) becomes a homogeneous function of degree 1 in the extensive variables. With the definitions (11) we have,

$$
m(s, q) = \frac{s}{2\sqrt{\pi}}\left[1 - \frac{q^\lambda \pi^{\frac{\lambda}{2}}}{s^\lambda}\right]. \tag{12}
$$

This procedure was performed explicitly in the context of GTD in [31]. Under this consideration the first law of the thermodynamics can be written as,

$$dm = Tds + \phi dq \,, \tag{13}$$

with $m = m(s, q)$. According with equation (13), the expression for the temperature T and the potential ϕ are given by the thermodynamic equilibrium conditions: $T = \left(\frac{\partial m}{\partial s}\right)_q$ and $\phi = \left(\frac{\partial m}{\partial q}\right)_s$, which lead to the following results,

$$T = \frac{1}{2\sqrt{\pi}}\left[1 - \frac{q^{\lambda}\pi^{\frac{\lambda}{2}}(1-\lambda)}{s^{\lambda}}\right], \tag{14}$$

$$\phi = -\frac{1}{2\pi^{\frac{1-\lambda}{2}}}\left[\frac{q}{s}\right]^{\lambda-1}. \tag{15}$$

It is easy to show that the temperature (14) coincides with the Hawking temperature. The heat capacity at constant values of q is given as

$$C_q = T\left(\frac{\partial s}{\partial T}\right)_q = \left(\frac{\frac{\partial M}{\partial s}}{\frac{\partial^2 M}{\partial s^2}}\right)_q, \tag{16}$$

where the subscript indicate that we should compute keeping the charge constant. Using the fundamental equation (12) we get,

$$C_q = \frac{s\left[s^{\lambda} - q^{\lambda}\pi^{\frac{\lambda}{2}}(1-\lambda)\right]}{q^{\lambda}\pi^{\frac{\lambda}{2}}(1-\lambda)}. \tag{17}$$

According to Davies [30], second order phase transitions take place at those points where the heat capacity diverges, i. e., for

$$q^{\lambda}\pi^{\frac{\lambda}{2}}(1-\lambda) = 0\,, \tag{18}$$

in this case, because $\lambda \neq 1$ there are no points where the heat capacity becomes singular; therefore, this thermodynamic system does not present second order phase transitions. We can also observe that the the thermodynamic variables T and ϕ are homogeneous functions of zero order, as in the standard thermodynamics. It is worth noticing in the case of more general black hole solutions of massive gravity, the investigation of the heat capacity leads to a non trivial phase transition structure [27]. In the present case, however, no phase transitions are found.

3. Brief review of geometrothermodynamics

Geometrothermodynamics (GTD) [16] is a formalism that has been applied to different thermodynamic systems(ordinary systems like the Van der Waals gas or exotic systems like black holes [32, 33, 34, 35, 36, 37, 38, 39] in order to get consistent results to describe geometrically the phases transitions and thermodynamic interaction using Legendre invariant metrics.

In GTD we work with a $(2n+1)$-dimensional manifold $\mathcal{T}$ with a set of coordinates Z^A which allow us to define a non-degenerate Legendre invariant metric G together with a linear differential 1-form Θ which fulfills the condition $\Theta \wedge (d\Theta)^n \neq 0$, where n is the number of thermodynamic degrees of freedom, $\wedge$ represents the exterior product and d the exterior derivative. In GTD we also have the space of thermodynamic equilibrium states which is a submanifold $\mathcal{E} \subset \mathcal{T}$ defined by means of a smooth embedding mapping $\varphi : \mathcal{E} \longrightarrow \mathcal{T}$ such that the pullback $\varphi^*(\Theta) = 0$. By means of $\varphi^*(G) = g$ a metric g is induced in $\mathcal{E}$, giving a Riemannian structure to this space. Therefore, in GTD the physical properties of a thermodynamic system in a state of equilibrium are described in terms of the geometric properties of the corresponding space $\mathcal{E}$.

If we consider the $(2n+1)$-dimensional space $\mathcal{T}$ coordinatized by the set $Z^A = \Big\{\Phi\,, E^a\,, I^a\Big\}$ where $A = 0, \ldots, 2n$ and $a = 1, \ldots n$, the 1-form Θ will be,

$$\Theta = d\Phi - I_a dE^a\,. \tag{19}$$

We choose now the subset E^a as coordinates of $\mathcal{E}$. Then the mapping φ is given by,

$$\varphi : (E^a) \longrightarrow (\Phi\,, E^a\,, I^a)\,, \tag{20}$$

and the condition,

$$\varphi^*(\Theta) = \varphi^*(d\Phi - \delta_{ab} I^a dE^b) = 0\,, \tag{21}$$

leads to the standard conditions of the thermodynamic equilibrium and the first law of thermodynamics,

$$\frac{\partial \Phi}{\partial E^a} = I_a\,, \qquad\qquad d\Phi = I_a dE^a\,. \tag{22}$$

The second law of the thermodynamics under this formalism is written as,

$$\frac{\partial^2 U}{\partial E^a \partial E^b} \geq 0\,, \qquad ; \qquad \frac{\partial^2 S}{\partial F^a \partial F^b} \leq 0\,, \tag{23}$$

where U and S represent the energy and entropy for each of the corresponding thermodynamic systems. Here E^a (F^a) represent all the extensive variables other than U (S).

In GTD the only requirement for defining a metric G of the space $\mathcal{T}$ is that it fulfills the condition of Legendre invariance. Therefore, we have many possibilities of constructing a metric with these features, one of them is the following,

$$G = \Theta^2 + (\delta_{ab} E^a I^b)(\eta_{cd} dE^c dI^d)\,, \tag{24}$$

where $\delta_{ab} = \mathrm{diag}(1, 1, \ldots, 1)$ and $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$. It can be shown that the metric (24) is invariant with respect to a total Legendre transformation which changes the variables without tilde $\{\Phi, E^a, I^a\}$ to the tilde variables $\{\tilde{\Phi}, \tilde{E^a}, \tilde{I^a}\}$ using the following algebraic rules,

$$\Phi = \tilde{\Phi} - \tilde{E}_a \tilde{I}^a\,, \qquad E^a = -\tilde{I}^a\,, \qquad I^a = \tilde{E}^a\,. \tag{25}$$

Applying the pullback φ^* to the metric (24), we obtain the corresponding thermodynamic metric g,

$$g^{GTD} = \varphi^*(G) = \Big(E^c \frac{\partial \Phi}{\partial E^c}\Big)\Big(\eta_{ab}\delta^{bc} \frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^a dE^d\Big)\,, \tag{26}$$

which depends only of the fundamental potential $\Phi = \Phi(E^a)$. If we know the fundamental potential of the thermodynamic system that we want to study, the corresponding metric g can be computed explicitly and the relations between thermodynamic and geometry can be also studied.

4. Geometrothermodynamics of the black hole solution in Lorentz non-invariant massive gravity

According with [40], the geometrothermodynamic metric that we should use to investigate the thermodynamic properties of black holes is given by the expression (26). If we define $\Phi = m$ and we assume that $E^a = \{s, q\}$, then we have a system with two thermodynamic degrees of freedom. Under these considerations we get,

$$g^{GTD} = \Big(s\frac{\partial m}{\partial s} + q\frac{\partial m}{\partial q}\Big)\Big(-\frac{\partial^2 m}{\partial s^2}ds^2 + \frac{\partial^2 m}{\partial q^2}dq^2\Big)\,. \tag{27}$$

Using the expressions for the m, as was given in Eq. (12), we obtain explicitly the GTD metric coefficients which can be written as,

$$g^{GTD}_{ss} = \frac{\lambda(\lambda-1)}{4\pi^{\frac{2-\lambda}{2}}}\frac{q^{\lambda}\Big[q^{\lambda}\pi^{\frac{\lambda}{2}} - s^{\lambda}\Big]}{s^{2\lambda}}\,, \tag{28}$$

$$g^{GTD}_{qq} = -\frac{\lambda(\lambda-1)}{4\pi^{\frac{2-\lambda}{2}}}\frac{q^{\lambda-2}\Big[q^{\lambda}\pi^{\frac{\lambda}{2}} - s^{\lambda}\Big]}{s^{2\lambda-2}}\,, \tag{29}$$

and the equation (27) takes the form,

$$g^{GTD} = f(s,q)\Big[\frac{ds^2}{s^2} - \frac{dq^2}{q^2}\Big]\,, \tag{30}$$

where,

$$f(s,q) = \frac{\lambda(\lambda-1)}{4\pi^{\frac{2-\lambda}{2}}}\frac{q^{\lambda}\Big[q^{\lambda}\pi^{\frac{\lambda}{2}} - s^{\lambda}\Big]}{s^{2\lambda-2}}\,. \tag{31}$$

The curvature scalar corresponding to the metric (30) takes the form,

$$R^{GTD} = 0\,. \tag{32}$$

This result tell us that the curvature scalar of the space of thermodynamic equilibrium states does not have points where becomes singular. Also, in accordance with the geometrothermodynamics, the system does not have interaction thermodynamics [40].

5. Conclusions

In this work, we studied the geometrothermodynamics of a black hole solution in massive gravity. Using this formalism we found the geometric properties of the corresponding manifold of equilibrium states. We found that the corresponding thermodynamic curvature turned out to be zero, indicating that there is not thermodynamics interaction. We also show that the GTD correctly describes the fact that this thermodynamic system has not phase transitions, since they are characterized by divergencies of the heat capacity which are described in GTD by curvature singularities.

It is interesting that, according to GTD, the black hole considered in this work does not posses thermodynamic interaction. In fact, this intriguing behavior has been also found in the case of topological black holes in Horava-Lifshitz gravity [41] and in cosmological models [42]. This means that in GTD the presence of gravitational interaction does not necessarily imply the presence of thermodynamic interaction. We expect to study this intriguing relationship in future works.

We assumed in this work Davies's proposal to solve the problem of the lack of homogeneity of the fundamental equation. We have considered the definition (11), with which the fundamental equation (9) becomes a homogeneous function of degree 1 in the extensive variables in order to have consistency with the physical meaning of the intensive variables which must be homogenous of degree zero.

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