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Finite time calculations for hard parton production relevant to the quark-gluon plasma

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Abstract. AdS/CFT computations have been used to describe the energy loss of QCD-like particles moving through a strongly coupled plasma, but little is understood regarding the initial conditions of these jets. We use the Schwinger-Keldysh finite-time formalism applied to an interacting scalar field theory to derive perturbative expressions detailing the system which exists during the initial stages of a high energy collision.

In this paper we calculate $\langle\phi\rangle$(x) for a scalar Yukawa model, demonstrate the finiteness of the energy momentum tensor for $\lambda\phi^4$ to leading order, and derive an expression for the conditional expectation value of operators to aid in the description jet-like behaviour in interacting theories.

1. Introduction

The intention of this research is to understand the behaviour of particles in an interacting theory as a function of spacetime, during the initial moments after a high energy collision. Asymptotic freedom in QCD suggests that a perturbative approach with weak coupling will be well justified in this regime. A natural object to study is the expectation value of the energy momentum tensor. This can be done in the Schwinger-Keldysh formalism, matching the full interacting theory states to asymptotically free states at $t = \infty$ in the interaction picture. We will refer to these asymptotic states as $|\text{in}\rangle$ states. For some operator $\hat{O}(t, \vec{x})$ the expectation value is given by

$$\left\langle \hat{O}_{\text{Heisenberg}} \right\rangle (t) = \left\langle \underset{\rightarrow}{T} \exp\left(i \int_{-\infty}^{t} dz_1\, \hat{H}_{\text{I}}^{-}(z_1) \right) \hat{O}_{\text{I}}(t) \underset{\leftarrow}{T} \exp\left(-i \int_{-\infty}^{t} dz_1\, \hat{H}_{\text{I}}^{+}(z_1) \right) \right\rangle \tag{1}$$

The operator here is spacetime dependent, and so information regarding the evolution of the system can be retrieved. The $+$ and $-$ superscripts are used to disintguish the Hamiltonians of the time ordered and anti-time ordered exponentials, but the same Hamiltonians are used. It can be shown that this object is equivalent to the contour ordered exponential

$$\left\langle T_C \left(e^{-i \int_{-\infty}^{\infty} dz_1 \left(\hat{H}_I^{+}(z_1) - \hat{H}_I^{-}(z_1) \right)} \hat{O}_I(t) \right) \right\rangle, \tag{2}$$

where T_C is the contour ordering operator which orders the fields by their position along the Schwinger-Keldysh contour given below.[1]

[1] The superscript $+$ indicates a field on the top path of the contour, which will be evaluated before any field with index $-$ which is located on the bottom path.

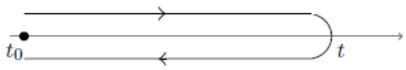


Figure 1: Time evolution indicated by the Schwinger-Keldysh contour.

This system can be solved in a similar way to usual diagrammatic calculations, now with four possible propagators due to contractions between $+$ and $-$ field operators. For some field $\phi(x)$ the contour ordered contractions $\langle 0|T_C\{\phi^i(x_1)\phi^j(x_2)\}|0\rangle$ will give propagator contributions of the form

$$D^{ij}(x_1,x_2) = \begin{cases} \langle 0|\underset{\leftarrow}{T}\{\phi(x_1)\phi(x_2)\}|0\rangle & \text{if } i,j=+,+ \\ \langle 0|\{\phi(x_2)\phi(x_1)\}|0\rangle & \text{if } i,j=+,- \\ \langle 0|\{\phi(x_1)\phi(x_2)\}|0\rangle & \text{if } i,j=-,+ \\ \langle 0|\underset{\rightarrow}{T}\{\phi(x_1)\phi(x_2)\}|0\rangle & \text{if } i,j=-,- \end{cases}$$

Important relations $D_R(x_1-x_2) = \sum_{m=\{+,-\}} mD^{+m}(x_1,x_2)$ and $\sum_{n,m=\{+,-\}} nmD^{nm}(x_1,x_2) = 0$ can be derived through this definition.

2. Expectation of $\phi(x)$

To gain an intuition for spacetime dependent Schwinger-Keldysh calculations, we find $\langle\phi(x)\rangle$. For now we will drop the T_C path ordering operator and assume that every expectation value is contour ordered. First we note that $\langle \text{in}_1|\text{in}_2\rangle \sim \delta_{\text{in}_1,\text{in}_2}$ in Schwinger-Keldysh, which can be seen by finding the expectation of the identity operator. In general we can choose the operator $\hat{O}$ to be evaluated on the $+$ or $-$ contour without changing the result. Choosing $\phi(x)$ on the $+$ contour, we will have

$$\langle\phi(x)\rangle_{\text{Heisenberg}} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \langle \phi^+(x) \prod_{i=1}^{n} \sum_{m_i=\{+,-\}} m_i H^{m_i} \rangle_{\text{interaction}} \tag{3}$$

If we choose the incoming states to be the same $|\text{in}_1\rangle$ then contracting $\phi(x)$ with an external state will allow us to factorize the operator out. Thus the rest of diagram will depend on the expectation $\langle \text{in}_1|\mathbb{I}|\text{in}_2\rangle$ with $|\text{in}_1\rangle \neq |\text{in}_2\rangle$. Thus these diagrams will contribute 0, and $\phi(x)$ will only contract with $\phi(z_i)$ operators from the interaction Hamiltonians.

Let $H^{m_i} = \int d^4z \left(g\psi\phi\psi^{m_i} - \delta_g\psi\phi\psi^{m_i} + \frac{1}{2}\psi\left(\delta_\psi\partial^2 - \delta_{m_\psi}\right)\psi^{m_i} + \frac{1}{2}\phi\left(\delta_\phi\partial^2 - \delta_{m_\phi}\right)\phi^{m_i}\right)$. The counter terms here simply cancel the divergences that come from the loop integrals, so we will not consider their effect in too much detail. Focusing on the $g\psi\phi\psi^{m_i}$ term, we explicitly write out the possible contractions.

$$\langle\phi(x)\rangle_{\text{Heisenberg}} = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \langle \phi^+(x) \int d^4z_1 ... d^4z_n \sum_{m_i=\{+,-\}} m_1\psi\phi\psi^{m_1} ... \sum_{m_n=\{+,-\}} m_n\psi\phi\psi^{m_n}\rangle$$

$$= \sum_{n=0}^{\infty} \frac{(-ig)^n n}{n!} \int d^4z_n \langle \sum_{m_n=\{+,-\}} m_n \overline{\phi^+(x)\phi^{m_n}}(z_n)\left(\psi\psi^{m_n}(z_n)\right) \prod_{i}^{n-1} \int d^4z_i \sum_{m_i=\{+,-\}} m_i\psi\phi\psi^{m_i}\rangle$$

$$= (-ig)\int d^4z \left(\sum_{m=\{+,-\}} mD^{+m}(x,z)\right) \sum_{n=0}^{\infty} \frac{(-ig)^{n-1}}{(n-1)!} \langle\left(\psi\psi^m(z)\right) \prod_{i}^{n-1} \int d^4z_i \sum_{m_i=\{+,-\}} m_i\psi\phi\psi^{m_i}\rangle$$

$$= (-ig)\int d^4z \left(\sum_{m=\{+,-\}} mD^{+m}(x,z)\right) \langle\psi\psi^m(z)\rangle_{\text{Heisenberg}}$$

Including the counter terms in this analysis gives us the same result, now with all loop contributions being renormalized. $\langle\psi\psi^m(z)\rangle_{\text{Heisenberg}}$ is just the expectation value of some operator. It does not matter if this operator is evaluated on the $+$ or $-$ contour, the answer will be the same through the m sum.

$$\begin{aligned}\langle\phi(x)\rangle_{\text{Heisenberg}} &= (-ig)\int d^4z\left(\sum_{m=\{+,-\}} mD^{+m}(x,z)\right)\langle\psi\psi(z)\rangle_{\text{Heisenberg}}\\ &= (-ig)\int d^4z\, D_R(x-z)\langle \text{in}|\psi\psi(z)|\text{in}\rangle_{\text{Heisenberg}}\end{aligned}$$

2.1. Leading order emission from a single particle
The above result will hold in arbitrary dimensions.

$$\langle\phi(x)\rangle = -ig\int d^{n+1}z\, D_R(x-z)\langle\psi\psi(z)\rangle \tag{4}$$

Notice that this is the solution to the equations of motion given by classically coupled lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 - g\phi\langle\psi\psi\rangle$. In general if we are able to determine the expectation $\langle\psi\psi\rangle$, we will have this expectation exactly. This gives this quantity the interpretation of being a propagation of the ϕ field from the current given by the expectation of the square of the ψ. In weak coupling this case be considered as a statistical ensamble of the possible virtual/real particles off which the ϕ field can be emitted.

Suppose we have $|\text{in}\rangle = |\psi\rangle$. To leading order we will find $\langle\psi|\psi\psi|\psi\rangle(z) = 2\langle\psi|\psi\psi|\psi\rangle(z)$ Using wavepackets $\tilde{\psi}(k)$ we can represent the assumptotic states as

$$|\psi\rangle = \int \frac{d^nk}{(2\pi)^{n/2}\sqrt{2E_k}}\tilde{\psi}(k)|\vec{k}\rangle. \tag{5}$$

with normalisation condition $\int \frac{d^nk}{(2\pi)^n}|\tilde{\psi}(k)|^2 = 1$. We find

$$\langle\psi|\psi\psi|\psi\rangle(z) = \int \frac{d^nk_1d^nk_2}{(2\pi)^n\sqrt{E_{k_1}E_{k_2}}}\tilde{\psi}(k_1)\tilde{\psi}^*(k_2)e^{-iz(k_1-k_2)}. \tag{6}$$

Then

$$\langle\phi\rangle(x) = g\int \frac{d^nk_1d^nk_2}{(2\pi)^n\sqrt{E_{k_1}E_{k_2}}}\tilde{\psi}(k_1)\tilde{\psi}^*(k_2)\frac{e^{-ix(k_1-k_2)}}{(k_1-k_2)^2 - m_\phi^2 + i\epsilon(k_1^0-k_2^0)} \tag{7}$$

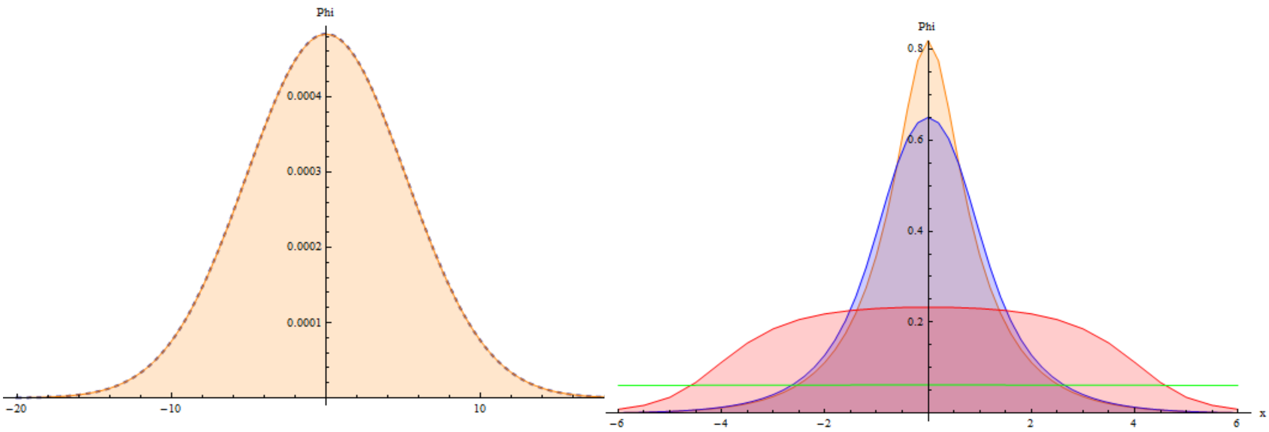
For an analytic result we take $\tilde{\psi}(k) = \frac{(2\pi)^{n/2}}{(2\pi\alpha^2)^{n/4}}e^{-\frac{1}{4}(\vec{k})^2/\alpha^2}$ Effectively there is only a significant contribution to the integral when $\vec{k}_1^2 + \vec{k}_2^2 << \alpha^2$. As $\alpha \to 0$ the wavepackets become sharply peaked and this becomes a smaller region. If $m_\psi >> \alpha^2$ then $E_q \approx m_\psi$ through the integral.[2] Thus

$$\langle\psi|\psi\psi|\psi\rangle(z) \approx \frac{1}{m_\psi}\left|\int \frac{d^nk}{(2\pi)^{n/2}}\tilde{\psi}(k)e^{-i\vec{z}\cdot\vec{k}}\right|^2 = \frac{1}{m_\psi}|\tilde{\psi}(\vec{z})|^2 \tag{8}$$

Note that where the time is small in comparison to m_ψ, the time dependence drops out of the expression. We are left with a static, stable source over this time interval. Thus

$$\langle\phi(x)\rangle \approx -i\frac{g}{m_\psi}\int d^{n+1}z\, D_R(x-z)|\tilde{\psi}(\vec{z})|^2. \tag{9}$$

[2] This argument will hold true for any peaked wavepacket with width parameterized by α.



(a) Comparison between numerical and analytic results for $\langle\phi(x)\rangle$ in $1+1$ dimensions, for large m_ψ

(b) Time evolution of $\langle\phi(x)\rangle$ in $1+1$ dimensions

Figure 2: Numerical Solutions in 1+1 D for $\langle\phi(x)\rangle$.

For our chosen wavepackets, $|\tilde{\psi}(\vec{z})|^2 = 2^n\sqrt{2\pi\alpha^2}^n e^{-2\alpha^2\vec{z}^2}$. Making the substitution $2\alpha^2 = \frac{1}{4\tau}$, choosing $n=3$ and integrating the expression over z^0, we will find

$$\langle\phi(x)\rangle \approx (2\pi)^3 \frac{g}{m_\psi}\int d^3z\, \frac{e^{-m_\phi|\vec{z}|}}{4\pi|\vec{z}|}\frac{e^{-\frac{(\vec{z}-\vec{x})^2}{4\tau}}}{\sqrt{4\pi\tau}^3} \tag{10}$$

We can think of this as the solution of the heat equation with initial condition given by $\frac{e^{-m_\phi|\vec{x}|}}{4\pi|\vec{x}|}$, the Yukawa potential. τ is some function of the width of the asymptotic wavepackets, as we localize this momenta (or spread out the concentration of the source in position space) we smear out the concentration of $\langle\phi(x)\rangle$ given at the origin. For non-zero τ this contribution will be finite. Similar arguments will hold in arbitrary dimensions. Far from the origin, the "diffusion" has not yet had a noticable effect, and the result will approach what we would expect in the limit $\tau = 0$. In $n=1$ the result can be written in closed form.

$$\langle\phi(x)\rangle \approx \frac{g}{m_\phi m_\psi}\frac{\pi}{2}e^{\frac{m_\phi^2}{8\alpha^2}}\left(e^{-m_\phi|\vec{x}|}\mathrm{Erfc}\left(\frac{m_\phi - 4|\vec{x}|\alpha^2}{2\sqrt{2\alpha^2}}\right) + e^{m_\phi|\vec{x}|}\mathrm{Erfc}\left(\frac{m_\phi + 4|\vec{x}|\alpha^2}{2\sqrt{2\alpha^2}}\right)\right) \tag{11}$$

3. Brief look at the improved energy momentum tensor

In general, expecations of operators in QFT are not finite, and in particular $\langle T_{\mu\nu}\rangle = \langle\sum_i \partial_\mu\phi_i\partial_\nu\phi_i - g_{\mu\nu}\mathcal{L}\rangle$ diverges order by order even after standard renormalization techniques. One must instead compute the improved energy momentum tensor[4] $\langle\Theta_{\mu\nu}\rangle = \langle T_{\mu\nu}\rangle + \langle\frac{1}{4}\frac{n-2}{n-1}\left(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2\right)\sum_i\phi_i^2\rangle$ for a quantity that is finite to at least one loop order.[3] Notice that this quantity is still satisfies $\partial^\mu\Theta_{\mu\nu} = 0_\nu$ and will produce the same result as $T_{\mu\nu}$ when integrated over all space.

Consider scalar $\lambda\phi^4$ theory with $|\mathrm{in}\rangle = |\phi\rangle$. The divergent contribution (when $\Theta_{\mu\nu}$ is evaluated on the $+$ contour) is given exclusively by the time ordered propagators, so we will ignore all other contributions when showing finiteness. Define $M_{\mu\nu}$ as the sum of all time ordered momentum space diagrams contributing to $\langle\Theta_{\mu\nu}\rangle$ at order λ. Then we can find

$$M_{\mu\nu} = 24i\lambda\int\frac{d^dq}{(2\pi)^4}\frac{(q-k)_\mu q_\nu - \frac{1}{2}g_{\mu\nu}((q-k)q - m^2) + \frac{1}{4}\frac{n-2}{n-1}(k_\mu k_\nu - g_{\mu\nu}k^2)}{((q-k)^2 - m^2)(q^2 - m^2)} \tag{12}$$

[3] n here is the number of spatial dimensions.

With $k = k_1 - k_2$ being the different in momenta of the bra and ket asymptotic states, and m the mass of the ϕ field. Working through this expression we can find the divergent part to be given by

$$\frac{24\lambda}{(4\pi)^{d/2}} \int_0^1 dx \left(\left(x(1-x) - \frac{1}{6} \right) (k_\mu k_\nu - g_{\mu\nu} k^2) \right) \frac{1}{2 - \frac{d}{2}} = 0. \tag{13}$$

Thus we can make sense of the $\Theta_{\mu\nu}$ operator to leading order in λ.

4. Conditional Expectation Value

To understand $\langle \Theta_{\mu\nu} \rangle(x)$ for a jet we propose choosing an initial state of two particles. After a two-particle collision, an infinite number of final states are kinematically accessible. Thus $\langle \Theta_{\mu\nu} \rangle$ will be some average value of these possible states. To restrict this expectation value with additional information we define the conditional expectation value. This is the expectation given both the initial and final states.

4.1. Derivation

Let r_n be an eigenvalue of the operator $\hat{R}$ and Δ a set of some of these eigenvalues. The probability that a measurement of $\hat{R}$ on some state will yield r_n is 1, what is the probability that it will yield $r_n \in \Delta$? We define

$$M_R(\Delta) = \sum_{r_n \in \Delta} |r_n\rangle\langle r_n| \tag{14}$$

This is the projection operator onto states given by Δ. Define density matrix $\rho = \sum_b \rho_b |b\rangle\langle b|$ with property $\sum_b \rho_b = 1$. The probability that a measurement will yield a result in Δ is the average value of the projection operator. This can be motivated by noting that if Δ_c is a complete set, $M_R(\Delta_c) = 1$. Then $\mathrm{Tr}\{\rho M_R(\Delta_c)\} = 1$. If Δ is a smaller subset, the expectation of this operator now excludes the contribution from the missing states. Thus we say that

$$P(\{R \in \Delta\}|\rho) = \mathrm{Tr}\{\rho M_R(\Delta)\} \tag{15}$$

We now want to consider two observables Q and R represented respectively by operators $\hat{Q}$ and $\hat{R}$ with eigenvectors given by $|q_n\rangle$ and $|r_n\rangle$ with $q_n \in \Gamma$ and $r_n \in \Delta$.

To define the conditional probability, to need to know the probability that a state corresponding to $q_n \in \Gamma$ will be measured, given that a state corresponding to $r_n \in \Delta$ has been measured. This is simply

$$P(\{Q \in \Gamma\}|\tilde{\rho}) = \mathrm{Tr}\{\tilde{\rho} M_Q(\Gamma)\} \tag{16}$$

where $\tilde{\rho}$ is the state of the system after a measurement of $\hat{R}$. How do we find $\tilde{\rho}$? If a state $\tilde{\rho}$ is such that a measurement will yield $r_n \in \Delta$ with certainty, then $\tilde{\rho} \sim M_R(\Delta)\hat{A}M_R(\Delta)$. Enforcing that $\mathrm{Tr}\{\tilde{\rho}\} = 1$ provides a normalization of $\mathrm{Tr}\{M_R(\Delta)\hat{A}M_R(\Delta)\}$.

Suppose we collapsed the state into $\tilde{\rho}(0)$ immediately after preparing the state in $\rho(0)$. Then we would find $\hat{A} = \rho(0)$. Define $\hat{\rho}(t)$ as the time evolved $\hat{\rho}(0)$ state. Then

$$\tilde{\rho}(t) = \frac{M_R(\Delta)\hat{\rho}(t)M_R(\Delta)}{\mathrm{Tr}\{M_R(\Delta)\hat{\rho}(t)M_R(\Delta)\}} = \frac{M_R(\Delta)\hat{\rho}(t)M_R(\Delta)}{\mathrm{Tr}\{\hat{\rho}(t)M_R(\Delta)\}} \tag{17}$$

where we have used the cyclicity of the trace and the fact that $M_R(\Delta)$ is a projection operator. Now we can write

$$P(\{Q \in \Gamma\}|\tilde{\rho}) = \frac{\mathrm{Tr}\{M_R(\Delta)\hat{\rho}(t)M_R(\Delta)M_Q(\Gamma)\}}{\mathrm{Tr}\{\hat{\rho}(t)M_R(\Delta)\}} = \frac{\mathrm{Tr}\{\hat{\rho}(t)M_R(\Delta)M_Q(\Gamma)M_R(\Delta)\}}{\mathrm{Tr}\{\hat{\rho}(t)M_R(\Delta)\}}. \tag{18}$$

This is the conditional probability that a state corresponding to $q_n \in \Gamma$ will be measured, given that a state corresponding to $r_n \in \Delta$ has been measured.

The conditional expecation value of an operator $\hat{\Theta}$ is defined

$$E(\hat{\Theta}, \{R \in \Delta\}|\rho) = \sum_{q_n \in \Gamma} \Theta_{q_n} P(Q \in \{q_n\}|\tilde{\rho}) \tag{19}$$

the sum of the eigenvalues Θ_{q_n} of $\hat{\Theta}$ weighted by the conditional probability. For $\Theta = 1$ this reduces to $E(1, \{R \in \Delta\}|\rho) = P(Q \in \Gamma|\tilde{\rho})$ as expected. We will take Γ to be a complete set so that any eigenvalue of $\hat{\Theta}$ is accessible.

We can write $\hat{\Theta} = \sum_{q_n \in \Gamma} \Theta_{q_n} |q_n\rangle\langle q_n|$, allowing us to express the conditional probability as

$$E(\hat{\Theta}, \{R \in \Delta\}|\rho) = \sum_{q_n \in \Gamma} \Theta_{q_n} \frac{\mathrm{Tr}\{\hat{\rho}(s) M_R(\Delta)|q_n\rangle\langle q_n|M_R(\Delta)\}}{\mathrm{Tr}\{\hat{\rho}(s) M_R(\Delta)\}} = \frac{\mathrm{Tr}\{\hat{\rho}(s) M_R(\Delta)\hat{\Theta}(t) M_R(\Delta)\}}{\mathrm{Tr}\{\hat{\rho}(s) M_R(\Delta)\}} \tag{20}$$

For the situations in which we are interested, we will set $s = \infty$ (the time at which the state $M_R(\Delta)$ is measured). We can consider $\hat{\Theta}$ to be given at some arbitrary time t because we have used a general eigenbasis given by $|q_n(t)\rangle$. Writing the density matrix in terms of some wavefunction for the system we will have

$$E(\hat{\Theta}, \{R \in \Delta\}|\rho) = \frac{\langle\Psi|M_R(\Delta)\hat{\Theta}(t)M_R(\Delta)|\Psi\rangle(s)}{\langle\Psi|M_R(\Delta)|\Psi\rangle(s)} \tag{21}$$

For an interacting theory, this expectation value can again be related to the asymptotic free field eigenvectors through the interaction picture, and perturbation theory can be done in the usual way. The new contour is given in the diagram provided.

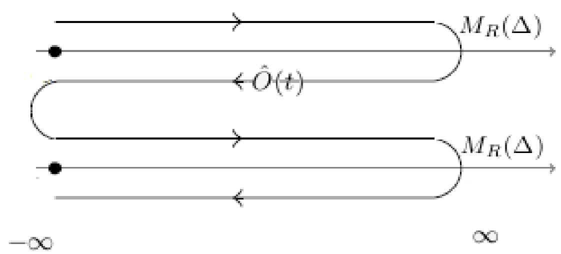


Figure 3: Time evolution indicated by a modified contour with various measurements

5. Conclusion

We have made progress towards building a spacetime description phenomena in quantum field theory, in particular towards finding the structure of jets in the perturbative regime. We have shown how calculations can be done in the case of $\langle\phi(x)\rangle$ for a simple scalar theory, and have demonstrated how to make sense of the energy momentum tensor in QFT through the improvement term. To extend this work we intend to calculate the conditional expectation value of $\Theta_{\mu\nu}$ given a simple $\psi\psi \to \psi\psi$ scalar particle collision. We also plan to derive similar results for theories containing gauge fields and fermions.

References

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