

## Ball's motion, sliding friction, and internal load distribution in a high-speed ball bearing subjected to a combined radial, thrust, and moment load, applied to the inner ring's center of mass: *Numerical procedure*

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**Abstract.** In a companion paper of this was introduced a set of non-linear algebraic equations for ball's motion, sliding friction and internal loading distribution computation in a high-speed, single-row, angular-contact ball bearing, subjected to a known combined radial, thrust and moment load, which must be applied to the inner ring's center of mass. It was shown there that it is required the iterative solution of  $9Z + 3$  simultaneous non-linear equations – where  $Z$  is the number of balls – to yield exact solution for contact angles, ball attitude angles, rolling radii, normal contact deformations and axial, radial, and angular deflections of the inner ring with respect the outer ring. The Newton-Raphson method is to be used to solve the problem. This paper deals with the numerical procedure description. The numerical results derived from the described procedure shall be published later.

### 1. Introduction

When an external load is applied to one of the rings of a rolling bearing it is transmitted through rolling elements to the other ring. Because the internal load distribution on the rolling elements is an important operating characteristic of a bearing a great number of authors have addressing the problem. A literature review on the subject can be found in [1] and [2], in which a mathematical model for necessary radial displacement between rings, and a mathematical model for external radial load, so that the  $q$ -th rolling element passes to participate in the load transfer were presented.

In [3] a model was developed, which enables a very simple determination of the number of active rolling elements participating in an external load transfer, depending on the bearing type and internal radial clearance.

In [4] the theoretical analysis of a single-row radial bearing with radial clearance under constant external radial load was presented. The analysis was focused on finding the rolling element deflection that allows determining the number of active rolling elements that participate in the load transfer. Taking into account the bearing internal geometry, a mathematical model to calculate the rolling elements deflections during the bearing rotation has been derived.

In [5], [6] and [7] capacitive probes were inserted into the fixed ring of the bearing such that forms with the raceway a capacitor with variable gap that depends on the transmitted load by the rolling element. A numerical model of this capacitor's capacitance as a function of transmitted load by the rolling element has been established. An experimental prototype has been established in order to

precisely measure the probe's capacitance. Finally, this technique has been generalized with a capacitive probe in front of each rolling element. Thus, knowing the load transmitted by each of the rolling elements, the external load on the bearing of the rotating machine can be easily reconstructed.

The evaluation of change in contact angle due to applied load is vital in order to study the load carrying capacity of large diameter bearings. Analytical and numerical procedures have been developed to calculate various design factors such as contact angle, contact stress and deformation. In [8] the change in contact angle of balls was determined by using FEA. The change in contact angle was compared with analytical, FEA and the Newton-Raphson method. The results show a good agreement with the values calculated using Hertz's relations for deformation. The FEA method was used to get the nodal solution of contact angle, contact Stress and deflection for various loading conditions.

In [9] the dynamic modeling of a centrally supported symmetrical disk-shaft bearing system has been analyzed using Timoshenko beam elements. Intermittent ball bearing contact forces and Muszynska's force [10] at seal-disk interface were considered in the model to simulate a real-time system. Results show that there was a marked effect of each type of nonlinear excitation on the overall system response.

In [11] a wheel bearing life prediction method, which considers the bearing dynamics characteristics, was proposed. The results were compared with existing formulas and static analyses results from structural dynamics commercial software.

In [12] a unidirectional compression spring was used to model the contact between a rolling element and the raceway of a heavy-duty slewing bearing accounting for the supporting structure flexibility and the plastic deformation of the bearing. The spring constant was determined by the load against elastic-plastic deformation relationship of a single rolling element, which was obtained by finite element contact method. The difference between the traditional Hertz contact results and the FEM results is very obvious for the slewing bearings with plastic deformation, such as contact deflection of the rolling elements and the raceway, load distribution on the rolling elements, stress in the raceway and contact pressure between the rolling elements and the raceway. Therefore, the method based on the Hertz contact mechanics theory is not applicable for the performance analysis of the heavy-duty slewing bearing.

The focus of this paper is to describe the numerical procedure for ball's motion, sliding friction and internal load distribution computation in a *high-speed*, single-row, angular contact ball bearing, subjected to a known combined radial, thrust and moment load, which must be applied to the inner ring's center of mass. In a companion paper of this [13] a set of non-linear algebraic equations was introduced in order to accomplish this task. It was shown there that it is required the iterative solution of  $9Z + 3$  simultaneous non-linear equations – where  $Z$  is the number of balls – to yield exact solution for contact angles, ball attitude angles, rolling radii, normal contact deformations and axial, radial, and angular deflections of the inner ring with respect the outer ring. The Newton-Raphson method is to be used to solve the problem.

## 2. Symbols

|     |   |               |  |                     |
|-----|---|---------------|--|---------------------|
| $a$ | Semimajor axis of the projected contact, m; elements of square matrix | $K$           | inertia, $\text{kgm}^2$<br>Load-deflection factor, $\text{N/m}^{3/2}$  | $m$ ; short for sin |
| $b$ | Semiminor axis of the projected contact, m                            | $R$           | Curvature radius of deformed surfaces, m   |                     |
| $c$ | Short for cos   | $\mathcal{R}$ | Radius to locus of raceway groove curvature centers, m   |                     |
| $d$ | Bearing pitch diameter, m   | $V$           | Radial projection of distance between ball center and outer raceway groove curvature center, m; slip velocity of the race on the ball, m/sec |                     |
| $D$ | Ball's diameter, m  |               |  |                     |
| $f$ | $r/D$   | $r$           | Raceway groove curvature radius, m   |                     |
| $F$ | External, friction or centrifugal forces, N                           | $r'$          | Rolling radius, m  |                     |
| $J$ | Ball's mass moment of inertia, $\text{kgm}^2$                         | $s, s$        | Distance between loci of inner and outer raceway groove curvature centers,   |                     |

|            |  |               |                                    |      |   |   |
|------------|--|---------------|------------------------------------|------|---|---|
| $W$        | Axial projection of distance between ball center and outer raceway groove curvature center, m                      | $\mu$         | Friction coefficient               | $o$  | Refers to outer raceway   |   |
| $Z$        | Number of rolling elements   | $\psi$        | Azimuth angle, rad, °              | $r$  | Refers to radial direction  |   |
| $\beta$    | Contact angle, rad, °  | $\omega$      | rotational speed, rad/sec          | $R$  | Refers to the axis through the ball center perpendicular to the line defining the contact angle; refers to ball angular speed axis about its own center |   |
| $\gamma$   | Angle of the race on the ball slip velocity with respect to the semiminor axes of the pressure ellipses, rad, °    | Superscripts: |                                    |      | $s$   | Refers to the normal at the center of the contact ellipse |
| $\delta$   | Normal contact deformation, m; relative displacement between inner and outer rings, m; vector of systems variables | Subscripts:   |                                    |      | $x$   | Refers to x-direction                                     |
| $\epsilon$ | Error function   | $a$           | Refers to axial direction          | $x'$ | Refers to x'-direction  |   |
| $\theta$   | Relative angular displacement between inner and  | $g, h$        | index numbers                      | $y$  | Refers to y-direction   |   |
|            |  | $i$           | Refers to inner raceway            | $y'$ | Refers to y'-direction  |   |
|            |  | $j$           | Refers to rolling element position | $z$  | Refers to z-direction   |   |
|            |  | $k$           | index numbers                      | $Z$  | Refers to number of rolling elements  |   |
|            |  | $m$           | Refers to pitch diameter           | $z'$ | Refers to z'-direction  |   |

### 3. Numerical procedure

Equations (16)-(18), (22)-(24), (27)-(29), and (54)-(56) of [13] may be written as

$$\epsilon_g(\delta_h) = 0, \quad g, h = 1, \dots, 9Z + 3, \quad (1)$$

in which  $\delta_1 = V_1, \dots, \delta_Z = V_Z, \delta_{Z+1} = W_1, \dots, \delta_{2Z} = W_Z, \delta_{2Z+1} = \delta_{o1}, \dots, \delta_{3Z} = \delta_{oZ}, \delta_{3Z+1} = \delta_{i1}, \dots, \delta_{4Z} = \delta_{iZ}, \delta_{4Z+1} = r'_{o1}, \dots, \delta_{5Z} = r'_{oZ}, \delta_{5Z+1} = r'_{i1}, \dots, \delta_{6Z} = r'_{iZ}, \delta_{6Z+1} = \omega_{x'1}, \dots, \delta_{7Z} = \omega_{x'Z}, \delta_{7Z+1} = \omega_{y'1}, \dots, \delta_{8Z} = \omega_{y'Z}, \delta_{8Z+1} = \omega_{z'1}, \dots, \delta_{9Z} = \omega_{z'Z}, \delta_{9Z+1} = \delta_a, \delta_{9Z+2} = \delta_r, \delta_{9Z+3} = \theta$ .

The first  $9Z$  equations from (1) must be solved simultaneously for  $\delta_1, \dots, \delta_{9Z}$  once values for  $\delta_{9Z+1}, \dots, \delta_{9Z+3}$  are assumed. If  $\delta_h^0, h = 1, \dots, 9Z$ , is a  $9Z$ -dimensional vector with the initial estimates of the variables  $\delta_1, \dots, \delta_{9Z}$ , improved values are given by

$$\delta'_h = \delta_h^0 - [\epsilon_g]^{-1} \{\epsilon_g\}, \quad (2)$$

in which  $\{\epsilon_g\}, g = 1, \dots, 9Z$ , is the  $9Z$ -dimensional vector with the first  $9Z$  errors functions from (1). The elements of the square  $9Z \times 9Z$ -matrix  $[\epsilon_g]$  are

$$a_{jh} = -2(s_{zj} - \delta_j) \frac{\partial \delta_j}{\partial \delta_h} - 2(s_{xj} - \delta_{Z+j}) \frac{\partial \delta_{Z+j}}{\partial \delta_h} - 2[(f_i - 0.5)D + \delta_{3Z+j}] \frac{\partial \delta_{3Z+j}}{\partial \delta_h}, \quad (3)$$

$$a_{(j+Z)h} = 2\delta_j \frac{\partial \delta_j}{\partial \delta_h} + 2\delta_{Z+j} \frac{\partial \delta_{Z+j}}{\partial \delta_h} - 2[(f_o - 0.5)D + \delta_{2Z+j}] \frac{\partial \delta_{2Z+j}}{\partial \delta_h}, \quad (4)$$

$$a_{(j+2Z)h} = 2\delta_{6Z+j} \frac{\partial \delta_{6Z+j}}{\partial \delta_h} + 2\delta_{7Z+j} \frac{\partial \delta_{7Z+j}}{\partial \delta_h} + 2\delta_{8Z+j} \frac{\partial \delta_{8Z+j}}{\partial \delta_h}, \quad (5)$$

$$a_{(j+3Z)h} = \frac{\left\{ \left( F_{x0} \frac{\partial \delta_j}{\partial \delta_h} - K_{0j} \delta_{2Z+j}^{\frac{1}{2}} \frac{\partial \delta_{2Z+j}}{\partial \delta_h} - 1.5 K_{0j} \delta_{2Z+j}^{\frac{0.5}{2}} \delta_{Z+j} \frac{\partial \delta_{2Z+j}}{\partial \delta_h} + \delta_j \frac{\partial F_{x0j}}{\partial \delta_h} \right) \times \left\{ \left[ F_{x1j} \frac{\partial \delta_j}{\partial \delta_h} - K_{1j} \delta_{3Z+j}^{\frac{1}{2}} \frac{\partial \delta_{3Z+j}}{\partial \delta_h} + 1.5 K_{1j} \delta_{3Z+j}^{\frac{0.5}{2}} (s_{xj} - \delta_{Z+j}) \frac{\partial \delta_{3Z+j}}{\partial \delta_h} - (s_{xj} - \delta_j) \frac{\partial F_{x1j}}{\partial \delta_h} \right] \times \right\}}{\left[ (f_o - 0.5)D + \delta_{2Z+j} \right]^2} - \left( F_{x0j} \delta_j - K_{0j} \delta_{2Z+j}^{\frac{1}{2}} \frac{\partial \delta_{2Z+j}}{\partial \delta_h} \right) \frac{\partial \delta_{2Z+j}}{\partial \delta_h}}{\left[ (f_i - 0.5)D + \delta_{3Z+j} \right]^2} + \frac{\left\{ \left[ F_{x1j} \frac{\partial \delta_j}{\partial \delta_h} - K_{1j} \delta_{3Z+j}^{\frac{1}{2}} \frac{\partial \delta_{3Z+j}}{\partial \delta_h} + 1.5 K_{1j} \delta_{3Z+j}^{\frac{0.5}{2}} (s_{xj} - \delta_{Z+j}) \frac{\partial \delta_{3Z+j}}{\partial \delta_h} - (s_{xj} - \delta_j) \frac{\partial F_{x1j}}{\partial \delta_h} \right] \times \right\}}{\left[ (f_i - 0.5)D + \delta_{3Z+j} \right]^2}, \quad (6)$$

$$a_{(j+4Z)h} = \frac{\left\{ \left( K_{0j} \delta_{2Z+j}^{\frac{1}{2}} \frac{\partial \delta_j}{\partial \delta_h} + F_{x0j} \frac{\partial \delta_{Z+j}}{\partial \delta_h} + 1.5 K_{0j} \delta_{2Z+j}^{\frac{0.5}{2}} \delta_j \frac{\partial \delta_{2Z+j}}{\partial \delta_h} + \delta_{Z+j} \frac{\partial F_{x0j}}{\partial \delta_h} \right) \times \left\{ \left[ K_{1j} \delta_{3Z+j}^{\frac{1}{2}} \frac{\partial \delta_j}{\partial \delta_h} + F_{x1j} \frac{\partial \delta_{Z+j}}{\partial \delta_h} - 1.5 K_{1j} \delta_{3Z+j}^{\frac{0.5}{2}} (s_{zj} - \delta_j) \frac{\partial \delta_{3Z+j}}{\partial \delta_h} - (s_{xj} - \delta_{Z+j}) \frac{\partial F_{x1j}}{\partial \delta_h} \right] \times \right\}}{\left[ (f_o - 0.5)D + \delta_{2Z+j} \right]^2} - \left( K_{0j} \delta_{2Z+j}^{\frac{1}{2}} \delta_j + F_{x0j} \delta_{Z+j} \right) \frac{\partial \delta_{2Z+j}}{\partial \delta_h}}{\left[ (f_i - 0.5)D + \delta_{3Z+j} \right]^2} + \frac{\left\{ \left[ K_{1j} \delta_{3Z+j}^{\frac{1}{2}} (s_{zj} - \delta_j) + F_{x1j} (s_{xj} - \delta_{Z+j}) \right] \frac{\partial \delta_{3Z+j}}{\partial \delta_h} \right\}}{\left[ (f_i - 0.5)D + \delta_{3Z+j} \right]^2} - \frac{\partial F_{z'j}}{\partial \delta_h}, \quad (7)$$

$$a_{(j+5Z)h} = \frac{\partial F_{y0j}}{\partial \delta_h} + \frac{\partial F_{y1j}}{\partial \delta_h}, \quad (8)$$

$$a_{(j+6z)h} = \frac{\left\{ \left( M_{Roj} \frac{\partial \delta_j}{\partial \delta_h} + M_{soj} \frac{\partial \delta_{2z+j}}{\partial \delta_h} + \delta_j \frac{\partial M_{Roj}}{\partial \delta_h} + \delta_{2z+j} \frac{\partial M_{soj}}{\partial \delta_h} \right) \times \right.}{\left[ (f_o - 0.5)D + \delta_{2z+j} \right]^2} + \frac{\left\{ \left[ M_{Rij} \frac{\partial \delta_j}{\partial \delta_h} + M_{sij} \frac{\partial \delta_{2z+j}}{\partial \delta_h} - (s_{xj} - \delta_{2z+j}) \frac{\partial M_{sij}}{\partial \delta_h} - (s_{zj} - \delta_j) \frac{\partial M_{Rij}}{\partial \delta_h} \right] \times \right.}{\left[ (f_i - 0.5)D + \delta_{3z+j} \right]^2}, \quad (9)$$

$$a_{(j+7z)h} = \frac{\left\{ \left( M_{soj} \frac{\partial \delta_j}{\partial \delta_h} - M_{Roj} \frac{\partial \delta_{2z+j}}{\partial \delta_h} + \delta_j \frac{\partial M_{soj}}{\partial \delta_h} - \delta_{2z+j} \frac{\partial M_{Roj}}{\partial \delta_h} \right) \times \right.}{\left[ (f_o - 0.5)D + \delta_{2z+j} \right]^2} + \frac{\left\{ \left[ M_{sij} \frac{\partial \delta_j}{\partial \delta_h} - M_{Rij} \frac{\partial \delta_{2z+j}}{\partial \delta_h} - (s_{zj} - \delta_j) \frac{\partial M_{sij}}{\partial \delta_h} + (s_{xj} - \delta_{2z+j}) \frac{\partial M_{Rij}}{\partial \delta_h} \right] \times \right.}{\left[ (f_i - 0.5)D + \delta_{3z+j} \right]^2} + \frac{\partial M_{z'j}}{\partial \delta_h}, \quad (10)$$

$$a_{(j+8z)h} = \frac{\partial M_{y'j}}{\partial \delta_h} - \frac{\partial M_{yij}}{\partial \delta_h} - \frac{\partial M_{yoj}}{\partial \delta_h}. \quad (11)$$

The forces  $F_{z'j}$ ,  $F_{xij}$  and  $F_{xoj}$  to be used in (6) and (7) are given by (32), (38) and (39) of the companion paper of this [13] and their differentiation with respect to  $\delta_h$ ,  $h = 1, \dots, 9Z$ , yields

$$\frac{\partial F_{z'j}}{\partial \delta_h} = m\omega^2 \left[ \left( \frac{\omega_{mj}}{\omega} \right)^2 \frac{\partial \delta_j}{\partial \delta_h} + d_{mj} \left( \frac{\omega_{mj}}{\omega} \right) \frac{\partial \left( \frac{\omega_{mj}}{\omega} \right)}{\partial \delta_h} \right], \quad (12)$$

$$\frac{\partial F_{xij}}{\partial \delta_h} = \frac{3\mu K_{ij} \delta_{3z+j}^{1.5}}{2\pi a_{ij} b_{ij}} \int_{-a_{ij}}^{b_{ij}} \sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2}} \sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2} - \frac{y_{ij}^2}{b_{ij}^2}} \cos \gamma_{ij} \frac{\partial \gamma_{ij}}{\partial \delta_h} dy_{ij} dx_{ij} + \frac{3}{2} \frac{F_{xij}}{\delta_{3z+j}} \frac{\partial \delta_{3z+j}}{\partial \delta_h}, \quad (13)$$

$$\frac{\partial F_{xoj}}{\partial \delta_h} = \frac{3\mu K_{oj} \delta_{2z+j}^{1.5}}{2\pi a_{oj} b_{oj}} \int_{-a_{oj}}^{b_{oj}} \sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2}} \sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2} - \frac{y_{oj}^2}{b_{oj}^2}} \cos \gamma_{oj} \frac{\partial \gamma_{oj}}{\partial \delta_h} dy_{oj} dx_{oj} + \frac{3}{2} \frac{F_{xoj}}{\delta_{2z+j}} \frac{\partial \delta_{2z+j}}{\partial \delta_h}. \quad (14)$$

The forces  $F_{yij}$  and  $F_{yoj}$  to be used in (8) are given by (40) and (41) of the companion paper of this [13] and their differentiation with respect to  $\delta_h$ ,  $h = 1, \dots, 9Z$ , yields

$$\frac{\partial F_{yij}}{\partial \delta_h} = -\frac{3\mu K_{ij} \delta_{3z+j}^{1.5}}{2\pi a_{ij} b_{ij}} \int_{-a_{ij}}^{b_{ij}} \sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2}} \sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2} - \frac{y_{ij}^2}{b_{ij}^2}} \sin \gamma_{ij} \frac{\partial \gamma_{ij}}{\partial \delta_h} dy_{ij} dx_{ij} + \frac{3}{2} \frac{F_{yij}}{\delta_{3z+j}} \frac{\partial \delta_{3z+j}}{\partial \delta_h}, \quad (15)$$

$$\frac{\partial F_{yoj}}{\partial \delta_h} = -\frac{3\mu K_{oj} \delta_{2z+j}^{1.5}}{2\pi a_{oj} b_{oj}} \int_{-a_{oj}}^{b_{oj}} \sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2}} \sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2} - \frac{y_{oj}^2}{b_{oj}^2}} \sin \gamma_{oj} \frac{\partial \gamma_{oj}}{\partial \delta_h} dy_{oj} dx_{oj} + \frac{3}{2} \frac{F_{yoj}}{\delta_{2z+j}} \frac{\partial \delta_{2z+j}}{\partial \delta_h}. \quad (16)$$

The moments  $M_{sij}$ ,  $M_{soj}$ ,  $M_{rij}$ ,  $M_{roj}$  and  $M_{z'j}$  to be used in (9) and (10) are given by (48), (49), (52), (53) and (37) of [13] and their differentiation with respect to  $\delta_h$ ,  $h = 1, \dots, 9Z$ , yields

$$\frac{\partial M_{sij}}{\partial \delta_h} = \frac{-3\mu K_{ij} \delta_{3z+j}^{1.5}}{2\pi a_{ij} b_{ij}} \int_{-a_{ij}}^{b_{ij}} \sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2}} \sqrt{x_{ij}^2 + y_{ij}^2} \sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2} - \frac{y_{ij}^2}{b_{ij}^2}} \sin \left( \gamma_{ij} - \tan^{-1} \frac{y_{ij}}{x_{ij}} \right) \frac{\partial \gamma_{ij}}{\partial \delta_h} dy_{ij} dx_{ij} + \frac{3}{2} \frac{M_{sij}}{\delta_{3z+j}} \frac{\partial \delta_{3z+j}}{\partial \delta_h}, \quad (17)$$

$$\frac{\partial M_{soj}}{\partial \delta_h} = \frac{-3\mu K_{oj} \delta_{2z+j}^{1.5}}{2\pi a_{oj} b_{oj}} \int_{-a_{oj}}^{b_{oj}} \sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2}} \sqrt{x_{oj}^2 + y_{oj}^2} \sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2} - \frac{y_{oj}^2}{b_{oj}^2}} \sin \left( \gamma_{oj} - \tan^{-1} \frac{y_{oj}}{x_{oj}} \right) \frac{\partial \gamma_{oj}}{\partial \delta_h} dy_{oj} dx_{oj} + \frac{3}{2} \frac{M_{soj}}{\delta_{2z+j}} \frac{\partial \delta_{2z+j}}{\partial \delta_h}, \quad (18)$$

$$\frac{\partial M_{rij}}{\partial \delta_h} = \frac{-3\mu K_{ij} \delta_{3z+j}^{1.5}}{2\pi a_{ij} b_{ij}} \int_{-a_{ij}}^{b_{ij}} \sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2}} \left( \sqrt{R_i^2 - x_{ij}^2} - \sqrt{R_i^2 - a_{ij}^2} + \sqrt{\left( \frac{D}{2} \right)^2 - a_{ij}^2} \right) \sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2} - \frac{y_{ij}^2}{b_{ij}^2}} \sin \gamma_{ij} \frac{\partial \gamma_{ij}}{\partial \delta_h} dy_{ij} dx_{ij} + \frac{3}{2} \frac{M_{rij}}{\delta_{3z+j}} \frac{\partial \delta_{3z+j}}{\partial \delta_h}, \quad (19)$$

$$\frac{\partial M_{Roj}}{\partial \delta_h} = \frac{-3\mu K_{oj}\delta_{2Z+j}^{1.5}}{2\pi a_{oj}b_{oj}} \int_{-a_{oj}}^{a_{oj}} \int_{-b_{oj}}^{b_{oj}} \sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2}} \left( \sqrt{R_0^2 - x_{oj}^2} - \sqrt{R_0^2 - a_{oj}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{oj}^2} \right) \sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2} - \frac{y_{oj}^2}{b_{oj}^2}} \sin \gamma_{oj} \frac{\partial \gamma_{oj}}{\partial \delta_h} dy_{oj} dx_{oj} + \frac{3}{2} \frac{M_{Roj}}{\delta_{2Z+j}} \frac{\partial \delta_{2Z+j}}{\partial \delta_h}, \quad (20)$$

$$\frac{\partial M_{z'j}}{\partial \delta_h} = -J\omega^2 \left\{ \frac{\delta_{7Z+j}}{\sqrt{\delta_{6Z+j}^2 + \delta_{7Z+j}^2 + \delta_{8Z+j}^2}} \left[ \left( \frac{\omega_{Rj}}{\omega} \right) \frac{\partial \left( \frac{\omega_{mj}}{\omega} \right)}{\partial \delta_h} + \frac{\partial \left( \frac{\omega_{Rj}}{\omega} \right)}{\partial \delta_h} \left( \frac{\omega_{mj}}{\omega} \right) \right] - \frac{\delta_{6Z+j}\delta_{7Z+j} \frac{\partial \delta_{6Z+j}}{\partial \delta_h} - (\delta_{6Z+j}^2 + \delta_{7Z+j}^2) \frac{\partial \delta_{7Z+j}}{\partial \delta_h} + \delta_{7Z+j}\delta_{8Z+j} \frac{\partial \delta_{8Z+j}}{\partial \delta_h}}{(\delta_{6Z+j}^2 + \delta_{7Z+j}^2 + \delta_{8Z+j}^2)^{\frac{3}{2}}} \left( \frac{\omega_{Rj}}{\omega} \right) \left( \frac{\omega_{mj}}{\omega} \right) \right\}. \quad (21)$$

The moments  $M_{yij}$ ,  $M_{yoy}$  and  $M_{y'j}$  to be used in (11) are given by (50), (51) and (36) of the companion paper of this [13] and their differentiation with respect to  $\delta_h$ ,  $h = 1, \dots, 9Z$ , yields

$$\frac{\partial M_{yij}}{\partial \delta_h} = \frac{3\mu K_{ij}\delta_{3Z+j}^{1.5}}{2\pi a_{ij}b_{ij}} \int_{-a_{ij}}^{a_{ij}} \int_{-b_{ij}}^{b_{ij}} \sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2}} \left( \sqrt{R_i^2 - x_{ij}^2} - \sqrt{R_i^2 - a_{ij}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{ij}^2} \right) \sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2} - \frac{y_{ij}^2}{b_{ij}^2}} \cos \gamma_{ij} \frac{\partial \gamma_{ij}}{\partial \delta_h} dy_{ij} dx_{ij} + \frac{3}{2} \frac{M_{yij}}{\delta_{3Z+j}} \frac{\partial \delta_{3Z+j}}{\partial \delta_h}, \quad (22)$$

$$\frac{\partial M_{yoy}}{\partial \delta_h} = \frac{3\mu K_{oj}\delta_{2Z+j}^{1.5}}{2\pi a_{oj}b_{oj}} \int_{-a_{oj}}^{a_{oj}} \int_{-b_{oj}}^{b_{oj}} \sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2}} \left( \sqrt{R_0^2 - x_{oj}^2} - \sqrt{R_0^2 - a_{oj}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{oj}^2} \right) \sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2} - \frac{y_{oj}^2}{b_{oj}^2}} \cos \gamma_{oj} \frac{\partial \gamma_{oj}}{\partial \delta_h} dy_{oj} dx_{oj} + \frac{3}{2} \frac{M_{yoy}}{\delta_{2Z+j}} \frac{\partial \delta_{2Z+j}}{\partial \delta_h}, \quad (23)$$

$$\frac{\partial M_{y'j}}{\partial \delta_h} = J\omega^2 \left\{ \frac{\delta_{8Z+j}}{\sqrt{\delta_{6Z+j}^2 + \delta_{7Z+j}^2 + \delta_{8Z+j}^2}} \left[ \left( \frac{\omega_{Rj}}{\omega} \right) \frac{\partial \left( \frac{\omega_{mj}}{\omega} \right)}{\partial \delta_h} + \frac{\partial \left( \frac{\omega_{Rj}}{\omega} \right)}{\partial \delta_h} \left( \frac{\omega_{mj}}{\omega} \right) \right] - \frac{\delta_{6Z+j}\delta_{8Z+j} \frac{\partial \delta_{6Z+j}}{\partial \delta_h} + \delta_{7Z+j}\delta_{8Z+j} \frac{\partial \delta_{7Z+j}}{\partial \delta_h} - (\delta_{6Z+j}^2 + \delta_{7Z+j}^2) \frac{\partial \delta_{8Z+j}}{\partial \delta_h}}{(\delta_{6Z+j}^2 + \delta_{7Z+j}^2 + \delta_{8Z+j}^2)^{\frac{3}{2}}} \left( \frac{\omega_{Rj}}{\omega} \right) \left( \frac{\omega_{mj}}{\omega} \right) \right\}. \quad (24)$$

As in (38)-(41) and (48)-(53) of [13], also in (13)-(20), (22)-(23)  $\gamma_{ij}$ ,  $\gamma_{oj}$  are given by (42) of [13]. The derivatives of  $\gamma_{ij}$ ,  $\gamma_{oj}$  with respect  $\delta_h$ ,  $h = 1, \dots, 9Z$ , to be used in (13)-(20), (22)-(23) are given by

$$\frac{\partial \gamma_{ij}}{\partial \delta_h} = \frac{-\left(x_{ij} + \frac{V_{xij}}{\omega_{sij}}\right) \frac{\partial \left(\frac{V_{xij}}{\omega_{sij}}\right)}{\partial \delta_h} - \left(y_{ij} - \frac{V_{yij}}{\omega_{sij}}\right) \frac{\partial \left(\frac{V_{yij}}{\omega_{sij}}\right)}{\partial \delta_h}}{\left(y_{ij} - \frac{V_{yij}}{\omega_{sij}}\right)^2 + \left(x_{ij} + \frac{V_{xij}}{\omega_{sij}}\right)^2}, \quad \frac{\partial \gamma_{oj}}{\partial \delta_h} = \frac{-\left(x_{oj} + \frac{V_{xoj}}{\omega_{soj}}\right) \frac{\partial \left(\frac{V_{xoj}}{\omega_{soj}}\right)}{\partial \delta_h} - \left(y_{oj} - \frac{V_{yoy}}{\omega_{soj}}\right) \frac{\partial \left(\frac{V_{yoy}}{\omega_{soj}}\right)}{\partial \delta_h}}{\left(y_{oj} - \frac{V_{yoy}}{\omega_{soj}}\right)^2 + \left(x_{oj} + \frac{V_{xoj}}{\omega_{soj}}\right)^2}, \quad (25)$$

in which  $V_{xij}/\omega_{sij}$ ,  $V_{yij}/\omega_{sij}$ ,  $V_{xoj}/\omega_{soj}$ ,  $V_{yoy}/\omega_{soj}$  are given by (43)-(46) of [13]. The derivatives of  $V_{xij}/\omega_{sij}$ ,  $V_{yij}/\omega_{sij}$ ,  $V_{xoj}/\omega_{soj}$  and  $V_{yoy}/\omega_{soj}$  with respect  $\delta_h$ ,  $h = 1, \dots, 9Z$ , to be used in (25) are given by

$$\frac{\partial \left(\frac{V_{xij}}{\omega_{sij}}\right)}{\partial \delta_h} = \left( \sqrt{R_i^2 - x_{ij}^2} - \sqrt{R_i^2 - a_{ij}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{ij}^2} \right) \times \left\{ \left( \frac{\delta_{5Z+j}}{d_{mj}} \left[ \left( 1 + \frac{s_{zj} - \delta_j}{d_{mj}} \right) \frac{\partial \delta_j}{\partial \delta_h} - c\psi_j \frac{\partial \delta_{9Z+3}}{\partial \delta_h} + \mathcal{R}_i(c\psi_j | s\delta_{9Z+3}) \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right] + \frac{\partial \delta_{9Z+j}}{\partial \delta_h} \frac{s_{zj} - \delta_j}{d_{mj}} \frac{\partial \delta_{5Z+j}}{\partial \delta_h} \right) \delta_{7Z+j} + \left[ (f_i - 0.5)D + \delta_{3Z+j} - \frac{s_{zj} - \delta_j}{d_{mj}} \delta_{3Z+j} \right] \frac{\partial \delta_{7Z+j}}{\partial \delta_h} \right\} \left\{ \delta_{6Z+j}(s_{xj} - \delta_{2Z+j}) - \delta_{8Z+j} \left[ s_{zj} - \delta_j - \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} \delta_{5Z+j} \right] \right\} - \left[ (f_i - 0.5)D + \delta_{3Z+j} - \frac{s_{zj} - \delta_j}{d_{mj}} \delta_{3Z+j} \right] \delta_{7Z+j} \left\{ \delta_{6Z+j} \left( \frac{\partial \delta_{9Z+3}}{\partial \delta_h} + \mathcal{R}_i(c\psi_j | s\delta_{9Z+3}) \frac{\partial \delta_{9Z+3}}{\partial \delta_h} - \frac{\partial \delta_{9Z+j}}{\partial \delta_h} \right) + (s_{xj} - \delta_{2Z+j}) \frac{\partial \delta_{6Z+j}}{\partial \delta_h} - \delta_{8Z+j} \left[ s_{zj} - \delta_j - \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} \delta_{5Z+j} \right] \right\} - \left[ \frac{\delta_{5Z+j}}{d_{mj}} \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} - 1 \right] \frac{\partial \delta_j}{\partial \delta_h} \frac{\delta_{5Z+j}}{d_{mj}} \frac{\partial \delta_{3Z+j}}{\partial \delta_h} \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} \frac{\delta_{5Z+j}}{d_{mj}} \left\{ s_{zj} - \delta_j - \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} \delta_{5Z+j} \right\} \frac{\partial \delta_{8Z+j}}{\partial \delta_h} \right\} \left\{ \delta_{6Z+j}(s_{xj} - \delta_{2Z+j}) - \delta_{8Z+j} \left[ s_{zj} - \delta_j - \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} \delta_{5Z+j} \right] \right\}^2, \quad (26)$$

$$\frac{\partial \left(\frac{V_{yij}}{\omega_{sij}}\right)}{\partial \delta_h} = \left\{ \left( \sqrt{R_i^2 - x_{ij}^2} - \sqrt{R_i^2 - a_{ij}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{ij}^2} \right) \left[ \delta_{6Z+j} \left( c\psi_j \frac{\partial \delta_{9Z+3}}{\partial \delta_h} - \mathcal{R}_i(c\psi_j | s\delta_{9Z+3}) \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right) + (s_{zj} - \delta_j) \frac{\partial \delta_{6Z+j}}{\partial \delta_h} + \delta_{8Z+j} \left( \frac{\partial \delta_{9Z+3}}{\partial \delta_h} + \mathcal{R}_i(c\psi_j | s\delta_{9Z+3}) \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right) \right] \right. \\ \left. + \frac{\partial \delta_{9Z+j}}{\partial \delta_h} \frac{\partial \delta_{3Z+j}}{\partial \delta_h} + (s_{xj} - \delta_{2Z+j}) \frac{\partial \delta_{8Z+j}}{\partial \delta_h} - [\delta_{6Z+j}(s_{zj} - \delta_j) + \delta_{8Z+j}(s_{xj} - \delta_{2Z+j})] \frac{\partial \delta_{5Z+j}}{\partial \delta_h} \right\} \left\{ \delta_{6Z+j}(s_{xj} - \delta_{2Z+j}) - \delta_{8Z+j} \left[ s_{zj} - \delta_j - \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} \delta_{5Z+j} \right] \right\} \\ - \left( \sqrt{R_i^2 - x_{ij}^2} - \sqrt{R_i^2 - a_{ij}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{ij}^2} \right) \left[ \delta_{6Z+j}(s_{zj} - \delta_j) + \delta_{8Z+j}(s_{xj} - \delta_{2Z+j}) \right] \left\{ \delta_{6Z+j} \left( \frac{\partial \delta_{9Z+3}}{\partial \delta_h} + \mathcal{R}_i(c\psi_j | s\delta_{9Z+3}) \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right) + (s_{xj} - \delta_{2Z+j}) \frac{\partial \delta_{6Z+j}}{\partial \delta_h} - \delta_{8Z+j} \left[ s_{zj} - \delta_j - \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} \delta_{5Z+j} \right] \right\} \\ \times \left\{ c\psi_j \frac{\partial \delta_{9Z+3}}{\partial \delta_h} - \mathcal{R}_i(c\psi_j | s\delta_{9Z+3}) \frac{\partial \delta_{9Z+3}}{\partial \delta_h} + \left[ \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} \frac{\delta_{5Z+j}}{d_{mj}} - 1 \right] \frac{\partial \delta_j}{\partial \delta_h} \frac{\delta_{5Z+j}}{d_{mj}} \frac{\partial \delta_{3Z+j}}{\partial \delta_h} \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} \frac{\delta_{5Z+j}}{d_{mj}} \right\} - \left[ s_{zj} - \delta_j - \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} \delta_{5Z+j} \right] \frac{\partial \delta_{8Z+j}}{\partial \delta_h} \right\} \\ \left\{ \delta_{6Z+j}(s_{xj} - \delta_{2Z+j}) - \delta_{8Z+j} \left[ s_{zj} - \delta_j - \frac{(f_i - 0.5)D + \delta_{3Z+j}}{d_{mj}} \delta_{5Z+j} \right] \right\}^2, \quad (27)$$

(28)

(29)

For the outer race to be stationary  $\omega_{mj}/\omega$  and  $\omega_{Rj}/\omega$  are given by (33)-(34) of [13]. The derivatives of (33) and (34) of [13] with respect  $\delta_h$ ,  $h = 1, \dots, 9Z$ , to be used in (12), (21) and (24) are given by

(30)

(31)

Likewise, for the inner race to be stationary,

(32)



and  $\partial(\omega_{Rj}/\omega)/\partial\delta$  is given by (31) with the opposite sign.

The last three equations from (1) must be solved simultaneously for  $\delta_{9Z+1}, \dots, \delta_{9Z+3}$  after obtaining updated values for:  $\beta_{ij}, \beta_{oj}, K_{ij}, K_{oj}, s_{xj}, s_{zj}, F_{xij}, F_{yij}, F_{xoj}, F_{yij}, M_{sij}, M_{soj}, M_{Rij}, M_{Roj}, \delta_{kZ+j}, k = 0, \dots, 8; j = 1, \dots, Z$ . If  $\delta_h^0, h = 9Z+1, \dots, 9Z+3$ , is a 3-dimensional vector with the initial estimates of the variables  $\delta_{9Z+1}, \dots, \delta_{9Z+3}$ , in that order, improved values are given by (2), in which  $\{\epsilon_g\}, g = 9Z+1, \dots, 9Z+3$ , is the 3-dimensional vector with the errors functions, in that order, from (1). The elements of the  $3 \times 3$ -matrix  $[a_{gh}]$  are

$$a_{(9Z+1)h} = \frac{\partial F_a}{\partial \delta_h} - \sum_{j=1}^Z \frac{\left\{ \begin{aligned} &[(f_i-0.5)D+\delta_{3Z+j}] \left[ K_{ij} \delta_{3Z+j}^{1.5} \left( \frac{\partial \delta_{9Z+1}}{\partial \delta_h} + \mathcal{R}_i c \psi_j c \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} - \frac{\partial \delta_{Z+j}}{\partial \delta_h} \right) + 1.5 K_{ij} \delta_{3Z+j}^{0.5} (s_{xj}-\delta_{Z+j}) \frac{\partial \delta_{3Z+j}}{\partial \delta_h} \right. \\ &\left. F_{xij} \left( c \psi_j \frac{\partial \delta_{9Z+2}}{\partial \delta_h} - \mathcal{R}_i |c \psi_j| s \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} - \frac{\partial \delta_j}{\partial \delta_h} \right) - (s_{zj}-\delta_j) \frac{\partial F_{xij}}{\partial \delta_h} \right] - [K_{ij} \delta_{3Z+j}^{1.5} (s_{xj}-\delta_{Z+j}) - F_{xij} (s_{zj}-\delta_j)] \frac{\partial \delta_{3Z+j}}{\partial \delta_h} \end{aligned} \right\}}{[(f_i-0.5)D+\delta_{3Z+j}]^2}, \quad (33)$$

$$a_{(9Z+2)h} = \frac{\partial F_r}{\partial \delta_h} - \sum_{j=1}^Z \frac{\left\{ \begin{aligned} &[(f_i-0.5)D+\delta_{3Z+j}] \left[ K_{ij} \delta_{3Z+j}^{1.5} \left( c \psi_j \frac{\partial \delta_{9Z+2}}{\partial \delta_h} - \mathcal{R}_i |c \psi_j| s \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} - \frac{\partial \delta_j}{\partial \delta_h} \right) + 1.5 K_{ij} \delta_{3Z+j}^{0.5} (s_{xj}-\delta_j) \frac{\partial \delta_{3Z+j}}{\partial \delta_h} \right. \\ &\left. F_{xij} \left( \frac{\partial \delta_{9Z+1}}{\partial \delta_h} + \mathcal{R}_i c \psi_j c \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} - \frac{\partial \delta_{Z+j}}{\partial \delta_h} \right) + (s_{xj}-\delta_{Z+j}) \frac{\partial F_{xij}}{\partial \delta_h} \right] - [K_{ij} \delta_{3Z+j}^{1.5} (s_{zj}-\delta_j) + F_{xij} (s_{xj}-\delta_{Z+j})] \frac{\partial \delta_{3Z+j}}{\partial \delta_h} \end{aligned} \right\}}{[(f_i-0.5)D+\delta_{3Z+j}]^2} c \psi_j, \quad (34)$$

$$a_{(9Z+3)h} = \frac{\partial M}{\partial \delta_h} - \sum_{j=1}^Z \left\{ \mathcal{R}_i \left( K_{ij} \delta_{3Z+j}^{1.5} c \beta_{ij} + F_{xij} s \beta_{ij} \right) \left[ \frac{(s_{zj}-\delta_j) \left( \frac{\partial \delta_{9Z+1}}{\partial \delta_h} + \mathcal{R}_i c \psi_j c \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} - \frac{\partial \delta_{Z+j}}{\partial \delta_h} \right) - (s_{xj}-\delta_{Z+j}) \left( c \psi_j \frac{\partial \delta_{9Z+2}}{\partial \delta_h} - \mathcal{R}_i |c \psi_j| s \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} - \frac{\partial \delta_j}{\partial \delta_h} \right)}{(s_{zj}-\delta_j)^2 + (s_{xj}-\delta_{Z+j})^2} \right. \right. \\ \left. \left. - c \psi_j \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right] + 1.5 K_{ij} \delta_{3Z+j}^{0.5} s \beta_{ij} \frac{\partial \delta_{3Z+j}}{\partial \delta_h} - (c \beta_{ij} - r_i / \mathcal{R}_i) \frac{\partial F_{xij}}{\partial \delta_h} \right\} c \psi_j - \left\{ (F_{yij} r_i c \beta_{ij} + M_{sij} s \beta_{ij}) \left[ \frac{(s_{zj}-\delta_j) \left( \frac{\partial \delta_{9Z+1}}{\partial \delta_h} + \mathcal{R}_i c \psi_j c \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} - \frac{\partial \delta_{Z+j}}{\partial \delta_h} \right) - (s_{xj}-\delta_{Z+j}) \left( c \psi_j \frac{\partial \delta_{9Z+2}}{\partial \delta_h} - \mathcal{R}_i |c \psi_j| s \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} - \frac{\partial \delta_j}{\partial \delta_h} \right)}{(s_{zj}-\delta_j)^2 + (s_{xj}-\delta_{Z+j})^2} \right. \right. \\ \left. \left. - c \psi_j \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right] + 1.5 K_{ij} \delta_{3Z+j}^{0.5} s \beta_{ij} \frac{\partial \delta_{3Z+j}}{\partial \delta_h} - c \psi_j \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right\} + r_i s \beta_{ij} \frac{\partial F_{yij}}{\partial \delta_h} - c \beta_{ij} \frac{\partial M_{sij}}{\partial \delta_h} \left\{ s \psi_j \right\} \quad (35)$$

Differentiating (16)-(18), (22)-(24) and (27)-(29) of [13] with respect  $\delta_h, h = 9Z+1, \dots, 9Z+3, 27Z$  simultaneous linear equations in  $\partial \delta_{kZ+j} / \partial \delta_h, k = 0, \dots, 8; j = 1, \dots, Z$ , results, which are

$$(s_{zj}-\delta_j) \frac{\partial \delta_j}{\partial \delta_h} + (s_{xj}-\delta_{Z+j}) \frac{\partial \delta_{Z+j}}{\partial \delta_h} + [(f_i-0.5)D+\delta_{3Z+j}] \frac{\partial \delta_{3Z+j}}{\partial \delta_h} = (s_{xj}-\delta_{Z+j}) \left( \frac{\partial \delta_{9Z+1}}{\partial \delta_h} + \mathcal{R}_i c \psi_j c \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right) + (s_{zj}-\delta_j) \left( c \psi_j \frac{\partial \delta_{9Z+2}}{\partial \delta_h} - \mathcal{R}_i |c \psi_j| s \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right), \quad (36)$$

$$\delta_j \frac{\partial \delta_j}{\partial \delta_h} + \delta_{Z+j} \frac{\partial \delta_{Z+j}}{\partial \delta_h} - [(f_o-0.5)D+\delta_{2Z+j}] \frac{\partial \delta_{2Z+j}}{\partial \delta_h} = 0, \quad (37)$$

$$\delta_{6Z+j} \frac{\partial \delta_{6Z+j}}{\partial \delta_h} + \delta_{7Z+j} \frac{\partial \delta_{7Z+j}}{\partial \delta_h} + \delta_{8Z+j} \frac{\partial \delta_{8Z+j}}{\partial \delta_h} = 0, \quad (38)$$

$$\left[ \frac{F_{xoj}}{(f_o-0.5)D+\delta_{2Z+j}} + \frac{F_{xij}}{(f_i-0.5)D+\delta_{3Z+j}} \right] \frac{\partial \delta_j}{\partial \delta_h} + \frac{\delta_j}{(f_o-0.5)D+\delta_{2Z+j}} \frac{\partial F_{xoj}}{\partial \delta_h} - \frac{(s_{xj}-\delta_{Z+j})}{(f_i-0.5)D+\delta_{3Z+j}} \frac{\partial F_{xij}}{\partial \delta_h} - \left[ \frac{K_{oj} \delta_{2Z+j}^{1.5}}{(f_o-0.5)D+\delta_{2Z+j}} + \frac{K_{ij} \delta_{3Z+j}^{1.5}}{(f_i-0.5)D+\delta_{3Z+j}} \right] \frac{\partial \delta_{Z+j}}{\partial \delta_h} - \left\{ \frac{1.5 K_{oj} \delta_{2Z+j}^{0.5} \delta_{Z+j} [(f_o-0.5)D+\delta_{2Z+j}] + F_{xoj} \delta_j - K_{oj} \delta_{2Z+j}^{1.5} \delta_{Z+j}}{[(f_o-0.5)D+\delta_{2Z+j}]^2} \right\} \frac{\partial \delta_{2Z+j}}{\partial \delta_h} + \left\{ \frac{1.5 K_{ij} \delta_{3Z+j}^{0.5} (s_{xj}-\delta_{Z+j}) [(f_i-0.5)D+\delta_{3Z+j}] + F_{xij} (s_{zj}-\delta_j) - K_{ij} \delta_{3Z+j}^{1.5} (s_{xj}-\delta_{Z+j})}{[(f_i-0.5)D+\delta_{3Z+j}]^2} \right\} \frac{\partial \delta_{3Z+j}}{\partial \delta_h} = \frac{-K_{ij} \delta_{3Z+j}^{1.5} \left( \frac{\partial \delta_{9Z+1}}{\partial \delta_h} + \mathcal{R}_i c \psi_j c \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right) + F_{xij} \left( c \psi_j \frac{\partial \delta_{9Z+2}}{\partial \delta_h} - \mathcal{R}_i |c \psi_j| s \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right)}{(f_i-0.5)D+\delta_{3Z+j}}, \quad (39)$$

$$\left[ \frac{K_{oj} \delta_{2Z+j}^{1.5}}{(f_o-0.5)D+\delta_{2Z+j}} + \frac{K_{ij} \delta_{3Z+j}^{1.5}}{(f_i-0.5)D+\delta_{3Z+j}} \right] \frac{\partial \delta_j}{\partial \delta_h} + \frac{\delta_{Z+j}}{(f_o-0.5)D+\delta_{2Z+j}} \frac{\partial F_{xoj}}{\partial \delta_h} - \frac{(s_{xj}-\delta_{Z+j})}{(f_i-0.5)D+\delta_{3Z+j}} \frac{\partial F_{xij}}{\partial \delta_h} + \left[ \frac{F_{xoj}}{(f_o-0.5)D+\delta_{2Z+j}} + \frac{F_{xij}}{(f_i-0.5)D+\delta_{3Z+j}} \right] \frac{\partial \delta_{Z+j}}{\partial \delta_h} + \left\{ \frac{1.5 K_{oj} \delta_{2Z+j}^{0.5} \delta_j [(f_o-0.5)D+\delta_{2Z+j}] - K_{oj} \delta_{2Z+j}^{1.5} \delta_j - F_{xoj} \delta_{Z+j}}{[(f_o-0.5)D+\delta_{2Z+j}]^2} \right\} \frac{\partial \delta_{2Z+j}}{\partial \delta_h} - \left\{ \frac{1.5 K_{ij} \delta_{3Z+j}^{0.5} (s_{zj}-\delta_j) [(f_i-0.5)D+\delta_{3Z+j}] - K_{ij} \delta_{3Z+j}^{1.5} (s_{zj}-\delta_j) - F_{xij} (s_{xj}-\delta_{Z+j})}{[(f_i-0.5)D+\delta_{3Z+j}]^2} \right\} \frac{\partial \delta_{3Z+j}}{\partial \delta_h} - \frac{\partial F_{xij}}{\partial \delta_h} = \frac{K_{ij} \delta_{3Z+j}^{1.5} \left( c \psi_j \frac{\partial \delta_{9Z+2}}{\partial \delta_h} - \mathcal{R}_i |c \psi_j| s \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right) + F_{xij} \left( \frac{\partial \delta_{9Z+1}}{\partial \delta_h} + \mathcal{R}_i c \psi_j c \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right)}{(f_i-0.5)D+\delta_{3Z+j}}, \quad (40)$$

$$\frac{\partial F_{yoj}}{\partial \delta_h} + \frac{\partial F_{yij}}{\partial \delta_h} = 0, \quad (41)$$

$$\left[ \frac{M_{Roj}}{(f_o-0.5)D+\delta_{2Z+j}} + \frac{M_{Rij}}{(f_i-0.5)D+\delta_{3Z+j}} \right] \frac{\partial \delta_j}{\partial \delta_h} + \frac{\delta_j}{(f_o-0.5)D+\delta_{2Z+j}} \frac{\partial M_{Roj}}{\partial \delta_h} + \frac{\delta_{Z+j}}{(f_o-0.5)D+\delta_{2Z+j}} \frac{\partial M_{soj}}{\partial \delta_h} + \left[ \frac{M_{soj}}{(f_o-0.5)D+\delta_{2Z+j}} + \frac{M_{sij}}{(f_i-0.5)D+\delta_{3Z+j}} \right] \frac{\partial \delta_{Z+j}}{\partial \delta_h} - \frac{(s_{xj}-\delta_{Z+j})}{(f_i-0.5)D+\delta_{3Z+j}} \frac{\partial M_{sij}}{\partial \delta_h} - \frac{(s_{zj}-\delta_j)}{(f_i-0.5)D+\delta_{3Z+j}} \frac{\partial M_{Rij}}{\partial \delta_h} - \frac{M_{Roj} \delta_j + M_{soj} \delta_{Z+j}}{[(f_o-0.5)D+\delta_{2Z+j}]^2} \frac{\partial \delta_{2Z+j}}{\partial \delta_h} + \frac{M_{sij} (s_{xj}-\delta_{Z+j}) + M_{Rij} (s_{zj}-\delta_j)}{[(f_i-0.5)D+\delta_{3Z+j}]^2} \frac{\partial \delta_{3Z+j}}{\partial \delta_h} = \frac{M_{sij} \left( \frac{\partial \delta_{9Z+1}}{\partial \delta_h} + \mathcal{R}_i c \psi_j c \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right) + M_{Rij} \left( c \psi_j \frac{\partial \delta_{9Z+2}}{\partial \delta_h} - \mathcal{R}_i |c \psi_j| s \delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right)}{(f_i-0.5)D+\delta_{3Z+j}}, \quad (42)$$

$$\left[ \frac{M_{soj}}{(f_o-0.5)D+\delta_{2Z+j}} + \frac{M_{sij}}{(f_i-0.5)D+\delta_{3Z+j}} \right] \frac{\partial \delta_j}{\partial \delta_h} + \frac{\delta_j}{(f_o-0.5)D+\delta_{2Z+j}} \frac{\partial M_{soj}}{\partial \delta_h} - \frac{\delta_{Z+j}}{(f_o-0.5)D+\delta_{2Z+j}} \frac{\partial M_{roj}}{\partial \delta_h} -$$

$$\left[ \frac{M_{roj}}{(f_o-0.5)D+\delta_{2Z+j}} + \frac{M_{rij}}{(f_i-0.5)D+\delta_{3Z+j}} \right] \frac{\partial \delta_{Z+j}}{\partial \delta_h} + \frac{(s_{xj}-\delta_{Z+j})}{(f_i-0.5)D+\delta_{3Z+j}} \frac{\partial M_{rij}}{\partial \delta_h} - \frac{(s_{Zj}-\delta_j)}{(f_i-0.5)D+\delta_{3Z+j}} \frac{\partial M_{sij}}{\partial \delta_h} - \frac{M_{soj}\delta_j - M_{roj}\delta_{Z+j}}{[(f_o-0.5)D+\delta_{2Z+j}]^2} \frac{\partial \delta_{2Z+j}}{\partial \delta_h} +$$

$$\frac{M_{sij}(s_{Zj}-\delta_j) - M_{rij}(s_{xj}-\delta_{Z+j})}{[(f_i-0.5)D+\delta_{3Z+j}]^2} \frac{\partial \delta_{3Z+j}}{\partial \delta_h} + \frac{\partial M_{zrj}}{\partial \delta_h} = \frac{M_{sij} \left( c\psi_j \frac{\partial \delta_{9Z+2}}{\partial \delta_h} - \mathcal{R}_i c\psi_j s\delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right) - M_{rij} \left( \frac{\partial \delta_{9Z+1}}{\partial \delta_h} + \mathcal{R}_i c\psi_j c\delta_{9Z+3} \frac{\partial \delta_{9Z+3}}{\partial \delta_h} \right)}{(f_i-0.5)D+\delta_{3Z+j}}, \quad (43)$$

$$\frac{\partial M_{y'j}}{\partial \delta_h} - \frac{\partial M_{yij}}{\partial \delta_h} - \frac{\partial M_{yoj}}{\partial \delta_h} = 0. \quad (44)$$

The derivatives of  $F_{z'j}$ ,  $F_{xij}$ ,  $F_{xoj}$ ,  $F_{yij}$ ,  $F_{yoj}$ ,  $M_{sij}$ ,  $M_{soj}$ ,  $M_{rij}$ ,  $M_{roj}$ ,  $M_{z'j}$ ,  $M_{yij}$ ,  $M_{yoj}$  and  $M_{y'j}$  with respect  $\delta_h$ ,  $h = 9Z+1, \dots, 9Z+3$ , to be used in (33)-(44) are given by (12)-(24). In (13)-(20), (22)-(23)  $\gamma_{ij}$ ,  $\gamma_{oj}$  are given by (42) of [13]. The derivatives of  $\gamma_{ij}$ ,  $\gamma_{oj}$  with respect  $\delta_h$ ,  $h = 9Z+1, \dots, 9Z+3$ , to be used in (13)-(20), (22)-(23) are given by (25), with  $V_{xij}/\omega_{sij}$ ,  $V_{yij}/\omega_{sij}$ ,  $V_{xoj}/\omega_{soj}$  and  $V_{yoj}/\omega_{soj}$  given by (43)-(46) of [13]. The derivatives of  $V_{xij}/\omega_{sij}$ ,  $V_{yij}/\omega_{sij}$ ,  $V_{xoj}/\omega_{soj}$  and  $V_{yoj}/\omega_{soj}$  with respect  $\delta_h$ ,  $h = 9Z+1, \dots, 9Z+3$ , to be used in (25) are given by (26)-(29).

For outer race to be stationary  $\omega_{mj}/\omega$  and  $\omega_{Rj}/\omega$  are given by (33)-(34) of [13] and for inner race to be stationary are given by (35)-(34) of [13], the last with opposite sign. The derivatives of (33)-(34) of [13] with respect  $\delta_h$ ,  $h = 9Z+1, \dots, 9Z+3$ , to be used in (12), (21) and (24) are given by (30)-(31) and for (32) and (31), the last with opposite sign, are given by (32)-(31), the last with opposite sign.

The (36)-(44) linear system's solutions  $-\partial \delta_{kZ+j}/\partial \delta_h$ ,  $k = 0, \dots, 8$ ;  $j = 1, \dots, Z$  – are to be used in (33)-(35) for the new estimates of  $\delta_{9Z+1}$ ,  $\delta_{9Z+2}$  and  $\delta_{9Z+3}$ .

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