

# Exact closed-form solution for the electrostatic interaction of two equal-sized charged conducting spheres

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**Abstract.** We provide an exact closed-form solution for the electrostatic interaction of two equal-sized conducting spheres. We calculate the capacitance coefficients for the spheres in terms of the q-analogue of the digamma function. In the near limit, when the two spheres are about to touch, the closed-form exact solutions allow for much faster numerical calculations than the well-known infinite series solutions. By analyzing the exact solution in the near limit, we provide Taylor series expressions for the capacitance coefficients in terms of the surface-to-surface separation of the two spheres.

## 1. Introduction

The two sphere electrostatic interaction was originally analyzed by Kirchoff, Lord Kelvin, Poisson, Maxwell, and Russell [1, 2, 3] and continues to generate interest in the electrostatics community even today [4, 5, 6]. Here, we study the special case of equal-sized spheres.

The setup of the two equal sphere problem is shown in figure 1. Two equal conducting spheres  $A$  and  $B$  of radii  $a$  and  $b$  (equal to  $a$ ) respectively are separated by distance  $s$  between their centers. Sphere  $A$  is held at potential  $V_a$  and sphere  $B$  at potential  $V_b$ . The capacitance coefficients  $C_{aa}$ ,  $C_{ab}$ ,  $C_{ba}$ , and  $C_{bb}$  of the two spheres are defined through the relations

$$\begin{aligned} Q_a &\equiv C_{aa}V_a + C_{ab}V_b, \\ Q_b &\equiv C_{ba}V_a + C_{bb}V_b, \end{aligned} \quad (1)$$

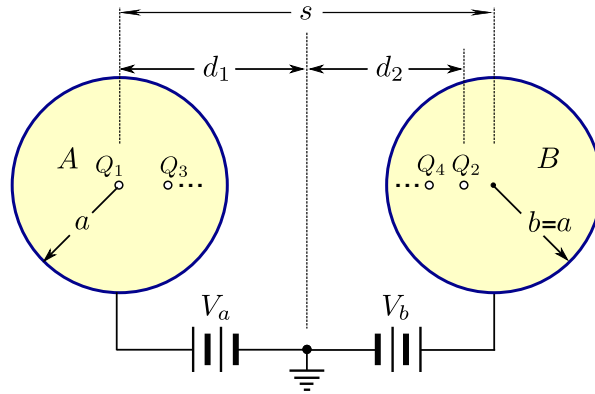
where  $Q_a$  and  $Q_b$  are the net charges on the two spheres. The coefficients  $C_{ab} = C_{ba}$  are always equal. For the special case of equal-sized spheres we have  $C_{aa} = C_{bb}$ .

The solution for the capacitance coefficients of the two spheres can be calculated using the method of images [7, 8]. By summing up the images charges inside each sphere and substituting  $a/s = \frac{1}{2}\text{sech } \alpha$  the capacitance coefficients defined above can be written as [8]

$$\begin{aligned} c_{aa} &= \frac{C_{aa}}{4\pi\epsilon a} = \sinh \alpha \sum_{n=0}^{\infty} \text{csch}(2n+1)\alpha, \\ c_{ab} &= \frac{C_{ab}}{4\pi\epsilon a} = -\sinh \alpha \sum_{n=1}^{\infty} \text{csch } 2n\alpha. \end{aligned} \quad (2)$$

The series in equation (2) were numerically analyzed by Pisler and Adhikari [9].





**Figure 1.** Two equal-sized conducting spheres are at center-to-center distance  $s$  from each other. Sphere  $A$  is maintained at voltage  $V_a$  and sphere  $B$  at voltage  $V_b$ . The image charges  $Q_1, Q_2, Q_3, Q_4, \dots$  shown are for the special case when  $B$  is grounded. The odd numbered images appear inside  $A$  and the even numbered images appear inside  $B$ .

## 2. Capacitance coefficients in closed form

Using the substitution  $e^{-\alpha} = x$  in equation (2) and noting that  $\sinh n\alpha = (e^{n\alpha} - e^{-n\alpha})/2$ , the series can be rewritten as

$$c_{aa} = \left( \frac{1-x^2}{x} \right) \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1-x^{4n+2}}, \quad (3)$$

$$c_{ab} = - \left( \frac{1-x^2}{x} \right) \sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{4n}}, \quad (4)$$

where the variable

$$x = \frac{s - \sqrt{s^2 - 4a^2}}{2a}. \quad (5)$$

In the limit of large separation between the sphere and the plane i.e.  $s \rightarrow \infty$ ,  $x$  approaches 0, and in the limit of small separation between the sphere and the plane i.e.  $s \rightarrow 2a$ , we have  $x \rightarrow 1$ . Writing the capacitance coefficients as a series in  $x$  allows us to reexpress equations (3, 4) as

$$c_{aa} = \left( \frac{1-x^2}{x} \right) \left[ \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} - 2 \sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}} + \sum_{n=1}^{\infty} \frac{x^{4n}}{1-x^{4n}} \right], \quad (6)$$

$$c_{ab} = - \left( \frac{1-x^2}{x} \right) \left[ \sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}} - \sum_{n=1}^{\infty} \frac{x^{4n}}{1-x^{4n}} \right]. \quad (7)$$

Noting that the expressions within the braces above are linear combinations of Lambert series in  $x$ ,  $x^2$ , and  $x^4$  [10], we can write the capacitance coefficients as

$$c_{aa} = \left( \frac{1-x^2}{x} \right) [L(x) - 2L(x^2) + L(x^4)], \quad (8)$$

$$c_{ab} = - \left( \frac{1-x^2}{x} \right) [L(x^2) - L(x^4)]. \quad (9)$$

The Lambert series  $L(x)$  can be expressed in closed form as

$$L(x) \equiv \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} = \frac{\psi_x(1) + \ln(1-x)}{\ln x}, \quad (10)$$

where  $\psi_x(1)$  is the  $q$ -digamma function [11],  $\psi_q(z)$ , evaluated at  $q = x$  and  $z = 1$ . For any  $0 < q < 1$  and  $z > 0$  the  $q$ -digamma function can be written as [12]

$$\psi_q(z) = -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{n+z}}{1-q^{n+z}}, \quad (11)$$

$$= -\ln(1-q) + \ln q \sum_{n=1}^{\infty} \frac{q^{nz}}{1-q^n}. \quad (12)$$

Using equation (10) the capacitance coefficients for the two sphere system can be expressed as

$$c_{aa} = \left( \frac{1-x^2}{x} \right) \frac{4\psi_x(1) - 4\psi_{x^2}(1) + \psi_{x^4}(1) - 4\tanh^{-1}x + 2\tanh^{-1}x^2}{4\ln x}, \quad (13)$$

$$c_{ab} = -\left( \frac{1-x^2}{x} \right) \frac{2\psi_{x^2}(1) - \psi_{x^4}(1) - 2\tanh^{-1}x^2}{4\ln x}. \quad (14)$$

These new closed-form results exactly reproduce the numerical calculations of the infinite series solutions in equation (2) by Pislser and Adhikari (see Table I in reference [9]).

### 3. Capacitance coefficients in the near limit

When the separation between the spheres is relatively small it is convenient to express their surface-to-surface separation in dimensionless form as

$$\xi \equiv \frac{s-2a}{2a} = \frac{(1-x)^2}{2x}. \quad (15)$$

As the spheres are about to touch we have  $\xi \rightarrow 0$  and  $x \rightarrow 1$ . Each term in the series in equations (6, 7) series diverges making the analysis in this limit difficult. Having the capacitances in closed form allows us to expand them in terms of  $\xi$  with  $\theta = \frac{1}{4} \ln \frac{8e^{2\gamma}}{\xi}$  to obtain

$$\begin{aligned} c_{aa} = & \theta + \frac{\xi}{3} \left( \theta + \frac{1}{24} \right) - \frac{\xi^2}{45} \left( \theta + \frac{21}{160} \right) + \frac{\xi^3}{189} \left( \theta - \frac{1171}{2520} \right) \\ & - \frac{23\xi^4}{14175} \left( \theta + \frac{1402133}{1854720} \right) + \frac{263\xi^5}{467775} \left( \theta - \frac{909776807}{233291520} \right) + \dots, \end{aligned} \quad (16)$$

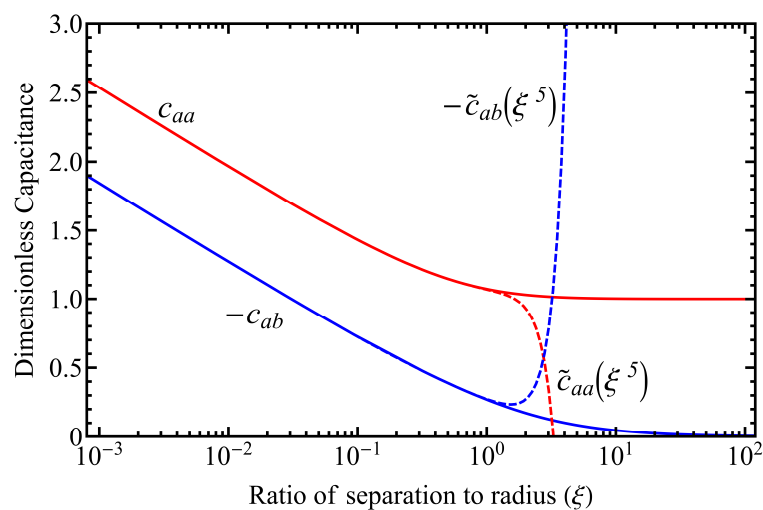
and with  $\phi = \frac{1}{4} \ln \frac{e^{2\gamma}}{2\xi}$  to obtain

$$\begin{aligned} c_{ab} = & -\phi - \frac{\xi}{3} \left( \phi + \frac{7}{24} \right) + \frac{\xi^2}{45} \left( \phi - \frac{149}{160} \right) - \frac{\xi^3}{189} \left( \phi - \frac{541}{2520} \right) \\ & + \frac{23\xi^4}{14175} \left( \phi - \frac{2619157}{1854720} \right) - \frac{263\xi^5}{467775} \left( \phi + \frac{762619303}{233291520} \right) + \dots, \end{aligned} \quad (17)$$

where  $\gamma = 0.577216\dots$  is the Euler's constant. The expansions in equations (16) and (17) indicate a logarithmic divergence of the capacitance coefficients and agree with Russell's results [3].

Along with the closed-form expressions in equations (13) and (14), the series expansions in equations (16) and (17) are the main results presented in this paper. The logarithmic dependence of capacitance in the near regime is well known but expansions are available only to order  $\xi^2$  [3, 4]. Having the capacitances in closed form allows us to expand the capacitance to higher orders through the analysis of the q-digamma function.

In figure 2 the capacitances in equations (13) and (14) are plotted against the relative sphere separation. Also plotted with dashed lines are the series expansions in equations (16) and (17) terminated at order  $\xi^5$ . At  $\xi = 1$  (i.e.  $s = 4a$ ), the terminated series for  $c_{aa}$  and  $c_{ab}$  are accurate to within 0.31% and 1.25% respectively.



**Figure 2.** The capacitance coefficients  $c_{aa}$  and  $c_{ab}$  in equations (13, 14) are plotted against the relative separation between spheres. Also plotted with dashed lines are  $\tilde{c}_{aa}(\xi^5)$  and  $\tilde{c}_{ab}(\xi^5)$ , the series expressions for capacitances in equations (16, 17) terminated at the  $\xi^5$  term.

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