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Quasi-normal Modes for Spin-$3/2$ Fields

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Abstract. With the Large Hadron Collider already able to produce collisions with an energy of 8 TeV, the formation of higher dimensional black holes may soon be possible. In order to determine if we are detecting these higher dimensional black holes we need to have a theoretical understanding of what the signatures of such black holes could be. As such we shall discuss quasi-normal modes (QNMs) for spin-3/2 fields as they travel through a black hole background. We will begin by studying possible QNMs for N-dimensional Schwarzschild black holes and extend by looking at N-dimensional Kerr black holes. We will use the Wentzel-Kramers-Brillouin approximation to determine the QNMs for the two types of black holes described above.

1. Introduction

There are currently many ways to indirectly detect black holes [1]. One possible direct way of detecting a black hole is through gravitational wave detection. These are oscillations in the space time and are emitted by black holes when they are perturbed through either black hole-black hole collisions, stellar collapse or other matter interactions. These perturbations cause non-radial oscillations on the surface of a black hole and are called quasi-normal modes (QNMs). These QNMs in turn effect the space-time around a black hole and result in quasi-normal frequencies being emitted from black holes, these frequencies would be detected as gravitational waves, or particles [2]. What is interesting is that the allowed quasi-normal frequencies of a black hole are directly related to the characteristics of the black holes. For instance, the allowed quasi-normal frequencies of a Schwarzschild black hole are given by the mass of the black hole, but for a Kerr black hole they are determined by the angular momentum, charge and mass of the black hole [2]. This means that we could not only directly detect black holes by detecting their gravitational emission, but we could also get an idea of the parameters of the black hole by using this. A more elaborate discussion on detecting these QNMs is given in Ref. [3]. It has been shown that the allowed QNMs of a black hole can be calculated using perturbative methods such as the Wentzel-Kramers-Brillouin (WKB) method [4] and the improved asymptotic iteration method (AIM) [5]. Where the validity for determining QNMs using these methods has been shown. We will use these methods to determine the allowed QNMs for spin-3/2 fields. We shall first give a brief overview of black holes and the metrics that we will be using, in order to describe the space time surrounding these black holes. We then give a brief discussion of QNMs and then give an overview of the WKB method and improved AIM.

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2. Black Holes

Black holes have been described as the hydrogen atom of general relativity, since they are relatively simple general relativistic objects which display all of the properties predicted by Einstein's field equations. Hence a better understanding of black holes would allow us to better understand general relativity, just as the hydrogen atom can help in our understanding of quantum mechanics. In this paper we will set c = 1.

2.1. Schwarzchild Black holes

The Schwarzschild metric can be used to describe the space time surrounding any non-rotating objects in a vacuum [6]. It is given as:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dx^2 + r^2\left(d\phi^2 + \sin(\theta)d\theta^2\right), \tag{1}$$

where $G = 7.426 \times 10^{-28} m/kg$, the mass ($M$) denotes the mass of the object and r denotes the radial distance between a point in the space time and the center of that object. This metric is valid for all values of r greater than zero. We can describe the space time surrounding a black hole by looking at systems for which the value of $2GM$ is large compared to r. When r approaches $2GM$ our metric will produce a singularity, where this singularity is in fact a "coordinate singularity", which means that there exists a problem with the coordinate system we have chosen rather than there being any actual physical singularity at that point in space-time [6]. The point $r = 2GM$ is in fact the radial location of the event horizon of the black hole. These types of black holes are the simplest types of black holes and provide the simplest case for studying QNMs.

2.2. Kerr black holes

Although the Schwarzschild black holes are simple non-trivial examples of black holes, we are more interested in studying possible QNMs for a spinning black hole; as these are more likely to form as it is unlikely that matter would collapse with zero angular momentum. As before, we will use a metric to describe the space-time surrounding a black hole [7]. Where the metric we will use is the Kerr metric, given as:

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GMr}{r^2 + a^2\cos^2(\theta))}\right)dt^2 + \left(\frac{r^2 + a^2\cos^2\theta}{r^2 - 2GMr + a^2}\right)dr^2 + \left(r^2 + a^2\cos^2\theta\right)d\theta^2 \\ &+ \left(r^2 + a^2 + \frac{2GMra^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta d\phi^2 - \left(\frac{4GMra\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)d\phi dt, \end{aligned} \tag{2}$$

where G, M and r represent the same quantities as they did for the Schwarzchild metric. The new term in this metric is the "a" term, which describes the rate of rotation for the black hole. If we take this "a" value to zero we see that the "$d\phi dt$" term goes to zero. So through a simple analysis we can see that this metric describes a black hole rotating in the ϕ plane.

3. Quasi-Normal Modes

As stated before, QNMs are the reaction to a black hole being perturbed in some way. A simple explanation of what a QNM is can be described by the tapping of a wine glass. If we tap the wine glass we are in fact perturbing the glass. This perturbation causes the glass to resonate at specific frequencies. The allowed frequencies are determined by the composition of the glass, and even the contents within the glass. If the glass continued to resonate forever then we would be generating a normal mode. This would be an example of the standing wave, as many have studied in first year physics courses. Since the glass does stop resonating we say it is a QNM. So when studying QNMs we are studying standing waves with some damping term attached. If we consider our wine glass example we should note that wine glasses with different amounts of water in them ring at different frequencies, so we could, in theory, determine the amount of water in a glass by the sound it makes when we tap the glass. A similar idea can be used for the emitted quasi-normal frequencies of a black hole. Although with a black hole energy is being radiated, not a sound, but rather as disruptions in the space time which we call gravitational waves.

3.1. Tortoise coordinate

It is common to use tortoise coordinates when studying black holes, where these are defined as follows [8]:

$$r* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right). \tag{3}$$

This then satisfies the following relation:

$$\frac{dr*}{dr} = \left(1 - \frac{2GM}{r}\right)^{-1}. \tag{4}$$

The consequence of using this coordinate system is that as r tends to $2GM$, $r*$ goes to negative infinity, and as r tends to infinity, $r*$ tends to positive infinity. These two points correspond to the event horizon and a point infinitely far from the black hole respectively. We will use tortoise coordinates in order to develop a mathematical description of QNMs in a black hole background.

3.2. A Mathematical description of QNMs

In order to develop a more mathematical understanding of QNMs we can look at the mathematical formula for standing waves since, as stated above, QNMs are merely damped standing waves (mathematically speaking). So we can represent these QNMs as follows [9]:

$$\frac{d^2}{dx^2}\Psi - \frac{d^2}{dt^2}\Psi - V\Psi = 0. \tag{5}$$

With Ψ describing some wavelike function where x and t denote space and time coordinates respectively. V is some x-dependent potential. We can solve the equation as we would a standing wave problem and so we can assume the following time dependency [3]:

$$\Psi(x,t) = e^{-i\omega t}\phi(x). \tag{6}$$

Plugging this into our equation for a QNM we get:

$$\frac{d^2\phi(x)}{dx^2} - (\omega^2 + V(x))\phi(x) = 0. \tag{7}$$

This is the general form for the QNMs we will be studying in this paper. In order to solve this equation we require specific boundary terms, similar to those required for solving the standing wave equation. The boundary conditions that we will impose are:

$$\begin{aligned} V &\to 0; x \to \infty, \\ V &\to 0; x \to -\infty. \end{aligned} \tag{8}$$

This means that particles fall into the black hole at the horizon, and those which move to infinity are no longer influenced by the black hole. In order to determine if a frequency of a field is in fact a QNM, we need to ensure that it obeys these requirements. In this project we will use both the WKB method, to 6th order [4] , and the improved AIM [5] to determine the solutions to our QNM equation.

4. Aproximation Methods

4.1. WKB Approximation

The WKB method is a well known approximation, and is usually used in quantum mechanics to determine the solutions of Schrödinger equations in systems with non-constant potentials. The WKB method can be used to determine the approximate solution to second order differential equations. A brief example of how the method is used to solve second order differential equations is given below.

We can use the WKB approximation to solve problems of the following form [10]:

$$\epsilon^2 \frac{d^2 y}{dx^2} = Q(x) y, \tag{9}$$

where $\epsilon \ll 1$ and positive. We can solve this equation by assuming the following solution:

$$y(x,\epsilon) = A(x,\epsilon) e^{\frac{iu(x)}{\epsilon}}. \tag{10}$$

Taking the first and second derivatives of this function we have that:

$$\begin{aligned}
y(x,\epsilon) &= A(x,\epsilon) e^{\frac{iu(x)}{\epsilon}}, \\
y'(x,\epsilon) &= A'(x,\epsilon) e^{\frac{iu(x)}{\epsilon}} + A(x,\epsilon)\left(\frac{iu'(x)}{\epsilon}\right) e^{\frac{iu(x)}{\epsilon}}, \\
y''(x,\epsilon) &= A''(x,\epsilon) e^{\frac{iu(x)}{\epsilon}} + A'(x,\epsilon)\left(\frac{iu'(x)}{\epsilon}\right) e^{\frac{iu(x)}{\epsilon}} + A'(x,\epsilon)\left(\frac{iu'(x)}{\epsilon}\right) e^{\frac{iu(x)}{\epsilon}} - A(x,\epsilon)\left(\frac{u'(x)}{\epsilon}\right)^2 e^{\frac{iu(x)}{\epsilon}} \\
&= \left(A'' + 2A'\frac{iu'(x)}{\epsilon} + A(\frac{iu''(x)}{\epsilon} - \left(\frac{u'(x)}{\epsilon}\right)^2\right) e^{\frac{iu(x)}{\epsilon}}.
\end{aligned} \tag{11}$$

Plugging the relevant equations Eq.(11) into Eq.(9) we get:

$$\epsilon^2 \left(A'' + 2A'\frac{iu'(x)}{\epsilon} + A(\frac{iu''(x)}{\epsilon} - \left(\frac{u'(x)}{\epsilon}\right)^2\right) - Q(x) A(x,\epsilon) = 0. \tag{12}$$

We now perform a series expansion on A as follows:

$$\begin{aligned}
A(x,\epsilon) &= A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + ... \\
A'(x,\epsilon) &= A_0'(x) + \epsilon A_1'(x) + \epsilon^2 A_2'(x) + ... \\
A''(x,\epsilon) &= A_0''(x) + \epsilon A_1''(x) + \epsilon^2 A_2''(x) + ...
\end{aligned} \tag{13}$$

so that Eq.(12) becomes

$$\begin{aligned}
&\epsilon^2 \left(\left(A_0''(x) + \epsilon A_1''(x) + \epsilon^2 A_2''(x) + ...\right) + 2\left(A_0'(x) + \epsilon A_1'(x) + \epsilon^2 A_2'(x) + ...\right)\frac{iu'(x)}{\epsilon}\right) + \\
&\epsilon^2 \left(\left(A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + ...\right)(\frac{iu''(x)}{\epsilon} - \left(\frac{u'(x)}{\epsilon}\right)^2 - Q(x)\left(A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + ...\right)\right) = 0.
\end{aligned} \tag{14}$$

We then group the terms in powers of ϵ to get our approximation. So for instance we can collect all terms of ϵ^0 to get a solution to $u(x)$

$$\begin{aligned} A_0(x)u'(x)^2 + A_0 Q(x) &= 0, \\ u'(x) &= \pm\sqrt{Q(x)}, \\ u(x) &= \pm\int_{x_0}^{x}\sqrt{Q(k)}dk. \end{aligned} \tag{15}$$

We then take orders of ϵ to get our first order approximation of A

$$\begin{aligned} -u'(x)^2 A_1 + iA_0 u''(x) + 2iA_0' u'(x) + Q(x)A_0 &= 0, \\ A_0 u''(x) + 2A_0' u'(x) &= 0, \\ \frac{A_0}{2\sqrt{Q(x)}} + \frac{A_0'}{\sqrt{Q(x)}} &= 0, \end{aligned} \tag{16}$$

$$\int \frac{1}{A_0} dA_0 = \int \frac{1}{2Q(x)} dx. \tag{17}$$

We can use the WKB method in order to study black hole perturbations since these equations reduce to the form of a wave equation, similar to the example above [3]. This equation, with the known potential, is then solved using the 6th order WKB approximation. The cases for Schwarzschild black holes and Reissner-Nordström black holes has been studied extensively [5], these studies have not considered the case of spin-3/2 fields. The Kerr black hole requires us to solve a coupled wave equation for the radial and the angular case, hence this situation has not been studied as extensively as the other cases.

4.2. Improved AIM

The theory of this method is given in Ref. [11]. The theorem of this method is given below: Given λ_0 and s_0 in $C_\infty(a,b)$, then the differential

$$y'' = \lambda_0(x)y' + s_0(x)y, \tag{18}$$

has the general solution of:

$$y(x) = exp\left(-\int^x \alpha dt\right)\left[C_2 + C_1\int^x exp\left(\int^t \left(\lambda_0(\tau) + 2\alpha(\tau)\right)d\tau\right)\right], \tag{19}$$

for some $n > 0$.

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} \equiv \alpha, \tag{20}$$

where $\lambda_k = \lambda_{k-1}' + s_{k-1} + \lambda_0\lambda_{k-1}$ and $s_k = s_{k-1}' + s_0\lambda_{k-1}$ for $k = 1, 2, ..., n$.
We again solve for the QNMs by first obtaining the potential energy term for a spin-3/2 particle in a black hole background, and then use the AIM approximation to obtain the allowed QNMs.

5. Concluding remarks

We are currently checking the results obtained in Ref. [12] using the WKB method, and shall expand on this work by checking these results against those obtained by using the improved AIM Ref. [5]. We can then extend this by considering potentials for a spin-3/2 particle in a Kerr black hole background. We would like to, with time permitting, calculate the allowed QNMs for a spin-3/2 particle in a Reissner-Nordström black hole, which is a stationary charged black hole.

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References

[1] Novello M, Visser M and Volovik G E 2002 *Artificial black holes* vol 5 (World Scientific)
[2] Leaver E W 1985 *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **402** 285–298
[3] Kokkotas K D and Schmidt B G 1999 *Living Rev. Rel* **2** 262
[4] Konoplya R 2003 *Physical Review D* **68** 024018
[5] Cho H, Cornell A, Doukas J and Naylor W 2010 *Classical and Quantum Gravity* **27** 155004
[6] MooreThomas 2013 *AGeneralRelativityWorkbook* (University Science Books)
[7] Chandrasekhar S 1983 *Research supported by NSF. Oxford/New York, Clarendon Press/Oxford University Press (International Series of Monographs on Physics. Volume 69), 1983, 663 p.* **1**
[8] Gui-hua T, Wang S k and Zhong S 2006
[9] Vishveshwara C 1970
[10] Griffiths D J and Harris E G 1995 *Introduction to quantum mechanics* vol 2 (Prentice Hall New Jersey)
[11] Ciftci H, Hall R L and Saad N 2003 *Journal of Physics A: Mathematical and General* **36** 11807
[12] Piedra O P F 2011 *Int.J.Mod.Phys.* **D20** 93–109 (*Preprint* `1006.3327`)