The pion distribution amplitude from SDE-BSE

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Abstract. A brief exposition of the Schwinger-Dyson–Bethe-Salpeter equations of Quantum Chromodynamics and their application to hadron physics is given. Results for the rainbow-ladder trucantion scheme are presented. The Pion distribution amplitude is calculated in the SDE-BSE approach to hadron physics employing a novel method of computation [28]. The SDE-BSE is a well founded continuum approach to nonperturbative hadron physics that unifies a range of hadron observables.

1. Introduction

Strong interaction phenomena poses a wealth of fundamental questions with profound significance of our understanding of Nature and the structure of matter of which we and our Universe are composed [34]. The field of hadron physics is the study of strong interacting matter in all of its manifestations and the understanding of its properties and interactions in terms of the underlying fundamental theory, Quantum Chromodynamics (QCD) [34]. QCD is the theory of quarks, gluons, and their interactions; it is a self-contained part of the Standar Model of particle physics whose only input parameters are the masses of the quarks and the coupling constant between these and the gluons; it is a consistent quantum field theory with a simple and elegant Lagrangian, based entirely on the invariance under the local non-Abelian $SU(3)$ colour gauge group, and renormalisability. However, the Lagrangin written on the blackboard does not by itself explain the data on strongly interacting matter, and it is not clear how the plethora of observed bound states, the hadrons, and their properties arise from the fundamental quark and gluon fields, and the parameters of QCD.

At large momentum transfer, due to the property of asymptotic freedom of QCD, one could use the familiar perturbative language, the Feynman diagrams and the like, to describe hadron interactions. At these small dinstances, hadrons and their interactions are described as bound states of weakly interacting quarks and gluons. This description, however, starts to break down at energy scales of around 1-2 GeV, and it is surely inadequate at length scales corresponding to the size of the nucleon. At such lenght scales the strong coupling constant is large enough to invalidate perturbation theory and ones has to employ different methods to deal with what is called strong QCD.

In order to go from the nonperturbative quarks and gluons to the study of hadron physics, we need special tools such QCD correlation functions, a bridge between theory and experiment. At present, our choices for the nonperturbative calculation of theses correlation functions are lattice-QCD and the Schwinger-Dyson equations, both of which have their own advantages and drawbacks. Lattice-QCD Monte Carlo methods, based on the discretisation of spacetime,

Journal of Physics: Conference Series **651** (2015) 012006 doi:10.1088/1742-6596/651/1/012006

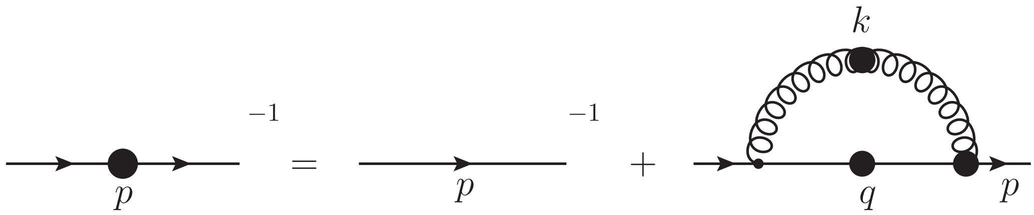

Figure 1. Quark Schwinger-Dyson Equation; filled circles indicate fully dressed objects.

include all the nonperturbative physics, and therefore are the only *ab initio* calculation method available so far. However, the simulations suffer from limitations at small momenta due to finite volume effects and one has to rely on extrapolation methods to obtain the infinite volume limit. Furthermore, calculations including quarks use unphysically large values for the quark masses, and extrapolations to the physical values are required. On the other hand, with the Schwinger-Dyson equations, which are the quantum equations of motion for the correlation functions of the theory, we can work in either Minkowski or Euclidean space for any value of the quark masses. However, here we are working with an infinite number of coupled integral equations, and in order to obtain a closed system of equations we must introduce ansatze for the higher correlation functions that are not expliclitly solved for, introducing a model dependence. However, this model dependence is limited to infrared momenta. Nevertheless, these two methods are entirely complementary in their strengths and weaknesses.

In this short talk we give a brief exposition of the Schwinger-Dyson and Bethe-Salpeter equations approach to hadron physics. The first few sections present generalities of this system of equations and of the rainbow-ladder truncation scheme. The main result is given in the last section where a novel method to obtain the pion distribution amplitude is presented [28]. An application of this distribution amplitude to the asymptotic form of the pion electromagnetic form factor is also shown. For a more comprehensive and up to date presentation see [35] and references therein.

2. The Schwinger-Dyson and Bethe-Salpeter Equations

The Schwinger-Dyson equations (SDEs) are the equations of motion of a quantum qield heory (QFT). They are derived from the full generating functional of the quantum theory. The starting point for the derivation of the SDEs (also called n-point functions) is the observation that the functional integral of a total derivative is zero:

$$\int \mathcal{D}\phi \frac{\delta}{\delta\phi} = 0. \tag{1}$$

In the Euclidean space formulation of QCD, with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\gamma_\mu^\dagger = \gamma_\mu$, and $a \cdot b = \sum_{i=1}^{4} a_i b_i$, the renormalised SDE for the dressed quark propagator (a 2-point function) for a particular quark flavour is

$$S^{-1}(p) = Z_2 i\gamma \cdot p + Z_4 m(\mu) + Z_1 \int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q) \frac{\lambda^a}{2} \gamma_\mu S(q) \Gamma^a_\nu(p,q), \tag{2}$$

where g is the renormalisation-point-dependent coupling constant, $D_{\mu\nu}$ is the renormalised dressed-gluon propagator, Γ^a_ν is the renormalised dressed-quark-gluon vertex, and $m(\mu)$ the renormalised current-quark mass; $Z_1(\mu,\Lambda)$ and $Z_2(\mu,\Lambda)$ are the quark-gluon vertex and the quark wave function renormalisation constants, respectively, which depend on the renormalisation point μ and regularisation mass scale Λ. This equation is depicted in Fig 1.

The dressed-quark propagator, as well as the dressed-gluon propagator, and dressed-quark-gluon vertex, depend on the renormalisation point at which they are defined; however obvervables do not depend on this.

Since both $D_{\mu\nu}$ and Γ^a_ν satisfy their own SDE, which in turn are coupled to higher n-point functions and so on *ad infinitum*, the quark SDE, Fig 1, explicitly shows that the SDEs form a coupled infinite set of nonlinear integral equations. A tractable problem is defined once a truncation scheme has been specified.

The general form of the renormalised dressed-quark propagator, obtained as the solution of Eq. (2), is given in terms of two Lorentz-scalar dressing functions, written in various forms as

$$\begin{aligned} S^{-1}(p) &= \mathrm{i}\gamma\cdot pA(p^2,\mu^2)+B(p^2,\mu^2) && (3)\\ &= Z^{-1}(p^2,\mu^2)\left(i\gamma\cdot p+M(p^2)\right), && (4)\end{aligned}$$

which are equivalent. In the last form, $Z(p^2,\mu^2)$ is known as the wave-function renormalisation, and $M(p^2)$ is the dressed quark mass function, which is independent of the renormalisation point if the quark propagator is renormalised multiplicatively.

The solution of the quark SDE Eq. (2) is subject to the renormalisation condition, for μ large and spacelike,

$$S^{-1}(p)\Big|_{p^2=\mu^2} = \mathrm{i}\gamma\cdot p + m(\mu), \tag{5}$$

where $m(\mu)$ is the flavour-dependent renormalised current-quark mass.

The best known truncation scheme of the SDEs is the weak coupling expansion, which reproduces every diagram in perturbation theory. It is a systematic-improvable truncation scheme, and a essential tool for the investigation of large momentum transfer phenomena because QCD is asymptotically free. However, it excludes the possibility of obtaining information about the low-energy regime relevant for the hadron structure and reactions, and the phenomena of dynamical chiral symmetry breaking (DCSB) and confinement, which are all essentially nonperturbative.

As an example, consider the chiral limit in perturbative QCD, defined by $Z_2 m_{\rm bare}\equiv 0$ for all $\Lambda \gg \mu$. In this case, the theory is chirally symmetric, and a perturbative evaluation of the quark propagator dressing functions, Eq. (3), gives [4, 7, 8]

$$B(p^2) = m\left[1-\frac{\alpha}{\pi}\ln\left(\frac{p^2}{m^2}\right)+\cdots\right], \tag{6}$$

where the ellipsis denote higher order terms in α. However, it is always true that at any order in perturbation theory

$$\lim_{m\to 0} B(p^2) = 0, \tag{7}$$

and hence dynamical chiral symmetry breaking is impossible in perturbation theory, and the quark SDE cannot generate a mass gap.

3. The Meson Bethe-Salpeter Equation

Meson bound states, whose flavour structure is given by a dressed quark-antiquark ($a\bar{b}$) pair, are described by the Bethe-Salpeter equation (BSE), depicted in Fig 2,

$$[\Gamma_H(p;P)]_{tu} = \int \frac{\mathrm{d}^4q}{(2\pi)^4}\,[K(p,q;P)]_{tu;rs}\left[S^a(q_+)\Gamma_H(q;P)S^b(q_-)\right]_{sr}, \tag{8}$$

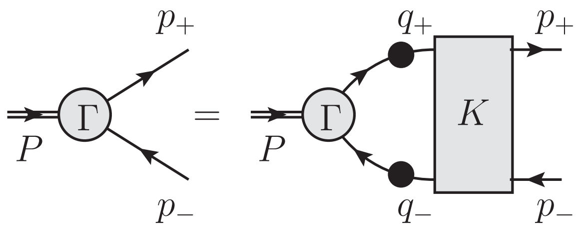

Figure 2. Bethe-Salpeter Equation: Γ is the fully-amputated quark-meson vertex or Bethe-Salpeter Amplitude; K is the fully-amputated, two-particle irreducible, scattering kernel; filled dots on the quark lines indicate quark propagators are fully dressed.

where $H = (a\bar{b})$ indicates the flavour structure. Here, $\Gamma_H(p;P)$ is the meson Bethe-Salpeter amplitude (BSA) describing the bound state, $S^f(q_\pm)$ is the propagator for a dressed quark, and $K(p,q;P)_{tu;rs}$ is the quark-antiquark scattering kernel. Latin indices indicate the colour, flavour, and Dirac structure. Poincaré covariance and momentum conservation entail $q_+ = q + \eta P$, $q_+ = q - (1-\eta)P$, and similarly for $p_\pm$, with $P = p_+ - p_-$. The parameter $\eta \in [0,1]$ describes the meson momentum sharing between the quark-antiquark pair. Observables, however, do not depend on this.

The Bethe-Salpeter equation, Eq. (8) is a homogeneous eigenvalue equation that admits solutions only for discrete values of the meson momenta $P^2 = -m_H^2$, where m_H is the mass of the meson under consideration. In a particular channel, the lowest mass solution corresponds to the physical ground state.

In the pseudoscalar channel ($J^P = 0^-$), the lowest mass solutions are the pion and kaon mesons, with flavour structure $u\bar{d}$ and $u\bar{s}$, respectively. The general form of the BSA in this channel is given by

$$\Gamma_H(p;P) = T^H\gamma_5\left[\mathrm{i}E_H(p;P) + \gamma\cdot PF_H(p;P) + \gamma\cdot p(p\cdot P)G_H(p;P) + \sigma_{\mu\nu}p_\mu P_\nu H_H(p;P)\right], \quad (9)$$

with $H = \pi,\ K$. For mesons that are eigenstates of the charge conjugation operation C, such as the π^0, there is an additional constraint on the Bethe-Salpeter amplitude to obtain a specified C-parity (In the isospin symmetric limit ($m_u = m_d$) we are working in, the mass and dressing functions of $\pi^\pm$ mesons will be equal to those of π^0). In Eq. (9), all the elements of the Lorentz-Dirac basis employed are even under C, and thus the only remaining quantity that can produce a definite C-parity is $p\cdot P$, which is odd under C. Therefore, a $C = +1\,(-1)$ solution will have dressing functions $E_H(p;P)$, etc that are even (odd) in $p\cdot P$. For mesons that are not eigenstates of C, each dressing function will contain both even an odd terms in $p\cdot P$. When using a Chebyshev expansion of $E_H(p;P)$ etc this means that for $C = +1\,(-1)$ we will only require even (odd) Chebyshev polynomials, and thus the odd (even) Chebyshev coefficients will vanish.
The pseudoscalar leptonic decay constant is calculated using

$$f_H P_\mu = Z_2\int\frac{\mathrm{d}^4q}{(2\pi)^4}\mathrm{Tr}\left[\frac{(T^H)^t}{2}\gamma_5\gamma_\mu S^a(q_+)\Gamma_H(q;P)S^b(q_-)\right],\ \text{ at }\ P^2 = -m_H^2, \quad (10)$$

where $H = \pi,\ K$, and the trace is over Dirac, colour, and flavour indices. In Eq. (10), $\Gamma_H(q;P)$ is the normalised BSA. [1]

In Eq. (8), $K(p,q;P)_{tu;rs}$ is the fully-amputated, two-particle irreducible, quark-antiquark scattering kernel. It is a four-point Schwinger function, obtained formally as the sum of a countable infinity of skeleton diagrams [9]. The complexity of $K(p,q;P)_{tu;rs}$ is one of the reasons why quantitative SDE and BSE studies employ modelling of $D_{\mu\nu}(k)$ and $\Gamma^a_\nu(k,p)$ [1], because $K(p,q;P)_{tu;rs}$ also appears in the SDE satisfied by $\Gamma^a_\nu(k,p)$. However, the lack of a full understanding of the interaction between quarks, through the complete knowledge of $K(p,q;P)_{tu;rs}$, does not prevent us from obtaining general results in hadron physics [2].

4. The Rainbow-Ladder Trucation and the Maris-Tandy Model

The kernel in the quark SDE, Eq. (2), is formed from the product of the dressed-gluon propagator and the dressed-quark-gluon-vertex, and therefore the structure of these quantities is largely responsible for the spectral properties of the dressed-quark propagator. However, in developing a truncation scheme for the quark SDE it is insufficient to concentrate only on these, as the SDE system forms an infinite tower. In the low-energy region of the strong interactions, dynamical chiral symmetry breaking is the dominant nonperturbative effect characterising the light hadron spectrum, in particular that of the octet of pseudoscalar mesons [10]. It is thus imperative that the implemented truncation scheme does not break chirally symmetry, and the pattern by which it is broken, if it is to be a viable approach.

In QCD, chiral symmetry and its breaking pattern are expressed through the axial-vector Ward-Takahashi identity [1, 2, 3]

$$\int \frac{\mathrm{d}^4q}{(2\pi)^4} K_{tu;rs}(k,q;P)\left[\gamma_5 T^H S(q_-) + S(q_+)\gamma_5 T^H\right]_{sr} = \left[\Sigma(k_+)\gamma_5 T^H + \gamma_5 T^H \Sigma(k_-)\right]_{tu}, \quad (11)$$

thus constrainig the content of the quark-antiquark scattering kernel $K(p,q;P)_{tu;rs}$ if an essential symmetry of the strong interactions, and its breaking pattern, is to be preserved.

From a practical point of view, Eq. (11) provides a way of obtaining the quark-antiquark scattering kernel if we can solve this constraint, given an expression for the quark self-energy. However, this is not always possible, see for example [11], and we must find an alternative way of preserving the chiral symmetry properties of the strong interactions. In principle, one may construct a quark-antiquark scattering kernel satisfying Eq. (11) from a functional derivative of the quark self-energy with respect to the quark propagator [12], within the framework of the effective action formalism for composite operators developed in [6].

Fortunately, for the rainbow truncation of the quark self-energy, Eq. (13), the axial-vector Ward-Takahashi identity can be easily satisfied. The quark-antiquark scattering kernel that is consistent with the rainbow truncation of the quark self-energy, in the sense that the axial-vector Ward-Takahashi identity, Eq. (11), is satisfied, is given by

$$K(p,q;P)_{tu;rs} = -\mathcal{G}(k^2) D^{\text{free}}_{\mu\nu}(k)\left[\frac{\lambda^a}{2}\gamma_\mu\right]_{ts}\left[\frac{\lambda^a}{2}\gamma_\nu\right]_{ru}, \quad (12)$$

where $k = p - q$, and $\mathcal{G}(k^2)$ is the effective coupling of Eq. (14). This is the so-called ladder truncation of the BSE. This equation corresponds to a single effective dressed-gluon exchange, and its solution corresponds to re-summing this gluon rung, thus providing the (infinite) ladder.

Despite their complexity, recent progress has been made in unveiling the nonperturbative structure of $D_{\mu\nu}$ and Γ^a_ν using their SDE [13, 14], as well the lattice formulation of QCD [15, 16]. Ultimately, the detailed infrared behaviour of these quantities should not materially affect the observable consequences of the quark SDE and meson BSE, as long as the infrared enhancement in the quark SDE implements the appropriate amount of dynamical chiral symmetry breaking and, explains the (pseudo)Goldstone character of the pion [5].

In modelling the quark SDE kernel we follow [1, 3], and make the following approximation in the self-energy, the so-called rainbow approximation,

$$Z_1 \int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(k) \frac{\lambda^a}{2} \gamma_\mu S(q) \Gamma^a_\nu(k,p) \to \int \frac{d^4q}{(2\pi)^4} \mathcal{G}(k^2) D^{\text{free}}_{\mu\nu}(k) \frac{\lambda^a}{2} \gamma_\mu S(q) \frac{\lambda^a}{2} \gamma_\nu, \tag{13}$$

where the phenomenological "effective" coupling $\mathcal{G}(k^2)$ contains information about the behaviour of the product $G(k^2, \mu^2)F^1(k, p, \mu)$, where $F^1(k, p, \mu)$ is the form factor associated with the γ_ν structure in the dressed-quark-gluon vertex Γ^a_ν, and $G(k^2, \mu^2)$ that in the dressed-gluon propagator. The model is completely defined once a form for $\mathcal{G}(k^2)$ has been specified. Note that we have actually made the replacement $\Gamma^a_\nu(k, p) \to (\lambda^a/2)\gamma_\nu$, and absorbed $Z_1 g^2$ into the effective coupling $\mathcal{G}(k^2)$. The function $\mathcal{G}(k^2)$ can be interpreted as an effective gluon dressing function. The solution of the resulting equation resumms an effectively-dressed gluon rainbow.

In principle, constraints on the form of $\mathcal{G}(k^2)$ come from the SDE satisfied by dressed-gluon propagator and the dressed-quark-gluon vertex. However, we know that the behaviour of the QCD running coupling $\alpha(k^2)$ in the ultraviolet, that is $k^2 > 2$-3 GeV^2, is well described by perturbation theory [17], and therefore the model dependence is restricted to the infrared region. On the other hand, the effective interaction in the quark SDE should exhibit sufficient infrared enhancement capable of triggering dynamical chiral symmetry breaking, through a nonzero quark condensate, and the generation of a momentum-dependent quark mass dressing function [18] that connects the current-quark mass to a constituent-like quark mass.

Various models for the effective interaction $\mathcal{G}(k^2)$ combining the ultraviolet behaviour known from perturbative QCD and an ansatz for the infrared part have been designed in the past. These have been applied to different detailed studies of dynamical chiral symmetry breaking, hadron structure and reactions [1, 3, 19, 20, 21].

In choosing a form for the effective coupling we follow reference [1, 3], and write the effective gluon dressing function $\mathcal{G}(k^2)$ as

$$\frac{\mathcal{G}(k^2)}{k^2} = 4\pi^2 \frac{D}{\omega^6} k^2 \exp\left(-\frac{k^2}{\omega^2}\right) + 4\pi \frac{\gamma_m \pi}{\frac{1}{2}\ln\left[\tau + \left(1 + \frac{k^2}{\Lambda_{\text{QCD}}}\right)^2\right]} \mathcal{F}(k^2), \tag{14}$$

where $\mathcal{F}(k^2) = \left(1 - \exp\left(-\frac{k^2}{4m_t^2}\right)\right)/k^2$, and fixed parameters $m_t = 0.5$ GeV, $\tau = e^2 - 1$, $N_f = 4$, $\gamma_m = 12/(33 - 2N_f)$, $\Lambda^{N_f}_{\text{QCD}} = 0.234$ GeV. The remaining parameters, ω and D, are phenomenological parameters fitted, together with the renormalised quark masses $m_u = m_d$ and m_s, to pion and kaon observables.

The detailed study of [3] found that pseudoscalar and vector meson ground state properties are practically insensitive to variations in the model parameters, with $\omega = 0.3$-0.5 GeV and $(\omega D)^{1/3} = 0.72$ GeV, as long as the integrated strength of the effective interaction is strong enough to generate an acceptable amount of chiral symmetry breaking, as required by the chiral quark condensate. However, this is not true for excited states made up of light quarks, where the long range part of the effective quark-quark interaction plays a significant role [23, 22, 24].

In Fig 3 we present numerical solutions to the mass dressing function $M(p^2) = B(p^2, \mu^2)/A(p^2, \mu^2)$, renormalised at $\mu = 19$ GeV, where various values of the renormalised current-quark mass are considered, while $\omega = 0.4$ GeV and $D = 0.93$ GeV^2 are kept fixed. It is evident that DCSB is a reality in the rainbow truncation with the effective coupling given by Eq. (14). At ultraviolet momenta, the magnitude of the mass function is determined from the renormalised current-quark mass. In the infrared, however, and specially for light-quarks, $M(p^2)$ is significantly enhanced. For light quarks, this enhancement is orders of magnitude larger than the mass present in the Lagrangian. Fig 3 also shows that the evolution from the current-quark mass to a constituent-like quark mass occurs at the scale of ≈ 1 GeV^2, as required

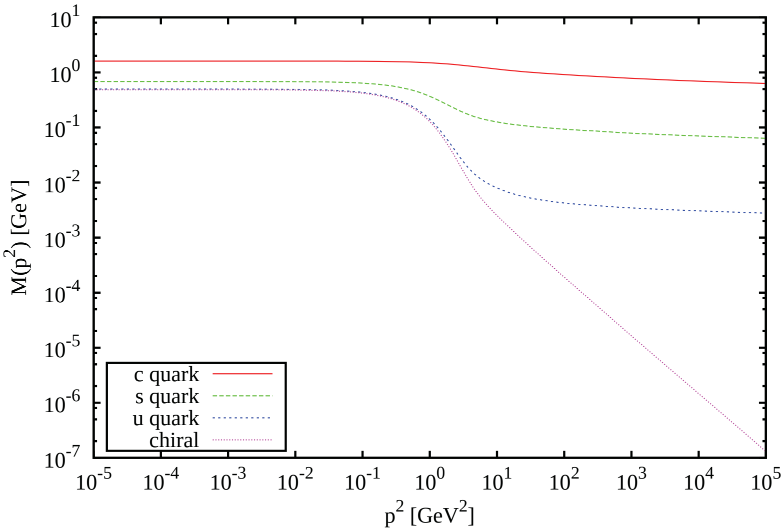

Figure 3. Quark mass function in the rainbow-ladder truncation with the Maris-Tandy dressing function for various values of the renormalised current-quark mass, with $\omega = 0.4$ GeV and $D = 0.93$ GeV^2.

from hadron phenomenology. Furthermore, this effect is particularly important for light quarks, that is the evolution and magnitude of their mass function in the infrared is dominated by the nonperturbative effect of DCSB. The domain in which the effect of DCSB is relevant decreases as the renormalised current-quark mass increases.

The behaviour of the quark mass function and wave function renormalisation of Fig 3 has also been confirmed semi-quantitatively in lattice simulations of QCD [36, 37, 38]. Agreement for a range of quark masses requires the effective interaction to be flavour dependent, and dressing the quark-gluon-vertex ensures this dependence, as pointed out in [39].

5. The Pion Parton Distribution Amplitude

The theoretical interest in the leading twist light-cone distribution amplitude (DA) of hadrons is due to their relevance in the description of exclusive reactions [25, 26, 27] from the fundamental accepted theory of the strong interactions, namely Quantum Chromodynamics. In terms of the Bethe-Salpeter wave functions these distributions are defined by keeping track of the momentum fraction x and integrating out the dependence on the transverse momentum $k_\perp$:

$$\phi(x,\mu) \sim \int_{k_\perp^2<\mu^2} d^2k_\perp \phi(x,k_\perp). \tag{15}$$

The DAs describe probability amplitudes to find the hadron in question in a state with minimum number of Fock constituents and at small transverse separation, which provides a ultraviolet cutoff. The dependence on the ultraviolet cutoff scale μ is given by the Efremov-Radyushkin–Brodsky-Lepage evolution equations [25, 27] and can be calculated in perturbative QCD, while the distribution amplitude at a certain low scale provides the necessary nonperturbative input for the rigorous QCD treatment of exclusive reactions with large momentum transfer [25, 26, 27].

In terms of dressed-quark propagators and the pion Bethe-Salpeter amplitude, the pion distribution amplitude is given by [28]

$$f_\pi \phi_\pi(x) = Z_2 N_c \mathrm{Tr} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \delta(n\cdot q_+ - x n \cdot P)\gamma_5 \gamma \cdot n S(q_+)\Gamma_\pi(q;P)S(q_- P), \tag{16}$$

where f_π is the pion decay constant; n is a light-like vector, $n^2 = 0$; P is the pion's four momentum, $P^2 = -m_\pi^2$ and $n \cdot P = -m_\pi$, with m_π being the pion's mass; Γ_π is the pion BSA; and S the dressed quark propagator. Due to Poincaré covarince, observables are independent of the momentum sharing parameter $\eta \in [0,1]$, with $q_+ = q + \eta P$ and $q_- = q - (1-\eta)P$. Using Eq. (16), the integer moments of the distribution $\phi_\pi(x)$, $\langle x^m \rangle = \int_0^1 dx x^m \phi_\pi(x)$, are given by [28]

$$f_\pi (n\cdot P)^{m+1} = Z_2 N_c \mathrm{Tr} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} (n \cdot q_+)^m \gamma_5 \gamma \cdot n S(q_+)\Gamma_\pi(q;P)S(q_- P). \tag{17}$$

The quark propagator S and pion BSA Γ_π are obtained as numerical solutions of their respective equations in the Rainbow-Ladder truncation using the Maris-Tandy interaction, which preserves the one-loop renormalisation group behaviour of QCD and guarantees that the quark mass function $M(p^2)$ is independent of the renormalisation point.

The computation of the moments in Eq. (17) is difficult with numerical solutions so we employ algebraic parametrisations of these that serve as interpolations in evaluating the moments. For the quark propagator a convenient representation is in terms of complex conjugate poles, a feature consistent with confinement . It is found that 2 complex conjugate poles is adequate [28]

$$S(p) = \sum_{l=1}^{2} \left(\frac{z_l}{\mathrm{i}\gamma \cdot p + m_l} + \frac{z_l^*}{\mathrm{i}\gamma \cdot p + m_l^*} \right). \tag{18}$$

The parameters z_l and m_l are numerically determined to provide the best fit to numerical solutions of the quark SDE. The quality of the fit is very good, see [28]. For the pion BSA, each scalar dressing function is expressed via a Nakanishi-like representation, with parameters fitted to that function's first two Chebyshev moments [28]

Using these representations it is now straigthforward to compute arbitrarily many moments using Feynman integral techniques and a subsequent numerical integrations over the Feynman parameters introduced. Once all needed moments are calculated (tipycally 50) a novel reconstruction procedure is implemented to obtain the distribution amplitude $\phi_\pi(x)$ [28]. In this procedure, the distribution amplitude is expanded in terms of the orthonormal set of Gegenbauer polynomials of order α, instead of the typical expansion in terms of Gegenbauer polynomials of order $\alpha = 3/2$; that is with α fixed. With $\alpha_- = \alpha - 1/2$ we write

$$\phi_\pi(x,\mu) = x^{\alpha_-}(1-x)^{\alpha_-} \left[1 + \sum_{j=2,4,\ldots} a_j^\alpha(\mu) C_j^\alpha(2x-1) \right]. \tag{19}$$

The parameter α and expasion coefficients $a_j^\alpha(\mu)$ are determined from a best fit to the numerical moments Eq. (17). Once these are obtained one may project out the result onto an $\alpha = 3/2$ basis, as dictated by the solution of the ERBL evolution equation. However, this requires many nonzero coefficient $\{a_j^{3/2}\}$, and introduce spurious oscillations that are typical of Fourier-like approximations to a simple function.

The plot in Fig 4 shows the RL result with the Maris-Tandy interaction, with $D\omega = (0.87\ \mathrm{GeV})^3$, and $\omega = 0.3$. It is described by

$$\phi_\pi^{\mathrm{RL}}(x,\mu) = 1.74 x^{\alpha_-^{RL}}(1-x)^{\alpha_-^{\mathrm{RL}}} \left[1 + a_2^{\alpha^{\mathrm{RL}}}(\mu) C_j^{\alpha^{\mathrm{RL}}}(2x-1) \right], \tag{20}$$

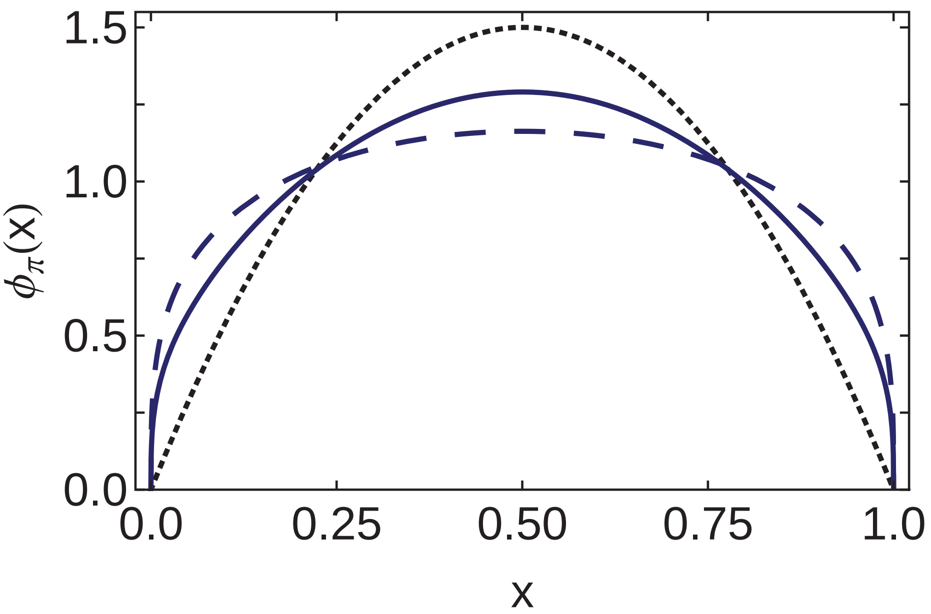

Figure 4. Computed distribution amplitude at $\mu = 2$ GeV. Curves: dashed, rainbow-ladder; solid: DCSB-improved kernel (see [28] and references therein); and dotted, asymptotic distribution.

with $\alpha^{\rm RL} = 0.79$, and $a_2^{\alpha^{\rm RL}} = 0.0029$. Projected onto a $\alpha = 3/2$ basis this corresponds to $a_2^{3/2} = 0.23, \ldots, a_{14}^{3/2} = 0.022$, etc. That $j \geq 14$ is required before $a_j^{3/2} < 0.1 a_2^{3/2}$ highlights the merit of reconstruction via Gegenbauer polynomials of order α at any reasonable scale μ. The merit is still greater still if one oly has access to a single nontrivial moment [30], as in lattice QCD [31]. For the moments $\langle (2x-1)^2 \rangle$ lattice-QCD gives $\langle (2x-1)^2 \rangle^{\text{lattice-QCD}} = 0.27 \pm 0.04$ [31], while the RL results gives $\langle (2x-1)^2 \rangle^{\rm RL} = 0.28$, and the asymptotic distribution gives $\langle (2x-1)^2 \rangle^{\rm asy} = 0.2$. Also shown in Fig 4 is the asymptotic QCD distribution amplitude $\phi_\pi^{\rm asy}(x) = 6x(1-x)$.

Various qualitative signficant results can be read from Fig 4. The most important being that DCSB is expressed in the pion PDA through a marked broadening with respect to $\phi_\pi^{\rm asy}$. This may be because the PDA has been computed at low renormalisation scale in the chiral limit, whereat the quark mass function owes entirely to DCSB; and on the domain $0 < p^2 < \mu^2$, the nonperturbative interactions responsible for DCSB produce significant structure in the dressed-quark's self-energy. The PDA is an integral of the pion's Bethe-Salpeter wave-function, whose pointwise behaviour is rigurously connected with that of the quark self-energy. Hence, the structure of the pion PDA at the hadronic scale is pure expression of DCSB. As the scale is removed to extremely large values, phase space growth diminishes the impact of nonperturbative DCSB interactions so that the PDA relaxes to its asymptotic form [28].

As a first application [29] of the calculated pion PDA, consider the perturbative QCD (pQCD) prediction for the asymptotic pion electromagnetic form factor $F_\pi(Q^2)$ [25, 26, 27]

$$Q^2 F_\pi(Q^2) \overset{Q^2\to\infty}{=} 16\pi\alpha_S(Q^2) f_\pi^2 \left| \frac{1}{3}\int dx \frac{1}{x}\phi_\pi(x)\right|^2 \quad (21)$$
$$= 16\pi\alpha_S(Q^2) f_\pi^2, \quad (22)$$

where in the $f_\pi = 92.2$ MeV and $\alpha_S(Q^2)$ is the strong running coupling constant at the scale Q^2, where Q is the four-momentum transfer in factorised electro-pion scattering. To get Eq. (22) we have used the asymptotic pion PDA, $\phi_\pi(x) = \phi_\pi^{\rm asy}(x)$. From the experimental data on $F_\pi(Q^2)$ [32, 33] there is a very weak suggestion that the condition that $Q^2F_\pi(Q^2) =$ constant is being approached. Furthermore the value at this should happen is not predicted by pQCD. Since the data is several times larger than the pQCD prediction, nonperturbative effects are still dominant. For example, for the largest experimental point [32, 33], $Q^2 = 2.45$ GeV^2, $Q^2F_\pi(Q^2) = 0.41$ GeV^2. Compare this with a larger Q^2 using the pQCD analysis. For $Q^2 = 4$ GeV^2 we have $Q^2F_\pi(Q^2) = 0.15$ GeV^2. These observations beg the question: Has $\phi_\pi(x)$ really reached its asymptotic form at currently accessible energies? At leading-log accuracy the Gegenbauer expansion coefficients evolve with scale according to [25, 27], with $\mu_1 = 2$ GeV,

$$a_n^{3/2}(\mu) = a_n^{3/2}(\mu_1)\left[\frac{\alpha_S(\mu_1)}{\alpha_S(\mu)}\right]^{\gamma_n^0/\beta_0}, \quad \alpha_S(Q^2) = \frac{4\pi}{\beta_0}\ln\left(\frac{Q^2}{\Lambda_{\rm QCD^2}}\right), \quad (23)$$

with

$$\beta_0 = 11 - (2/3)N_f, \quad \gamma_n^{(0)} = \frac{4}{3}\left[3 + \frac{2}{(n+1)(n+2)} - 4\sum_{k=1}^{n+1}\frac{1}{k}\right]. \quad (24)$$

Now comes the surprising result, see [29]: Using $N_f = 4$ quark flavours, and $\Lambda_{\rm QCD} = 0.234$ GeV it is necessary to evolve to $\mu = 100$ GeV before $a_2^{3/2}$ falls to 50 % of its value. This means that the asymptotic value $\phi_\pi^{\rm asy}(x)$ lies at very large momenta and the magnitude of $Q^2F_\pi(Q^2)$ reflect the scale of dynamical chiral symmetry breaking. The analysis in shows that hard contrubutions to the pion form factor dominate for $Q^2 \geq 8$ GeV^2. Therefore, in evaluating $Q^2F_\pi(Q^2)$, $\phi_\pi(x)$ has not yet reached its asymptotic value. Using the asymptotic form for $\phi_\pi(x)$ we have

$$\left|\frac{1}{3}\int dx\frac{1}{x}\phi_\pi^{\rm asy}(x)\right|^2 = 1. \quad (25)$$

On the other hand, using the RL result for $\phi_\pi(x)$ we have

$$\left|\frac{1}{3}\int dx\frac{1}{x}\phi_\pi^{\rm RL}(x)\right|^2 = 3.2. \quad (26)$$

This means that the pQCD analysis result has to be multiplied by a factor of 3.2 and the asymptotic analysis of various models has to be compared to this new result [29], since the asymptotic domain given by pQCD lies at very very large momenta .

6. Conclusions

The SDE-BSE is a well founded continuum approach to nonperturbative hadron physics. Indeed, they are a natural framework for the exploration of strong QCD since they provide access to infrared as well as ultraviolet momenta, thus giving a clear connection with processes that are well understood because QCD is asymptotically free. Moreover, the SDE are the generating tool for perturbation theory.

However, the SDE-BSE form an infinite tower of coupled n-point functions that must be truncated in order to define a tractable problem. That is, we have to make an ansatz for the n-point functions whose SDE are not explicitly solved for, thereby introducing a *model dependence* that is difficult to quantify. Furthermore, drawing a connection between QCD (in the form of the SDE for the nonperturbative quarks and gluons) and hadron observables (for example in the form of the meson BSE) is difficult, and that is why modelling remains a keystone is the SDE-BSE approach to hadron physics. Phenomenological input is proving useful in designing an effective quark-antiquark interaction, and quantitative comparisons between the SDE-BSE and lattice-QCD studies are today complementing the (SDE-BSE) approach.

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