Using a Classical Gluon Cascade to study the Equilibration of a Gluon-Plasma

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Abstract. Using a classical gluon cascade, we study the thermalisation of a gluon-plasma in a homogeneous box by considering the time evolution of the entropy, and in particular how the thermalisation time depends on the strong coupling α_s. We then partition the volume into cells with a linearly increasing temperature gradient in one direction, and homogeneous/isotropic in the the other two directions. We allow the gluons to stream in one direction in order to study how they then evolve spatially. We examine cases with and without collisions. We study the entropy as well as the flow-velocity in the z-direction and find that the system initially has a flow which dissipates over time as the gluons become distributed homogeneously throughout the box.

1. Introduction

Normal hadronic matter is composed of quarks in either three-quark or quark-antiquark bound states which are held together by the strong force. These quarks are said to be confined, as their strong attraction prevents them from being observed outside of these bound hadron states. However, under extremely high densities, it is possible for the quarks to become free of this strong attraction and become de-confined. When this happens the resultant state of matter is referred to as a Quark-Gluon Plasma (QGP). A QGP is a phase of Quantum Chromodynamics (QCD) that consists of free quarks and gluons in a state of matter that resembles (but is distinct from) a gas-like plasma. It is believed that up to a few milliseconds after the Big Bang that the universe was in a QGP state and thus its study can provide interesting insights into our earliest universe. In contrast to a gas-like plasma, the QGP behaves like a near-ideal Fermi liquid. The creation of the QGP is the main subject of the heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC) and at the Large Hadron Collider (LHC) at CERN. To recreate the conditions of the primordial universe, these powerful accelerators collide head-on heavy ions such as gold or lead. In these heavy ion collisions, hundreds of protons and neutrons that make up two such nuclei smash into each other at energies of upwards of a few trillion electronvolts each. This forms a miniscule fireball in which everything "melts" into a QGP.

The QGP can be described by the Boltzmann equation which gives a microscopic picture of particle systems. Thus in order to describe the QGP we solve the relatavistic Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{E}\nabla\right) f(\mathbf{x}, \mathbf{p}) = C_{22}, \tag{1}$$

where $f(\mathbf{x}, \mathbf{p})$ is the particle density in phase space, and C_{22} is the collision term. Unfortunately,

the Boltzmann equation has no analytic solution and must be solved using numerical methods. One such method is to solve it via Monte Carlo cascade simulations. We implement here such a simulation of a simplified picture of the QGP, and use it to study the thermalisation of a gluon only plasma in a volume. We will consider only elastic $gg \to gg$ collisions and will treat the gluons as classical Boltzmann particles. The cascade itself involves two distinct stochastic processes: to determine whether or not two gluons actually collide and to determine exactly how they collide.

2. Background

2.1. Scattering in the CoM Frame

It is often easier to work out the specifics of $2 \to 2$ scattering in the centre of mass (CoM) frame, which is defined as the frame wherein we have $\mathbf{p}_1 + \mathbf{p}_2 = 0$. We use a Lorentz boost to transform from the Lab frame to the CoM frame, work out the details of the scattering, before performing the inverse Lorentz boost back to the Lab frame. In the CoM Frame (quantities in the CoM Frame are primed), our pair of colliding massless gluons will have equal energies $E_1' = E_2' = \sqrt{\mathbf{p}_1' \cdot \mathbf{p}_1'}$ and thus equal three-momenta but in opposite directions $\mathbf{p}_1' = -\mathbf{p}_2'$. Mandelstam s is then

$$\begin{aligned} s &= (P_1' + P_2')^2 \\ &= (|\mathbf{p}_1'| + |\mathbf{p}_2'|)^2 \\ &= (2|\mathbf{p}_1'|)^2 = 4E_1'^2 = 4E_2'^2 \end{aligned} \tag{2}$$

Similarly Mandelstam t is

$$\begin{aligned} t &= (P_1' - P_3')^2 = 0 - (\mathbf{p}_1' - \mathbf{p}_3')^2 \\ &= -2\mathbf{p}_1'^2 + 2\mathbf{p}_1'\mathbf{p}_3' \cos\theta' \\ &= -2E_1'^2(1 - \cos\theta') = -\frac{s}{2}(1 - \cos\theta') \end{aligned} \tag{3}$$

Where the zero in the first line comes from conservation of four-momentum and θ describes to the scattering angle in the CoM frame. Note from Eq. (3) that since $\cos\theta' \in [-1, 1]$, that $t \in [-s, 0]$.

The three-momentum transfer is given by $\mathbf{q}^2 = (\mathbf{p}_1' - \mathbf{p}_3')^2$. Thus we have that $t = -\mathbf{q}^2$. The three-momenta in the CoM frame for a gluon i involved in the scattering is given by the Lorentz boost [3],

$$\mathbf{p}_i' = \mathbf{p}_i + \frac{\gamma - 1}{\mathbf{v}_{CoM}^2}(\mathbf{v}_{CoM} \cdot \mathbf{p}_i)\mathbf{v}_{CoM} - \gamma \mathbf{v}_{CoM} E_i \tag{4}$$

$$, \tag{5}$$

where the boost velocity and γ factor of the CoM are

$$\mathbf{v}_{CoM} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{|\mathbf{p}_1| + |\mathbf{p}_2|} \tag{6}$$

$$\gamma = \sqrt{\frac{1}{1 - \mathbf{v}^2}} \tag{7}$$

Our axes are defined by the units vectors

$$\hat{\mathbf{x}}' = \frac{\mathbf{p}_i'}{|\mathbf{p}_i'|}, \tag{8}$$

$$\hat{\mathbf{y}}' = N^{-1}[\mathbf{v}_{com} - (\mathbf{v}_{com} \cdot \hat{\mathbf{x}}')\hat{\mathbf{x}}']. \tag{9}$$

$$\hat{\mathbf{z}}' = \hat{\mathbf{x}}' \times \hat{\mathbf{y}}'. \tag{10}$$

Where N is a normalisation factor. Then the scattered momenta in terms of the scattering angles in the CoM frame are thus given by [2]

$$\mathbf{p}'_{i_{scattered}} = p'_i \left(\cos\theta'\hat{\mathbf{x}}' + \sin\theta'\sin\phi'\hat{\mathbf{y}}' + \sin\theta'\cos\phi'\hat{\mathbf{z}}'\right). \tag{11}$$

The scattering angles are determined via inverse transform sampling before performing the inverse Lorentz boost to get back to the Lab frame.

2.2. Stochatically determining if the Gluons Scatter

We derive the collision probability directly from the collision term of the Boltzmann equation (Eq. 1). The particle density in phase space is

$$f(x,p) = \frac{\Delta N}{\frac{1}{(2\pi)^3}\Delta^3 x \Delta^3 p_i}, \tag{12}$$

while (neglecting quantum effects) the collision term C_{22} is given by [1]

$$C_{22} = \frac{1}{2E_1}\int \frac{d^3p_2}{(2\pi)^3 2E_2}\frac{1}{\nu}\int \frac{d^3p'_1}{(2\pi)^3 2E'_1}\frac{d^3p'_2}{(2\pi)^3 2E'_2} f'_1 f'_2 \times |M_{1'2'\to 12}|^2 (2\pi)^4\delta^4(p'_1+p'_2-p_1-p_2)$$
$$-\frac{1}{2E_1}\times\int \frac{d^3p_2}{(2\pi)^3 2E_2}\frac{1}{\nu}\int \frac{d^3p'_1}{(2\pi)^3 E'_1}\frac{d^3p'_2}{(2\pi)^3 2E'_2} f_1 f_2 \times |M_{12\to 1'2'}|^2 (2\pi)^4\delta^{(4)}(p_1+p_2-p'_1-p'_2), \tag{13}$$

where ν is a factor related to double counting and $|M_{1'2'\to 12}|^2$ is the leading order pQCD interaction matrix element squared.

If we assume the two particles occupy a spatial volume $\Delta^3\mathbf{x}$ and have momenta in the range $(\mathbf{p}_1, \mathbf{p}_1+\Delta^3 p_1)$ and $(\mathbf{p}_2, \mathbf{p}_2+\Delta^3 p_2)$, then the collision rate per unit phase space for incoming particles p_1 and p_2 with $\Delta^3 p_1$ and $\Delta^3 p_2$ will be given by

$$\frac{\Delta N_{coll}}{\Delta t \frac{1}{(2\pi)^3}\Delta^3 x\Delta^3 p_1} = \frac{1}{2E_1}\frac{\Delta^3 p_2}{(2\pi)^3 2E_2} f_1 f_2 \times \frac{1}{2}\int \frac{d^3p'_1}{(2\pi)^3 2E'_1}\frac{d^3p'_2}{(2\pi)^3 2E'_2}|M_{12\to 1'2'}|^2(2\pi)^4\delta^{(4)}(p_1+p_2-p'_1-p'_2). \tag{14}$$

We can re-write Eq. (14) as [1]

$$\frac{\Delta N^{22}_{coll}}{\Delta t \frac{1}{(2\pi)^3}\Delta^3 x\Delta^3 p_1} = \frac{1}{2E_1}\frac{\Delta^3 p_2}{(2\pi)^3 2E_2} f_1 f_2 \times 2s \times \sigma_{22}. \tag{15}$$

Then using Eq. (12), we obtain an expression for the absolute collision probability in a unit box $\Delta^3 x$ and unit time Δt,

$$P_{22} = \frac{\Delta N^{22}_{coll}}{\Delta N_1 \Delta N_2} = v_{rel}\sigma_{22}\frac{\Delta t}{\Delta^3 x}, \tag{16}$$

where $v_{rel} = \frac{s}{E_1 E_2}$, and s is the invariant mass of the particle pair.

We will need the scattering cross-section in order to obtain the collision probabilities. For elastic gluon scattering, the differential cross-section is [1]

$$\frac{d\sigma^{gg\to gg}}{dt} = \frac{9\pi\alpha_s}{2s^2}\left(3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{st}{u^2}\right), \tag{17}$$

where s, t, and u are the Mandelstam invariants.

The collision probability requires a cross-section. The cross-section is large when t is small, corresponding to a small scattering angle which are overwhelmingly probable. The differential cross-section is then approximated by [1]

$$\frac{d\sigma}{dt} \approx \frac{9\pi\alpha_s}{t^2}. \tag{18}$$

To avoid infrared divergence, a Debye mass term is included to account for screening.

$$\frac{d\sigma}{dt} \approx \frac{9\pi\alpha_s^2}{(t-m_D^2)^2} \tag{19}$$

Integrating this over all t (i.e. from $-s$ to 0) then gives a regularised finite total cross-section

$$\sigma(s) = 9\pi\alpha_s^2 \frac{s}{m_D^2(s+m_D^2)}. \tag{20}$$

3. Simulation

With the background set out in the previous section, we are now equipped to simulate a gluon-plasma in a volume. Two separate configurations were examined. The first focuses on gluons colliding within a homogeneous box where the equilibrium temperature, number of gluons, strong coupling constant as well as the actual time period for which the gluon-plasma evolves are all variable parameters. Figure 1 shows an example of studying the thermalisation of the homogeneous box via the time evolution of its entropy. Through algebraic manipulation, we are able to fix the equilibrium temperature of the system through our choice of initial parameters. The system was simulated for a variety of initial phase space distributions including a delta function. After a characteristic time, called the thermalisation time, the system was observed to reach a state of dynamical equilibrium. We extracted the thermalisation time by modelling the difference of the entropy at each time step from its analytical equilibrium value as an exponential decay.

In the second configuration we partition the volume into cells and allow the gluons to stream in one direction. We consider two cases: one in which the gluons free-stream and the other where they are allowed to collide but only with other gluons in their cell. Figures 2 and 3 shows example plots of the flow-velocity in the z-direction (extracted using the energy-momentum tensor) as a function of time for free-streaming and including collisions respectively.

4. Conclusion

We found that the homogenous box system thermalises faster with increasing equilibrium temperature and particle number. We were then able to simulate the system for a variety of values for α_s and thus study how the thermalisation time is proportional to the variation of the QCD coupling constant. The thermalisation time was qualitatively shown to be inversely proportional to α_s.

We then added a spatial aspect to the model by dividing the volume into cells and allowing the gluons to stream in one direction. We examined a case where the gluons did not collide amongst themselves (free-streaming) and a case where they did. We implemented a linearly increasing temperature gradient for the cells and sampled the momenta of the gluons from a Boltzmann distribution. The entropy for both cases was seen to increase, with the free-streaming case gaining entropy from mixing, while the case which included collisions gained entropy from both mixing and thermalisation. We also studied how the gluons spatially distribute themselves throughout the volume. In both cases the gluon-plasma eventually reached a point where the gluons were distributed homogeneously throughout the volume. Though in the latter case, the

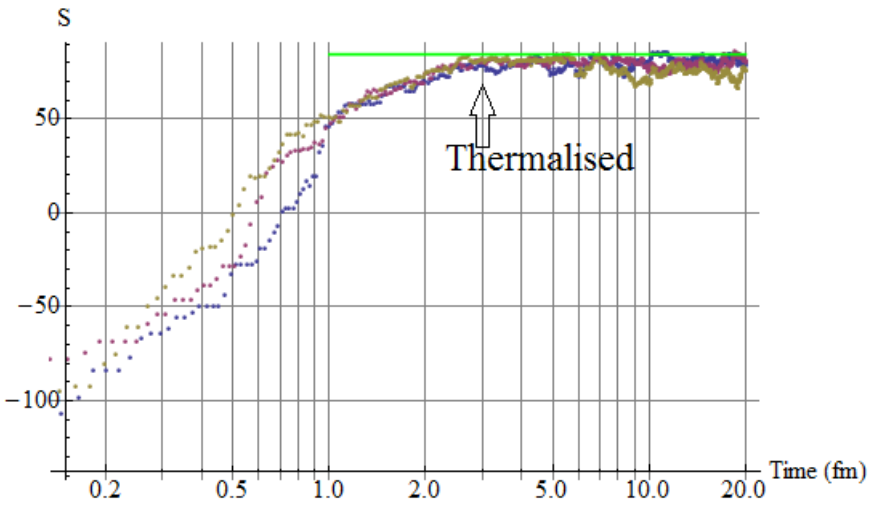

Figure 1: Time evolution of the entropy (initial phase space distribution is a delta function) for a plasma of 128 gluons with $T_{eq} = 300\ MeV$, $\alpha_s = 0.4$,and $dp = 0.2$. The solid line plotted alongside is the analytic value for the entropy at equilibrium.

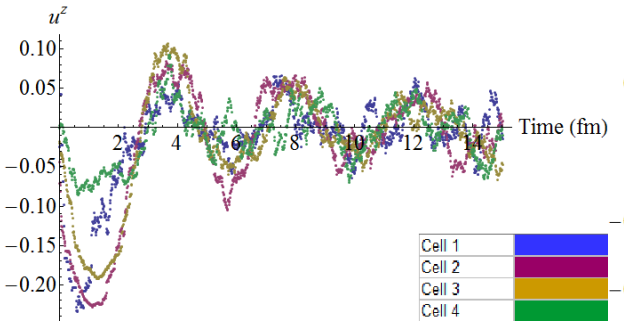

Figure 2: z-direction flow-velocity (free-streaming)

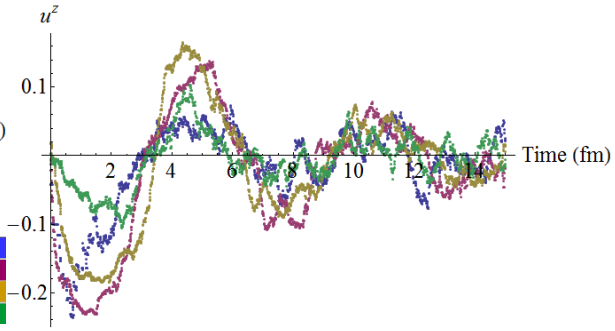

Figure 3: z-direction flow-velocity (collisions incl.)

end situation was noted to be a dynamic equilibrium which constantly fluctuated about its equilibrium values. We also derived from the energy-momentum tensor, the flow-velocity in the z-direction. We found that in both cases, the system initially had a flow-velocity in the negative z-direction which, after a characteristic time, reversed direction but also decreased in magnitude. This behaviour repeated until the flow-velocity went to zero (apart from fluctuations).

5. References

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