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Ball's motion, sliding friction, and internal load distribution in a high-speed ball bearing subjected to a combined radial, thrust, and moment load, applied to the inner ring's center of mass: *Mathematical model*

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Abstract. A set of non-linear algebraic equations, which must to be solved using a numerical procedure, for ball's motion, sliding friction and internal loading distribution computation in a high-speed, single-row, angular-contact ball bearing, subjected to a known combined radial, thrust and moment load, which must be applied to the inner ring's centre of mass, is introduced. For each step of the procedure it is required the iterative solution of $9Z + 3$ simultaneous non-linear equations – where Z is the number of the balls – to yield exact solution for contact angles, ball attitude angles, rolling radii, normal contact deformations and axial, radial, and angular deflections of the inner ring with respect the outer ring. While the focus of this work is obtaining the steady state forces and moments equilibrium conditions on the balls, under the selected loading, the numerical aspects of the procedure are treated in a companion paper. The numerical results derived from the described procedure shall be published later.

1. Introduction

Ball and roller bearings, generically called *rolling bearings*, are commonly used machine elements. They are employed to permit rotary motions of, or about, shafts in simple commercial devices such as bicycles, roller skates, and electric motors. They are also used in complex engineering mechanisms such as aircraft gas turbines, rolling mills, dental drills, gyroscopes, reaction and momentum wheels, and power transmissions.

The standardized forms of ball or roller bearings permit rotary motion between two machine elements and always include a complement of ball or rollers that maintain the shaft and a usually stationary supporting structure, frequently called *housing*, in a radially or axially spaced-apart relationship. Usually, a bearing may be obtained as a unit, which includes two steel rings each of which has a hardened raceway on which hardened balls or rollers roll. The balls or rollers, also called *rolling elements*, are usually held in an angularly spaced relationship by a *cage*, also called a *separator* or *retainer*.

There are many different kinds of rolling bearings. This work is concerned with *single-row angular-contact ball bearings* – see figure 1 – which are designed to support combined radial and thrust loads or heavy thrust loads depending on the *contact angle* magnitude. The bearings having large contact angle can support heavier thrust loads. The figure 1 shows bearings having small and

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large contact angles. The bearings generally have groove curvature radii in the range of 52-53% of the ball diameter. The contact angle does not usually exceed 40^o.

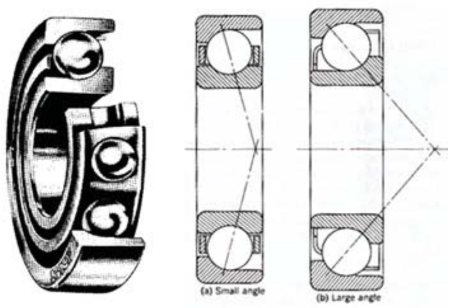

Figure 1. Angular-contact ball bearing.

This work is devoted to study of internal load distribution in a *high-speed* angular-contact ball bearing. Several researchers have studied the subject of internal load distribution in a *statically loaded* angular-contact ball bearing (see [1]-[10]). The methods developed by them to calculate distribution of load among the balls and rollers of rolling bearings can be used in most bearing applications because rotational speeds are usually slow to moderate. Under these speed conditions, the effects of rolling element centrifugal forces and gyroscopic moments are negligible. At high speeds of rotation these body forces become significant, tending to alter contact angles and clearance. Thus, they can affect the static load distribution to a great extension.

Harris [11] described methods for internal loading distribution in statically loaded bearings addressing pure radial; pure thrust (centric and eccentric loads); combined radial and thrust load, which uses radial and thrust integrals introduced by Sjoväll; and for ball bearings under combined radial, thrust, and moment load, initially due to Jones.

The first great contribution to the study of ball motion, sliding friction and internal load distribution in a *high-speed* angular-contact ball bearing must be credited to A B Jones [12]-[13]. Harris describes the orbital, pivotal and spinning ball's motions and load distribution in ball bearings, in general reproducing the Jones's developments. In this work the Jones's works is revisited and differences are introduced under the yoke of critical analysis, which will be detailed. Then, particularly, in this work, a set of non-linear algebraic equations, which must to be solved using a numerical procedure, for ball's motion, sliding friction and internal loading distribution computation in a high-speed, single-row, angular-contact ball bearing, subjected to a known combined radial, thrust and moment load, which must be applied to the inner ring center of mass, is introduced. For each step of the procedure it is required the iterative solution of $9Z + 3$ simultaneous non-linear equations – where Z is the number of the balls – to yield exact solution for contact angles, ball attitude angles, rolling radii, normal contact deformations and axial, radial, and angular deflections of the inner ring with respect the outer ring. While the focus of this work is obtaining the steady state forces and moments equilibrium conditions on the balls, under the selected external loading, the numerical aspects of the procedure are treated in a companion paper. The numerical results derived from the described procedure shall be published later.

2. Mathematical model

Having defined in other works analytical expressions for bearing geometry and the contact stress and deformations for a given ball or roller-raceway contact (point or line loading) in terms of load (see, e.g., [12]) it is possible to consider how the bearing load is distributed among the rolling elements. In this section a specific internal loading distribution resulting from a combined radial, thrust, and moment external load, which must be applied to the center of mass of the inner ring of a high speed ball bearing, is considered.

The figure 2 shows the displacements of an inner ring related to the outer ring due to a generalized loading system including radial, axial, and moment loads. The figure 3 shows the relative angular position of each ball in the bearing.

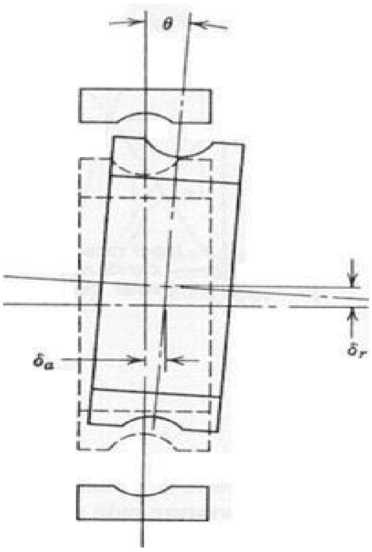

Figure 2. Displacements of an inner ring (outer ring fixed) due to a combined radial, axial, and moment external loading.

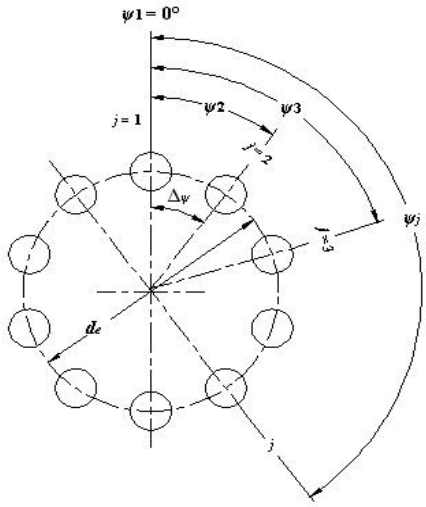

Figure 3. Ball angular positions in the radial plane that is perpendicular to the bearing's axis of rotation, $\Delta\psi = 2\pi/Z$, $\psi_j = 2\pi(j-1)/Z$, $j = 1 \ldots Z$, in which Z is the number of balls.

Let a ball bearing with Z balls, each with diameter D, symmetrically distributed about a pitch circle according to figure 3, to be subjected to a combined radial, thrust, and moment load applied to the inner ring's center of mass. Then, a *relative axial displacement*, δ_a, a *relative angular displacement*, θ, and a *relative radial displacement*, δ_r, between the inner and outer ring raceways may be expected according figure 2. Let $\psi = 0$ to be the angular position of the maximum loaded ball.

Under zero load the centers of raceway groove curvature radii are separated by a distance A given by

$$A = (f_o + f_i - 1)D, \tag{1}$$

in which f_o, f_i are the conformities for outer and inner raceways, respectively.

Under an applied static load, the distance s between centers will increase from A to A plus the amount of the contact deformation δ_i plus δ_o, as show by figure 4. The line of action between centers is collinear with A. If, however, a centrifugal force acts on the ball, then because the inner and outer raceway contact angles are dissimilar, the line of action between raceway groove curvature radii centers is not collinear with A, but is discontinuous as indicated by figure 5. It is assumed in figure 5 that the outer raceway groove curvature center is fixed in space and the inner raceway groove curvature center moves relative to that fixed center. Moreover, the ball center shifts by virtue of the dissimilar contact angles.

The figure 5 when compared with similar figures in [11] and [13] shows minor differences. The inner contact angle must be $\beta_{ij} + \theta\cos\psi_j$ rather than β_{ij}, to take into account the tilting of the rigid inner ring with respect the rigid outer ring, during the external loading application. Furthermore, since the problem is to be solved numerically, no makes sense to linearize the distances between the final and initial inner raceway groove curvature center positions, as done in previous works.

In accordance with figure 5 the distance between the fixed outer raceway groove curvature center and the final position of the ball center at any ball location j is

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$$\Delta_{oj} = r_o - \frac{D}{2} + \delta_{oj}. \quad (2)$$

Since $r_o = f_o D$,

$$\Delta_{oj} = (f_o - 0.5)D + \delta_{oj}. \quad (3)$$

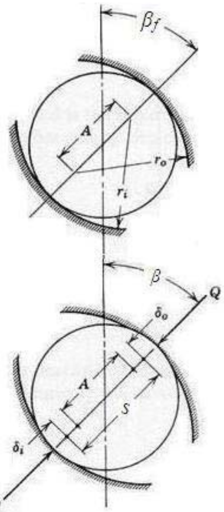

Figure 4. (a) Ball-raceway contact before loading; (b) Ball-raceway contact under load.

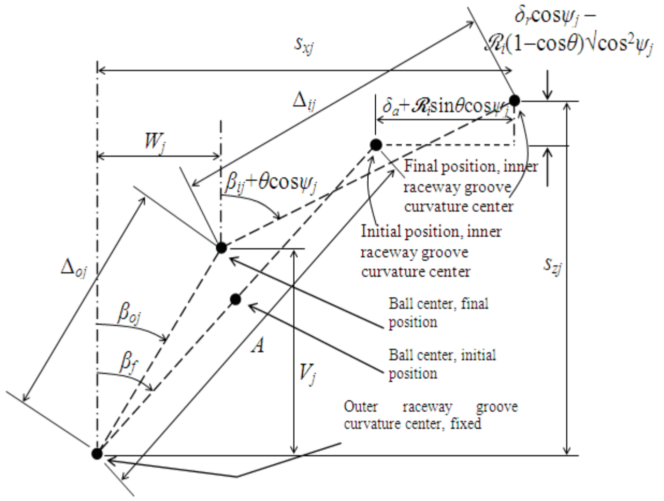

Figure 5. Positions of ball center and raceway groove curvature centers at angular position ψ_j with and without applied load.

Similarly, the distance between the moving inner raceway groove curvature center and the final position of the ball center at any ball location j is

$$\Delta_{ij} = (f_i - 0.5)D + \delta_{ij}, \quad (4)$$

in which δ_{oj} and δ_{ij} are the normal contact deformations at the outer and inner raceway contacts, respectively.

In accordance with the relative axial displacement between inner and outer rings mass centers, δ_a, and the relative angular displacement θ, the axial distance between inner and outer raceway groove curvature centers at ball position j is

$$s_{xj} = A\sin\beta_f + \delta_a + \mathscr{R}_i\sin\theta\cos\psi_j, \quad (5)$$

in which

$$\mathscr{R}_i = \tfrac{1}{2}d_e + (f_i - \tfrac{1}{2})D\cos\beta_f \quad (6)$$

is the radius to locus of inner raceway groove curvature centers, d_e is the unloaded pitch diameter, and β_f is the unloaded contact angle. Further, in accordance with the relative radial displacement between inner and outer rings mass centers, δ_r, and the relative angular displacement θ, the radial distance between inner and outer groove curvature centers at each ball location j is

$$s_{zj} = A\cos\beta_f + \delta_r\cos\psi_j - \mathscr{R}_i(1 - \cos\theta)\sqrt{\cos^2\psi_j}. \quad (7)$$

Since the iterative techniques of the Newton-Raphson method will be used to solve the associated nonlinear equations, the angles β_{oj} and β_{ij} are best stated in terms of the co-ordinates V and W, in figure 5. Then

$$\sin\beta_{oj} = \frac{W_j}{(f_o - 0.5)D + \delta_{oj}}, \quad (8)$$

$$\cos\beta_{oj} = \frac{V_j}{(f_o - 0.5)D + \delta_{oj}}, \quad (9)$$

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$$\sin(\beta_{ij} + \theta\cos\psi_j) = \frac{s_{xj} - W_j}{(f_i - 0.5)D + \delta_{ij}}, \tag{10}$$

$$\cos(\beta_{ij} + \theta\cos\psi_j) = \frac{s_{zj} - V_j}{(f_i - 0.5)D + \delta_{ij}}. \tag{11}$$

Similarly, the ball angular speed about its own center pitch and yaw angles, α_j and α'_j, are best stated in terms of the ball angular velocity components: $\omega_{x'j}$, $\omega_{y'j}$, and $\omega_{z'j}$; in which x', y', and z' are the axes of the coordinate frame whose origin is at the ball center; x' is parallel to the longitudinal axis of the bearing around which the balls circulate in its orbital motion, and z' is the radial axis. Then

$$\sin\alpha_j = \frac{\omega_{z'j}}{\sqrt{\omega_{x'j}^2 + \omega_{y'j}^2 + \omega_{z'j}^2}}, \tag{12}$$

$$\cos\alpha_j = \frac{\sqrt{\omega_{x'j}^2 + \omega_{y'j}^2}}{\sqrt{\omega_{x'j}^2 + \omega_{y'j}^2 + \omega_{z'j}^2}}, \tag{13}$$

$$\sin\alpha'_j = \frac{\omega_{y'j}}{\sqrt{\omega_{x'j}^2 + \omega_{y'j}^2}}, \tag{14}$$

$$\cos\alpha'_j = \frac{\omega_{x'j}}{\sqrt{\omega_{x'j}^2 + \omega_{y'j}^2}}. \tag{15}$$

Using the Pythagorean Theorem, it can be seen from figure 5 that

$$(s_{zj} - V_j)^2 + (s_{xj} - W_j)^2 - [(f_i - 0.5)D + \delta_{ij}]^2 = 0 = \epsilon_j, \tag{16}$$

$$V_j^2 + W_j^2 - [(f_o - 0.5)D + \delta_{oj}]^2 = 0 = \epsilon_{j+Z}. \tag{17}$$

From (12)-(15)

$$\omega_{x'j}^2 + \omega_{y'j}^2 + \omega_{z'j}^2 - \omega_{Rj}^2 = 0 = \epsilon_{j+2Z}, \qquad \omega_{Rj} = \sqrt{\omega_{x'j}^2 + \omega_{y'j}^2 + \omega_{z'j}^2}. \tag{18}$$

For steady state operation of a ball bearing at high speed, the forces and moments acting on each ball are as shown by figure 6.

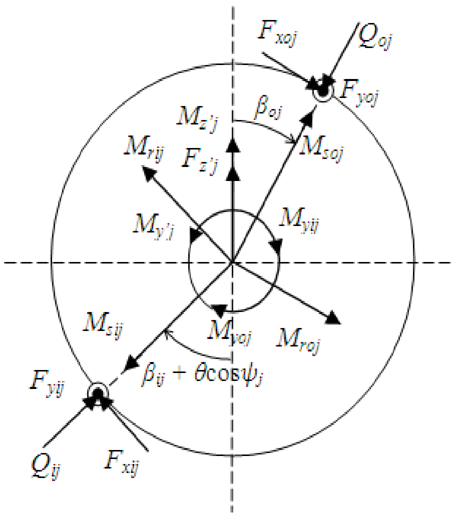

Figure 6. Ball loading at angular position ψ_j.

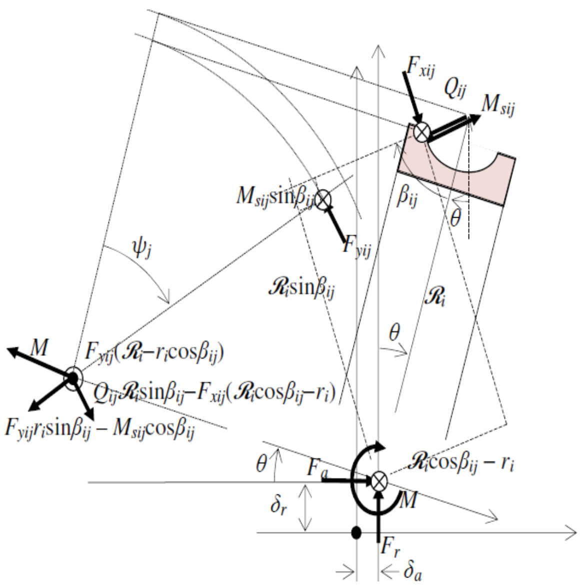

Figure 7. Forces and moments about the inner ring center of mass.

The normal ball loads are related to normal contact deformations by

$$Q_{oj} = K_{oj}\delta_{oj}^{1.5}, \qquad Q_{ij} = K_{ij}\delta_{ij}^{1.5}, \tag{19}$$

in which K_{oj} and K_{ij} are functions of contact angles [11].

From figure 6 considering the three axes equilibrium forces:

$$Q_{ij}\sin(\beta_{ij} + \theta\cos\psi_j) - Q_{oj}\sin\beta_{oj} - F_{xij}\cos(\beta_{ij} + \theta\cos\psi_j) + F_{xoj}\cos\beta_{oj} = 0, \tag{20}$$

$$Q_{ij}\cos(\beta_{ij} + \theta\cos\psi_j) - Q_{oj}\cos\beta_{oj} + F_{xij}\sin(\beta_{ij} + \theta\cos\psi_j) - F_{xoj}\sin\beta_{oj} + F_{z'j} = 0, \tag{21}$$

$$F_{yoj} + F_{yij} = 0 = \epsilon_{j+5Z}, \tag{22}$$

Substituting (8)-(11) and (19) into (20)-(21) yields

$$\frac{F_{xoj}V_j - K_{oj}\delta_{oj}^{1.5}W_j}{(f_o-0.5)D+\delta_{oj}} + \frac{K_{ij}\delta_{ij}^{1.5}(s_{xj}-W_j) - F_{xij}(s_{zj}-V_j)}{(f_i-0.5)D+\delta_{ij}} = 0 = \epsilon_{j+3Z}, \tag{23}$$

$$\frac{K_{oj}\delta_{oj}^{1.5}V_j + F_{xoj}W_j}{(f_o-0.5)D+\delta_{oj}} - \frac{K_{ij}\delta_{ij}^{1.5}(s_{zj}-V_j) + F_{xij}(s_{xj}-W_j)}{(f_i-0.5)D+\delta_{ij}} - F_{z'j} = 0 = \epsilon_{j+4Z}. \tag{24}$$

From figure 6 considering the three axes equilibrium moments:

$$-M_{sij}\sin(\beta_{ij} + \theta\cos\psi_j) + M_{soj}\sin\beta_{oj} - M_{Rij}\cos(\beta_{ij} + \theta\cos\psi_j) + M_{Roj}\cos\beta_{oj} = 0, \tag{25}$$

$$-M_{sij}\cos(\beta_{ij} + \theta\cos\psi_j) + M_{soj}\cos\beta_{oj} + M_{Rij}\sin(\beta_{ij} + \theta\cos\psi_j) - M_{Roj}\sin\beta_{oj} + M_{z'j} = 0, \tag{26}$$

$$M_{y'j} - M_{yij} - M_{yoj} = 0 = \epsilon_{j+8Z}. \tag{27}$$

Substituting (8)-(11) into (25)-(26) yields

$$\frac{M_{Roj}V_j + M_{soj}W_j}{(f_o-0.5)D+\delta_{oj}} - \frac{M_{sij}(s_{xj}-W_j) + M_{Rij}(s_{zj}-V_j)}{(f_i-0.5)D+\delta_{ij}} = 0 = \epsilon_{j+6Z}, \tag{28}$$

$$\frac{M_{soj}V_j - M_{Roj}W_j}{(f_o-0.5)D+\delta_{oj}} - \frac{M_{sij}(s_{zj}-V_j) - M_{Rij}(s_{xj}-W_j)}{(f_i-0.5)D+\delta_{ij}} + M_{z'j} = 0 = \epsilon_{j+7Z}. \tag{29}$$

The centrifugal force acting on the ball at angular position ψ_j is given by [11]

$$F_{z'j} = \frac{1}{2} m d_{mj}\omega_{mj}^2, \tag{30}$$

in which m is the mass of ball,

$$d_{mj} = d_e + 2\left[V_j - (f_o - ½)D\cos\beta_f\right] \tag{31}$$

is the operational ball's pitch diameter at position j, and ω_{mj} is the absolute orbital speed of the ball about of the bearing axis.

Substituting the identity $\omega_{mj}^2 = (\omega_{mj}/\omega)^2\omega^2$ in (30) gives

$$F_{z'j} = \frac{1}{2} m\omega^2 d_{mj}\left(\frac{\omega_{mj}}{\omega}\right)^2, \tag{32}$$

in which ω is the absolute angular velocity of the rotating ring.

For the outer race to be stationary $\omega_{mj} = -\omega_{oj}$, $\omega = \omega_{ij} + \omega_{mj}$,

$$\frac{\omega_{mj}}{\omega} = \frac{1}{1 + \dfrac{r'_{ij}\left\{\frac{d_{mj}}{2}\left[(f_o-0.5)D+\delta_{oj}\right] + r'_{oj}V_j\right\}\left[\omega_{x'j}(s_{zj}-V_j) + \omega_{z'j}(s_{xj}-W_j)\right]}{r'_{oj}\left\{\frac{d_{mj}}{2}\left[(f_i-0.5)D+\delta_{ij}\right] - r'_{ij}(s_{zj}-V_j)\right\}\left(\omega_{x'j}V_j + \omega_{z'j}W_j\right)}}, \tag{33}$$

and

$$\frac{\omega_{Rj}}{\omega} = \frac{-\sqrt{\omega_{x'j}^2 + \omega_{y'j}^2 + \omega_{z'j}^2}}{\dfrac{r'_{oj}\left(\omega_{x'j}V_j + \omega_{z'j}W_j\right)}{\frac{d_{mj}}{2}\left[(f_o-0.5)D+\delta_{oj}\right] + r'_{oj}V_j} + \dfrac{r'_{ij}\left[\omega_{x'j}(s_{zj}-V_j) + \omega_{z'j}(s_{xj}-W_j)\right]}{\frac{d_{mj}}{2}\left[(f_i-0.5)D+\delta_{ij}\right] - r'_{ij}(s_{zj}-V_j)}}, \tag{34}$$

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in which ω_{ij}, ω_{oj} are the angular velocities about the bearing axis of the inner and outer rings with respect to the ball at position j, and r'_{ij}, r'_{oj} are the inner and outer rolling radii [11].

Likewise, for the inner race to be stationary $\omega_{mj} = -\omega_{ij}$, $\omega = \omega_{oj} + \omega_{mj}$,

$$\frac{\omega_{mj}}{\omega} = \frac{1}{1+\frac{r'_{oj}\left\{\frac{d_{mj}}{2}\left[(f_i-0.5)D+\delta_{ij}\right]-r'_{ij}\left(s_{zj}-V_j\right)\right\}\left(\omega_{x'j}V_j+\omega_{z'j}W_j\right)}{r'_{ij}\left\{\frac{d_{mj}}{2}\left[(f_o-0.5)D+\delta_{oj}\right]+r'_{oj}V_j\right\}\left[\omega_{x'j}\left(s_{zj}-V_j\right)+\omega_{z'j}\left(s_{xj}-W_j\right)\right]}} \tag{35}$$

and ω_{Rj}/ω is given by (34) with opposite sign.

Similarly, the gyroscopic moments acting on the ball at angular position ψ_j are given by [11]

$$M_{y'j} = J\omega^2\left(\frac{\omega_{Rj}}{\omega}\right)\left(\frac{\omega_{mj}}{\omega}\right)\frac{\omega_{z'j}}{\sqrt{\omega_{x'j}^2+\omega_{y'j}^2+\omega_{z'j}^2}}, \tag{36}$$

and

$$M_{z'j} = -J\omega^2\left(\frac{\omega_{Rj}}{\omega}\right)\left(\frac{\omega_{mj}}{\omega}\right)\frac{\omega_{y'j}}{\sqrt{\omega_{x'j}^2+\omega_{y'j}^2+\omega_{z'j}^2}}, \tag{37}$$

in which J is the ball's mass moment of inertia.

The friction forces due to sliding in the x and y-directions of inner and outer ball-raceway elliptical contact areas are given by [11]

$$F_{xij} = \frac{3\mu K_{ij}\delta_{ij}^{1.5}}{2\pi a_{ij}b_{ij}}\int_{-a_{ij}}^{a_{ij}}\int_{-b_{ij}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}}}^{b_{ij}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}}}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}-\frac{y_{ij}^2}{b_{ij}^2}}\sin\gamma_{ij}dy_{ij}\,dx_{ij}, \tag{38}$$

$$F_{xoj} = \frac{3\mu K_{oj}\delta_{oj}^{1.5}}{2\pi a_{oj}b_{oj}}\int_{-a_{oj}}^{a_{oj}}\int_{-b_{oj}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}}}^{b_{oj}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}}}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}-\frac{y_{oj}^2}{b_{oj}^2}}\sin\gamma_{oj}dy_{oj}\,dx_{oj}, \tag{39}$$

$$F_{yij} = \frac{3\mu K_{ij}\delta_{ij}^{1.5}}{2\pi a_{ij}b_{ij}}\int_{-a_{ij}}^{a_{ij}}\int_{-b_{ij}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}}}^{b_{ij}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}}}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}-\frac{y_{ij}^2}{b_{ij}^2}}\cos\gamma_{ij}dy_{ij}\,dx_{ij}, \tag{40}$$

$$F_{yoj} = \frac{3\mu K_{oj}\delta_{oj}^{1.5}}{2\pi a_{oj}b_{oj}}\int_{-a_{oj}}^{a_{oj}}\int_{-b_{oj}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}}}^{b_{oj}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}}}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}-\frac{y_{oj}^2}{b_{oj}^2}}\cos\gamma_{oj}dy_{oj}\,dx_{oj}, \tag{41}$$

in which μ is the friction coefficient; a_{ij}, b_{ij}, a_{oj}, and b_{oj} are semimajor and semiminor-axes of inner and outer pressure ellipses; x_{ij}, y_{ij}, x_{oj}, y_{oj} are the co-ordinates of an element of area, $dydx$, inside the contact ellipse, which has a resultant velocity of slip V of the race on the ball acting at the angle γ with respect to the y-direction, which are given by

$$\gamma_{ij} = \tan^{-1}\frac{y_{ij}-\frac{V_{xij}}{\omega_{sij}}}{x_{ij}+\frac{V_{yij}}{\omega_{sij}}}, \qquad \gamma_{oj} = \tan^{-1}\frac{y_{oj}-\frac{V_{xoj}}{\omega_{soj}}}{x_{oj}+\frac{V_{yoj}}{\omega_{soj}}}. \tag{42}$$

V_{xij}, V_{xoj}, V_{yij}, V_{yoj}, ω_{sij}, and ω_{soj} are the relative linear and angular slip velocities of inner and outer races with respect to the ball located at position j. The terms involving these velocities for use in (42) are given by [11]

$$\frac{V_{xij}}{\omega_{sij}} = \frac{\left(\sqrt{R_i^2 - x_{ij}^2} - \sqrt{R_i^2 - a_{ij}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{ij}^2}\right)\left[(f_i - 0.5)D + \delta_{ij} - \frac{r_{ij}'}{\frac{d_{mj}}{2}}(s_{zj} - V_j)\right]\omega_{y'j}}{\omega_{x'j}(s_{xj} - W_j) - \omega_{z'j}\left[s_{zj} - V_j - \frac{(f_i - 0.5)D + \delta_{ij}}{\frac{d_{mj}}{2}} r_{ij}'\right]}, \tag{43}$$

$$\frac{V_{yij}}{\omega_{sij}} = \frac{\left(\sqrt{R_i^2 - x_{ij}^2} - \sqrt{R_i^2 - a_{ij}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{ij}^2} - r_{ij}'\right)\left[\omega_{x'j}(s_{zj} - V_j) + \omega_{z'j}(s_{xj} - W_j)\right]}{\omega_{x'j}(s_{xj} - W_j) - \omega_{z'j}\left[s_{zj} - V_j - \frac{(f_i - 0.5)D + \delta_{ij}}{\frac{d_{mj}}{2}} r_{ij}'\right]}, \tag{44}$$

$$\frac{V_{xoj}}{\omega_{soj}} = \frac{-\left(\sqrt{R_o^2 - x_{oj}^2} - \sqrt{R_o^2 - a_{oj}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{oj}^2}\right)\left[(f_o - 0.5)D + \delta_{oj} + \frac{r_{oj}'}{\frac{d_{mj}}{2}} V_j\right]\omega_{y'j}}{\omega_{x'j} W_j - \omega_{z'j}\left[V_j + \frac{(f_o - 0.5)D + \delta_{oj}}{\frac{d_{mj}}{2}} r_{oj}'\right]}, \tag{45}$$

$$\frac{V_{yoj}}{\omega_{soj}} = \frac{\left(\sqrt{R_o^2 - x_{oj}^2} - \sqrt{R_o^2 - a_{oj}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{oj}^2} - r_{oj}'\right)\left(\omega_{x'j} V_j + \omega_{z'j} W_j\right)}{\omega_{x'j} W_j - \omega_{z'j}\left[V_j + \frac{(f_o - 0.5)D + \delta_{oj}}{\frac{d_{mj}}{2}} r_{oj}'\right]}, \tag{46}$$

in which R_i and R_o are the curvature radii of deformed surfaces, given by

$$R_i = \frac{2f_i D}{2f_i + 1}, \qquad R_o = \frac{2f_o D}{2f_o + 1}. \tag{47}$$

The total frictional moments of the friction forces about the normal at the center of the contact ellipse are [11]

$$M_{sij} = \frac{3\mu K_{ij}\delta_{ij}^{1.5}}{2\pi a_{ij} b_{ij}} \int_{-a_{ij}}^{a_{ij}} \int_{-b_{ij}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}}}^{b_{ij}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}}} \sqrt{x_{ij}^2 + y_{ij}^2}\sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2} - \frac{y_{ij}^2}{b_{ij}^2}} \cos\left(\gamma_{ij} - \tan^{-1}\frac{y_{ij}}{x_{ij}}\right) dy_{ij}\, dx_{ij}, \tag{48}$$

$$M_{soj} = \frac{3\mu K_{oj}\delta_{oj}^{1.5}}{2\pi a_{oj} b_{oj}} \int_{-a_{oj}}^{a_{oj}} \int_{-b_{oj}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}}}^{b_{oj}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}}} \sqrt{x_{oj}^2 + y_{oj}^2}\sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2} - \frac{y_{oj}^2}{b_{oj}^2}} \cos\left(\gamma_{oj} - \tan^{-1}\frac{y_{oj}}{x_{oj}}\right) dy_{oj}\, dx_{oj}. \tag{49}$$

The moments of the friction forces about the y'-axis are [11]

$$M_{yij} = \frac{3\mu K_{ij}\delta_{ij}^{1.5}}{2\pi a_{ij} b_{ij}} \int_{-a_{ij}}^{a_{ij}} \int_{-b_{ij}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}}}^{b_{ij}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}}} \left(\sqrt{R_i^2 - x_{ij}^2} - \sqrt{R_i^2 - a_{ij}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{ij}^2}\right)\sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2} - \frac{y_{ij}^2}{b_{ij}^2}} \sin\gamma_{ij} dy_{ij}\, dx_{ij}, \tag{50}$$

$$M_{yoj} = \frac{3\mu K_{oj}\delta_{oj}^{1.5}}{2\pi a_{oj} b_{oj}} \int_{-a_{oj}}^{a_{oj}} \int_{-b_{oj}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}}}^{b_{oj}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}}} \left(\sqrt{R_o^2 - x_{oj}^2} - \sqrt{R_o^2 - a_{oj}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{oj}^2}\right)\sqrt{1 - \frac{x_{oj}^2}{a_{oj}^2} - \frac{y_{oj}^2}{b_{oj}^2}} \sin\gamma_{oj} dy_{oj}\, dx_{oj}. \tag{51}$$

The frictional moments about an axis through the ball center perpendicular to the line defining the contact angle, which line lies in the $x'z'$-plane, are [11]

$$M_{Rij} = \frac{3\mu K_{ij}\delta_{ij}^{1.5}}{2\pi a_{ij}b_{ij}} \int_{-a_{ij}}^{a_{ij}} \int_{-b_{ij}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}}}^{b_{ij}\sqrt{1-\frac{x_{ij}^2}{a_{ij}^2}}} \left(\sqrt{R_i^2 - x_{ij}^2} - \sqrt{R_i^2 - a_{ij}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{ij}^2}\right) \sqrt{1 - \frac{x_{ij}^2}{a_{ij}^2} - \frac{y_{ij}^2}{b_{ij}^2}} \cos\gamma_{ij} dy_{ij}\, dx_{ij}, \quad (52)$$

$$M_{Roj} = \frac{3\mu K_{oj}\delta_{oj}^{1.5}}{2\pi a_{oj}b_{oj}} \int_{-a_{oj}}^{a_{oj}} \int_{-b_{oj}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}}}^{b_{oj}\sqrt{1-\frac{x_{oj}^2}{a_{oj}^2}}} \left(\sqrt{R_0^2 - x_{0j}^2} - \sqrt{R_0^2 - a_{0j}^2} + \sqrt{\left(\frac{D}{2}\right)^2 - a_{0j}^2}\right) \sqrt{1 - \frac{x_{0j}^2}{a_{0j}^2} - \frac{y_{0j}^2}{b_{0j}^2}} \cos\gamma_{oj} dy_{oj}\, dx_{oj}. \quad (53)$$

Equations (16)-(18), (22)-(24) and (27)-(29) may be solved simultaneously for V_j, W_j, δ_{oj}, δ_{ij}, r'_{oj}, r'_{ij}, $\omega_{x'j}$, $\omega_{y'j}$, and $\omega_{z'j}$ at each ball angular location once values for δ_a, δ_r, and θ are assumed.

An iterative procedure is to be used to solve the equations simultaneously. Since K_{oj} and K_{ij} are functions of contact angle, equations (8)-(11) may be used to establish K_{oj} and K_{ij} values iteratively.

To find the values of δ_a, δ_r, and θ, it remains to establish the equilibrium conditions of forces and moments about the inner ring center of mass, as shown by figure 7, which are

$$F_a - \sum_{j=1}^{Z} \left[\frac{K_{ij}\delta_{ij}^{1.5}(s_{xj}-W_j) - F_{xij}(s_{zj}-V_j)}{(f_i-0.5)D+\delta_{ij}}\right] = 0 = \epsilon_{9Z+1}, \quad (54)$$

$$F_r - \sum_{j=1}^{Z} \left[\frac{K_{ij}\delta_{ij}^{1.5}(s_{zj}-V_j) + F_{xij}(s_{xj}-W_j)}{(f_i-0.5)D+\delta_{ij}}\right] \cos\psi_j = 0 = \epsilon_{9Z+2}, \quad (55)$$

$$M - \sum_{j=1}^{Z} \{\mathscr{R}_i[K_{ij}\delta_{ij}^{1.5}\sin\beta_{ij} - F_{xij}(\cos\beta_{ij} - r_i/\mathscr{R}_i)]\cos\psi_j - \left(F_{yij}r_i\sin\beta_{ij} - M_{sij}\cos\beta_{ij}\right)\sin\psi_j\} = 0 = \epsilon_{9Z+3}, \quad (56)$$

in which F_a, F_r, and M are external forces and moment applied to the inner ring center of mass.

Having computed values for V_j, W_j, δ_{oj}, δ_{ij}, r'_{oj}, r'_{ij}, $\omega_{x'j}$, $\omega_{y'j}$, and $\omega_{z'j}$ at each angular position and knowing F_a, F_r, and M as input conditions the values of δ_a, δ_r, and θ may be computed by equations (54)-(56). After obtaining the primary unknown quantities δ_a, δ_r, and θ, it is necessary to repeat the calculation of V_j, W_j, δ_{oj}, δ_{ij}, r'_{oj}, r'_{ij}, $\omega_{x'j}$, $\omega_{y'j}$, and $\omega_{z'j}$, until compatible values of primary unknown quantities δ_a, δ_r, and θ are obtained.

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