

# Lagrangian formulation for fermionic and supersymmetric non conservative systems

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**Abstract.** We show that a recent formulation of the principle of stationary action, compatible with generic non conservative interactions, is suitable for fermionic and supersymmetric systems. The main features of the mechanics contained in this new action principle are also encountered in the examples discussed in this work.

## 1. Introduction

The evolution of mechanical systems can be irreversible and non invariant under time reversal. Such systems are studied usually by means of their equations of motion. In two recent papers, Galley [1, 2] gives a systematic proposal for a lagrangian formulation for nonconservative systems. This proposal amounts to a modification of Hamilton's variational principle, introducing a nonconservative "potential". This formalism is inspired on the closed time path formalism, originally proposed by Schwinger [3] for field theory. Supersymmetry is a symmetry which has allowed consistent formulations of unification theories of all fundamental forces. It imposes rather strong constraints on the particle content, mass spectrum, and interactions. Thus we consider interesting to explore if supersymmetry is compatible with this formulation for nonconservative systems [4]. The generalization for supersymmetric systems is done in the superspace formalism. The boundary conditions must be also modified, and are given in terms of superfields. We illustrate the formalism by some examples.

## 2. Galley's mechanics for non conservative systems

Galley's formulation is based on the doubling of the degrees of freedom of a conservative system with Lagrangian  $L(q, \dot{q})$ ,  $q \rightarrow (q_1, q_2)$ . The action is defined as the integral of a conservative Lagrangian over two trajectories in configuration space, each one for one of the two copies of the original degrees of freedom. To this action, it is added the integral of a generalized potential  $K(q_1, \dot{q}_1, q_2, \dot{q}_2)$ , that couples these two trajectories. This potential is antisymmetric under the interchange  $q_1 \leftrightarrow q_2$ . Thus there is a new Lagrangian

$$\Lambda = L(q_1, \dot{q}_1, t) - L(q_2, \dot{q}_2, t) + K(q_1, \dot{q}_1, q_2, \dot{q}_2). \quad (1)$$

The variation of this action is given under the boundary conditions at the initial and final times

$$\delta q_1(t_i) = \delta q_2(t_i) = 0, \quad (2)$$

$$\delta q_1(t_f) = \delta q_2(t_f), \quad q_1(t_f) = q_2(t_f), \quad \dot{q}_1(t_f) = \dot{q}_2(t_f). \quad (3)$$



The last three equations define the so called *equality condition*. Thus, the equations of motion are

$$\frac{\partial}{\partial q_1}(L + K) - \frac{d}{dt} \frac{\partial}{\partial \dot{q}_1}(L + K) = 0, \quad (4)$$

$$\frac{\partial}{\partial q_2}(L - K) - \frac{d}{dt} \frac{\partial}{\partial \dot{q}_2}(L - K) = 0. \quad (5)$$

Taking into account the antisymmetry of the potential, and making the *physical limit*  $q_1 = q_2 = q$ , (17) and (14) coincide, giving the nonconservative equations of motion

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = F_K \equiv \left( \frac{\partial}{\partial q_2} - \frac{d}{dt} \frac{\partial}{\partial \dot{q}_2} \right) K(q_1, \dot{q}_1, q_2, \dot{q}_2) \Big|_{q_1=q_2=q}. \quad (6)$$

A convenient change of variables is given by

$$q_+ = \frac{1}{2}(q_1 + q_2), \quad (7)$$

$$q_- = q_1 - q_2. \quad (8)$$

In the physical limit, the variable  $q_-$  turns to zero and the variable  $q_+$  reduces to the physical coordinate  $q$ . In addition, the equality condition gets a more practical statement in terms of this coordinates.

### 2.1. Dissipative systems

The doubling of the degrees of freedom allows us to introduce terms of the form  $q_- \dot{q}_+^n$ , which allow to construct “generalized potentials”  $K(q_{\pm}, \dot{q}_{\pm})$  for dissipative forces. The form of the lagrangian for a general system is

$$\Lambda = L(q_1) - L(q_2) + K(q_{\pm}). \quad (9)$$

A simple, but illustrative example is the damped harmonic oscillator, with the nonconservative lagrangian

$$\Lambda = \dot{q}_+ \dot{q}_- - \omega^2 q_+ q_- + \frac{\epsilon}{2}(\dot{q}_- q_+ - \dot{q}_+ q_-), \quad (10)$$

which, after the variation as previously shown, leads to the well-known equation of motion

$$\ddot{q} + \epsilon \dot{q} + \omega^2 q = 0. \quad (11)$$

## 3. Fermionic degrees of freedom

Now we state formally Galley’s action principle for a fermionic system. We define the action  $S$  as follows

$$S[\psi_a^\alpha] = \int_{t_i}^{t_f} \Lambda(\psi_i^\alpha, \dot{\psi}_i^\alpha), \quad (12)$$

where the generalized fermionic lagrangian is, in analogy with the bosonic case, is given in terms of the doubled variables  $\psi_1$  and  $\psi_2$

$$\Lambda[\psi_1^\alpha, \psi_2^\alpha] = L(\psi_1^\alpha, \dot{\psi}_1^\alpha) - L(\psi_2^\alpha, \dot{\psi}_2^\alpha) + K(\psi_a^\alpha, \dot{\psi}_a^\alpha). \quad (13)$$

The boundary conditions are the same as in the previous section

$$\delta\psi_1(t_i) = \delta\psi_2(t_i) = 0, \quad (14)$$

$$\psi_1(t_f) = \psi_2(t_f), \quad \dot{\psi}_1(t_f) = \dot{\psi}_2(t_f), \quad \delta\psi_1(t_f) = \delta\psi_2(t_f). \quad (15)$$

As pointed out by Teitelboim [5], the boundary conditions must take into account that the equations of motion of fermions are first order, hence only one boundary condition is required. Thus, if  $\psi_1$  and  $\psi_2$  coincide at the final time, they must coincide also at the initial time, which should be added to (14) for consistency. Performing the variation of the action, the usual Euler-Lagrange equations are obtained

$$\frac{d}{dt} \left( \frac{\partial \Lambda}{\partial \dot{\psi}_a^\alpha} \right) - \frac{\partial \Lambda}{\partial \psi_a^\alpha} = 0. \quad (16)$$

### 3.1. Example: damped fermionic oscillator

Consider the lagrangian, given in terms of the  $\pm$  variables:

$$\Lambda = z^* \left( i\dot{\bar{\psi}}_+ \psi_- + i\dot{\psi}_- \bar{\psi}_+ \right) + z \left( i\dot{\bar{\psi}}_- \psi_+ + i\dot{\psi}_+ \bar{\psi}_- \right) + c \left( \bar{\psi}_- \psi_+ + \bar{\psi}_+ \psi_- \right), \quad (17)$$

where  $z = x + iy$  is a complex number. The equations of motion (taking the physical limit) are

$$2iz\dot{\psi} - \omega\psi = 0, \quad (18)$$

$$2iz^*\dot{\bar{\psi}} + \omega\bar{\psi} = 0, \quad (19)$$

whose solutions are given by the complex exponentials

$$\psi(t) = \psi_0 \exp \left[ \frac{\omega(-ix - y)}{2(x^2 + y^2)} t \right], \quad (20)$$

$$\bar{\psi}(t) = \bar{\psi}_0 \exp \left[ \frac{\omega(ix - y)}{2(x^2 + y^2)} t \right], \quad (21)$$

which correspond to a harmonic motion, with an exponentially decaying amplitude.

In terms of the coordinates  $q_{1,2}$  we have

$$\begin{aligned} \Lambda = x \left( i\dot{\bar{\psi}}_1 \psi_1 + i\dot{\psi}_1 \bar{\psi}_1 \right) + \omega \bar{\psi}_1 \psi_1 - x \left( i\dot{\bar{\psi}}_2 \psi_2 + i\dot{\psi}_2 \bar{\psi}_2 \right) - \omega \bar{\psi}_2 \psi_2 \\ + y \left[ \dot{\bar{\psi}}_2 \psi_1 - \dot{\bar{\psi}}_1 \psi_2 - \dot{\psi}_2 \bar{\psi}_1 + \dot{\psi}_1 \bar{\psi}_2 \right], \end{aligned} \quad (22)$$

where we can see that this lagrangian fits Galley's prescription by means of the identifications

$$L = x \left( i\dot{\bar{\psi}}\psi + i\dot{\psi}\bar{\psi} \right) + \omega \bar{\psi}\psi, \quad (23)$$

$$K = y \left[ \dot{\bar{\psi}}_2 \psi_1 - \dot{\bar{\psi}}_1 \psi_2 - \dot{\psi}_2 \bar{\psi}_1 + \dot{\psi}_1 \bar{\psi}_2 \right]. \quad (24)$$

As required, this generalized (fermionic-) potential is antisymmetric under  $\psi_1 \leftrightarrow \psi_2$ .

## 4. Supersymmetric action

The supersymmetric version of Galley's action principle is presented in the  $N = 2$  superspace formalism (some general remarks on superspace and notation conventions can be found in Appendix A). For simplicity, let us consider a system described by a single (superfield) supercoordinate  $\Phi(t, \theta, \bar{\theta})$ .

The non conservative supersymmetric lagrangian density is defined as

$$\Gamma(\Phi_1, \Phi_2) = \mathcal{L}(\Phi_1) - \mathcal{L}(\Phi_2) + \mathcal{K}(\Phi_1, \Phi_2). \quad (25)$$

where  $\mathcal{K}$  is the non conservative “superpotential”. The action is defined as

$$S = \int dX \Gamma(\Phi_1, \Phi_2). \quad (26)$$

In order to perform the variation of the action, the following boundary conditions are imposed

$$\delta\Phi_1|_{t_i} = \delta\Phi_2|_{t_i} = 0, \quad (27)$$

$$\Phi_1|_{t_f} = \Phi_2|_{t_f}, \quad D\Phi_1|_{t_f} = D\Phi_2|_{t_f}, \quad \bar{D}\Phi_1|_{t_f} = \bar{D}\Phi_2|_{t_f}, \quad \delta\Phi_1|_{t_f} = \delta\Phi_2|_{t_f}. \quad (28)$$

According to the discussion in the fermionic case, we should impose the additional condition  $\Phi_1(X, \epsilon)|_{t_i} = \Phi_2(X, \epsilon)|_{t_i}$ .

The variation of the action leads to the Euler-Lagrange equations

$$\frac{\partial\Gamma}{\partial\Phi_a} - D\frac{\partial\Gamma}{\partial D\Phi_a} - \bar{D}\frac{\partial\Gamma}{\partial\bar{D}\Phi_a} = 0. \quad (29)$$

In fact, the boundary terms of the previous variation are given by

$$\begin{aligned} B.T. &= \int dX \left\{ D \left( \eta_1 \frac{\partial\Gamma}{\partial D\Phi_1} - \eta_2 \frac{\partial\Gamma}{\partial D\Phi_2} \right) + \bar{D} \left( \eta_1 \frac{\partial\Gamma}{\partial\bar{D}\Phi_1} - \eta_2 \frac{\partial\Gamma}{\partial\bar{D}\Phi_2} \right) \right\} \\ &= i \neg D \left( \eta_1 \frac{\partial\Gamma}{\partial D\Phi_1} - \eta_2 \frac{\partial\Gamma}{\partial D\Phi_2} \right) \Big|_{t_i}^{t_f} - i \neg \bar{D} \left( \eta_1 \frac{\partial\Gamma}{\partial\bar{D}\Phi_1} - \eta_2 \frac{\partial\Gamma}{\partial\bar{D}\Phi_2} \right) \Big|_{t_i}^{t_f} \\ &= i \neg \left\{ (D\eta_1) \frac{\partial(\mathcal{L} + \mathcal{K})}{\partial D\Phi_1} + \eta_1 D \frac{\partial(\mathcal{L} + \mathcal{K})}{\partial D\Phi_1} - (D\eta_2) \frac{\partial(\mathcal{L} - \mathcal{K})}{\partial D\Phi_2} - \eta_2 D \frac{\partial(\mathcal{L} - \mathcal{K})}{\partial D\Phi_2} \right\} \Big|_{t_i}^{t_f} \\ &\quad - i \neg \left\{ (\bar{D}\eta_1) \frac{\partial(\mathcal{L} + \mathcal{K})}{\partial\bar{D}\Phi_1} + \eta_1 \bar{D} \frac{\partial(\mathcal{L} + \mathcal{K})}{\partial\bar{D}\Phi_1} - (\bar{D}\eta_2) \frac{\partial(\mathcal{L} - \mathcal{K})}{\partial\bar{D}\Phi_2} - \eta_2 \bar{D} \frac{\partial(\mathcal{L} - \mathcal{K})}{\partial\bar{D}\Phi_2} \right\} \Big|_{t_i}^{t_f}. \end{aligned} \quad (30)$$

In these expressions,  $\neg$  means setting  $\theta, \bar{\theta} = 0$  in the expressions to the right. Clearly,  $\neg(\Phi\Psi) = (\neg\Phi)(\neg\Psi)$ . With the last expression we can see that, because of the boundary conditions, all of these terms cancel or vanish.

#### 4.1. Example: Damped supersymmetric oscillator

The lagrangian density in the  $\pm$  variables is given by

$$\Gamma = \sqrt{m} [z^* \bar{D}\Phi_- D\Phi_+ + z \bar{D}\Phi_+ D\Phi_- - \omega \Phi_+ \Phi_-], \quad (31)$$

where  $z = \frac{1}{2} \left[ \sqrt{1 - \left(\frac{\epsilon}{2\omega}\right)^2} + i \frac{\epsilon}{2\omega} \right]$ .

The equation of motion, in the superspace language, is

$$[zD\bar{D} - z^*\bar{D}D]\Phi - \omega\Phi = 0. \quad (32)$$

In components, (32) is equivalent to the equations of motion of the damped (bosonic) oscillator (11), and those of the fermionic oscillators (18), (19). The solutions are

$$\psi(t) = e^{(-i\omega' - \frac{\epsilon}{2})t} \psi_0, \quad (33)$$

$$\bar{\psi}(t) = e^{(i\omega' - \frac{\epsilon}{2})t} \bar{\psi}_0, \quad (34)$$

$$q(t) = \left[ e^{i\omega't} q_{01} + e^{-i\omega't} q_{02} \right] e^{-\frac{\epsilon t}{2}}, \quad (35)$$

where  $\omega' = \omega \sqrt{1 - (\frac{\epsilon}{2\omega})^2}$ .

In terms of the original coordinates we have

$$\Gamma = \frac{\sqrt{m}}{2\omega} \left[ w' \bar{D}\Phi_1 D\Phi_1 - \omega^2 \Phi^2 - \left( w' \bar{D}\Phi_2 D\Phi_2 - \omega^2 \Phi_2^2 \right) + i \frac{\epsilon}{2} (\bar{D}\Phi_2 D\Phi_1 - \bar{D}\Phi_1 D\Phi_2) \right], \quad (36)$$

so that, we can identify the supersymmetric potential

$$\mathcal{K} = i \frac{\epsilon \sqrt{m}}{4\omega} (\bar{D}\Phi_2 D\Phi_1 - \bar{D}\Phi_1 D\Phi_2). \quad (37)$$

## 5. Conclusions

In this work we have applied Galley's principle of stationary action to fermionic and supersymmetric systems. Except for the additional boundary condition, the generalization is straightforward. Examples discussed fit the prescription for a non conservative lagrangian. The generalized potential can be written in superfield form.

## 6. Acknowledgments

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## Appendix A. Superspace Supersymmetry

$N = 2$  superspace contains a commuting variable  $t$ , which plays the role of time, and two anticommuting variables  $\theta$  and  $\bar{\theta}$ , which are complex conjugate of each other [6]. Superspace translations are defined

$$\delta\theta = \bar{\epsilon}, \quad (A.1)$$

$$\delta\bar{\theta} = \epsilon, \quad (A.2)$$

$$\delta t = -i(\epsilon\bar{\theta} + \bar{\epsilon}\theta). \quad (A.3)$$

The variation of a supercoordinate  $\Phi(t, \theta, \bar{\theta})$  is given by

$$\delta\Phi \equiv \epsilon Q\Phi + \bar{\epsilon}\bar{Q}\Phi, \quad (A.4)$$

where, the generators of super-translations are defined

$$Q = \partial_\theta - i\bar{\theta}\partial_t, \quad (A.5)$$

$$\bar{Q} = \partial_{\bar{\theta}} - i\theta\partial_t, \quad (A.6)$$

which satisfy the algebra of supersymmetry

$$\{Q, \bar{Q}\} = -2P, \quad (A.7)$$

$$QQ = \bar{Q}\bar{Q} = 0. \quad (A.8)$$

The covariant derivatives are

$$D = \partial_\theta + i\bar{\theta}\partial_t, \quad (A.9)$$

$$\bar{D} = \partial_{\bar{\theta}} + i\theta\partial_t, \quad (A.10)$$

which satisfy the algebra

$$\{D, D\} = 2P, \quad (A.11)$$

$$DD = \bar{D}\bar{D} = 0. \quad (A.12)$$

In order to get expressions in component form, we use the following expansion  $\Phi(t, \theta, \bar{\theta}) = q(t) + i\theta\bar{\psi} + i\bar{\theta}\psi - \theta\bar{\theta}A$ .

## References

- [1] Galley CR 2013. *Phys. Rev. Lett.* **110** 174301.
- [2] Galley v, Tsang D, Stein LC 2104. arXiv:1412.3082 [math-ph].
- [3] Schwinger J 1961, *J. Math. Phys.* **2** 407.
- [4] Martínez-Pérez NE and Ramírez C 2015, *Supersymmetric non conservative systems*, arXiv:1501.05018.
- [5] Teitelboim C, Galvao C.A.P. 1980 *J. Math. Phys.* **21** 1863-80.
- [6] Bellucci S(Ed) 2006. *Supersymmetric Mechanics - Vol. 1. Supersymmetry, Non commutativity and Matrix Models*. Lecture Notes in Physics. (Berlin: Springer) p 54.