

A study of optimal low-thrust limited-power transfers between coplanar orbits with small eccentricities

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Abstract. In this paper, an analytical solution, which includes the short periodic terms, for the problem of optimal time-fixed low-thrust limited power transfers (no rendezvous), in an inverse-square force field, between coplanar orbits with small eccentricities is described. This solution is derived through canonical transformation theory. Some numerical results based on the analytical solution are presented for a preliminary analysis of interplanetary missions.

1. Introduction

The study presented in the paper has been motivated by the renewed interest in the use of low-thrust propulsion systems in space missions in the last thirty years. Important space missions have made use of low-thrust propulsion systems: NASA-JPL Deep Space 1 and ESA-SMART1. Deep Space 1 was the first interplanetary spacecraft to utilize Solar Electric Propulsion. It was developed by NASA in the New Millennium program to testing new technologies for future Space and Earth science programs. It was launched on October 24, 1998. Deep Space 1 mission terminated on December 18, 2001, when its fuel supply exhausted. SMART-1 was the first of a series of ESA Small Missions for Advanced Research in Technology. It was used to test solar electric propulsion and other deep-space technologies. It was launched on September 27, 2003. SMART-1 mission ended on September 3, 2006, when the spacecraft, in a planned manoeuvre, impacted the lunar surface. Interesting details about these space missions can be found in [2], [11] and [12].

Low-thrust electric propulsion systems are characterized by high specific impulse and low-thrust capability (the ratio between the maximum thrust acceleration and the gravity acceleration on the ground is small, between 10^{-4} and 10^{-2}) and have their greatest benefits for high-energy planetary missions. Several researchers have obtained numerical and analytical solutions for several maneuvers involving specific initial and final orbits and specific thrust profiles ([1], [4], [5], [6], [7], [10]). In the analytical studies, averaging techniques are applied and solutions of the averaged equations are obtained such that only secular behavior of the optimal solutions is discussed. Few works discuss the inclusion of periodic terms which are, in general, considered only for transfers between close orbits.

In a previous work (Da Silva Fernandes and Carvalho, [4]), the authors presented a complete analytical solution, which includes the short periodic terms, for the problem of optimal low-thrust limited-power transfers between arbitrary elliptic coplanar orbits. But, this analytical solution becomes singular for circular orbits. So, the main purpose of this paper is to present a complete first order



analytical solution which eliminates the singularity for circular orbits and can be applied for time-fixed transfers between coplanar orbits with small eccentricities. This study is particularly interesting because the orbits found in practice often have a small eccentricity and the problem of transferring a vehicle from a low Earth orbit to a high Earth orbit is frequently found.

2. Formulation of the optimization problem

For completeness, in this section the general planar low-thrust limited power transfer problem is formulated as a Mayer problem of optimal control theory with the radial distance and the radial and circumferential components of the velocity as state variables, and, the maximum Hamiltonian function which governs the optimal trajectories is derived by applying the Pontryagin Maximum Principle (Pontryagin et al, [9]). This formulation of the transfer problem provides a straightforward physical interpretation to the optimal thrust acceleration in terms of the adjoint variables.

For a low-thrust limited-power propulsion system – LP system, the fuel consumption is described by the variable J defined as (Marec, [8])

$$J = \frac{1}{2} \int_{t_0}^t \gamma^2 dt, \quad (1)$$

where γ is the magnitude of the thrust acceleration vector γ , used as a control variable. The consumption variable J is a monotonic decreasing function of the mass m of the space vehicle,

$$J = P_{\max} \left(\frac{1}{m} - \frac{1}{m_0} \right),$$

where $P_{\max}$ is the maximum power and m_0 is the initial mass. The minimization of the final value of the fuel consumption, J_f , is equivalent to the maximization of m_f .

Consider the motion of a space vehicle M , powered by a limited-power engine in an inverse-square force field. At time t , the state of a space vehicle M is defined by the radial distance r from the center of attraction, the radial and circumferential components of the velocity, v_r and v_s , and the fuel consumption J .

The optimization problem can be formulated as a Mayer problem of optimal control as follows (Marec, [8]): It is proposed to transfer the space vehicle M from the initial state $(r_0, v_{r0}, v_{s0}, 0)$ at time t_0 to the final state $(r_f, v_{rf}, v_{sf}, J_f)$ at time t_f , such that the final consumption variable J_f is a minimum. The duration of the transfer $t_f - t_0$ is specified. In the two-dimensional formulation, the state equations are given by

$$\frac{dr}{dt} = v_r \quad \frac{dv_r}{dt} = \frac{v_s^2}{r} - \frac{\mu}{r^2} + R \quad \frac{dv_s}{dt} = -\frac{v_r v_s}{r} + S \quad \frac{dJ}{dt} = \frac{1}{2} (R^2 + S^2) \quad (2)$$

where μ is the gravitational parameter (for the Earth, $\mu = 3.986 \times 10^5 \text{ km}^3 / \text{s}^2$), R and S are the radial and circumferential components of the thrust acceleration vector, respectively. The performance index is then defined by

$$IP = J(t_f) \quad (3)$$

For LP system, it is assumed that there are no constraints on the thrust acceleration vector.

According to (Da Silva Fernandes et al, [3]), the optimal trajectories are governed by the maximum Hamiltonian H^* ,

$$H^* = v_r p_r + \left(\frac{v_s^2}{r} - \frac{\mu}{r^2} \right) p_{v_r} - \frac{v_r v_s}{r} p_{v_s} + \frac{1}{2} (p_{v_r}^2 + p_{v_s}^2), \quad (4)$$

with the optimal thrust acceleration given by

$$R^* = p_{v_r} \quad S^* = p_{v_s}. \quad (5)$$

Note that the components of the optimal thrust acceleration are given by the adjoint variables to the components of the velocity vector; in this case, the acceleration is modulated (Marec, [8]).

3. Analytical Solution

A first order analytical solution for the system of differential equations governed by the maximum Hamiltonian, which includes short periodic terms, can be derived through canonical transformation theory, as described in Da Silva Fernandes et al [3]. The optimal low-thrust power-limited trajectories for transfers between coplanar orbits with small eccentricities in an inverse-square force field are then described in a set of nonsingular elements by the following equations:

$$a''(t) = a'''(t) + \sqrt{\frac{a_0^{m5}}{\mu^3}} \left\{ [8h''' \sin \ell''' - 8k''' \cos \ell'''] a'''^2 p_{a''} + [4a''' \sin \ell''' - 2a''' k''' \cos 2\ell'''] \right. \\ \left. + 2a''' h''' \sin 2\ell''' \right] p_{h''} + [-4a''' \cos \ell''' - 2a''' h''' \cos 2\ell''' - 2a''' k''' \sin 2\ell'''] p_{k''} \Big\} \Big|_{t_0}^t \quad (6)$$

$$h''(t) = h'''(t) + \sqrt{\frac{a_0^{m5}}{\mu^3}} \left\{ [4a''' \sin \ell''' - 2a''' k''' \cos 2\ell''' + 2a''' h''' \sin 2\ell'''] p_{a''} \right. \\ \left. + \left[-2h''' \sin \ell''' - 2k''' \cos \ell''' + \frac{3}{4} \sin 2\ell''' + \frac{2}{3} h''' \sin 3\ell''' - \frac{2}{3} k''' \cos 3\ell''' \right] p_{h''} \right. \\ \left. + \left[2h''' \cos \ell''' - 2k''' \sin \ell''' - \frac{3}{4} \cos 2\ell''' - \frac{2}{3} h''' \cos 3\ell''' - \frac{2}{3} k''' \sin 3\ell''' \right] p_{k''} \right\} \Big|_{t_0}^t \quad (7)$$

$$k''(t) = k'''(t) + \sqrt{\frac{a_0^{m5}}{\mu^3}} \left\{ [-4a''' \cos \ell''' - 2a''' h''' \cos 2\ell''' - 2a''' k''' \sin 2\ell'''] p_{a''} \right. \\ \left. + \left[2h''' \cos \ell''' - 2k''' \sin \ell''' - \frac{3}{4} \cos 2\ell''' - \frac{2}{3} h''' \cos 3\ell''' - \frac{2}{3} k''' \sin 3\ell''' \right] p_{h''} \right. \\ \left. + \left[2h''' \sin \ell''' + 2k''' \cos \ell''' - \frac{3}{4} \sin 2\ell''' - \frac{2}{3} h''' \sin 3\ell''' + \frac{2}{3} k''' \cos 3\ell''' \right] p_{k''} \right\} \Big|_{t_0}^t \quad (8)$$

with a''' , ..., $p_{k''}$ given by

$$a'''(t) = \frac{a_0'''}{1 + \frac{4a_0'''}{\mu} \left(\frac{1}{2} \mathbf{E} t^2 - a_0''' p_{a_0'''} t \right)} \quad (9)$$

$$h'''(t) = h_0''' + \sqrt{\frac{5}{2}} \frac{p_{h''}}{C} \left\{ \tan^{-1} \left(\left(\frac{4\mu E}{5C^2 a_0'''} - 1 \right)^{1/2} \right) - \tan^{-1} \left(\left(\frac{4\mu E}{5C^2 a'''} - 1 \right)^{1/2} \right) \right\} \quad (10)$$

$$k'''(t) = k_0''' + \sqrt{\frac{5}{2}} \frac{p_{k''}}{C} \left\{ \tan^{-1} \left(\left(\frac{4\mu E}{5C^2 a_0'''} - 1 \right)^{1/2} \right) - \tan^{-1} \left(\left(\frac{4\mu E}{5C^2 a'''} - 1 \right)^{1/2} \right) \right\} \quad (11)$$

$$p_{a'''}^2 = \left(\frac{a_0'''}{a'''} \right)^3 p_{a_0'''}^2 + \frac{5}{8} C^2 \left(\frac{a_0'''}{a'''} - \frac{1}{a'''} \right) \quad (12)$$

$$p_{h''} = p_{h_0''}, \quad (13)$$

$$p_{k''} = p_{k_0''}, \quad (14)$$

with C and E given in terms of the initial conditions by

$$C^2 = p_{h_0''}^2 + p_{k_0''}^2, \quad (15)$$

$$4\mu E = a_0''' \left(8(a_0''' p_{a_0''}')^2 + 5(p_{h_0''}^2 + p_{k_0''}^2) \right). \quad (16)$$

The initial conditions for the state variables are $a'''(0) = a_0'''$, $h'''(0) = h_0'''$ and $k'''(0) = k_0'''$. Note that the mean latitude ℓ''' in equations above is given by

$$\ell'''(t) = \ell'''(t_0) + \int_{t_0}^t \left[\sqrt{\frac{\mu}{a'''}^3} + \frac{7}{4} \frac{a'''}{\mu} (k'' p_{h''} - h'' p_{k''}) \right] dt. \quad (17)$$

The set of nonsingular orbital elements are related to the classical orbital elements through the equations

$$h = e \cos \omega \quad k = e \sin \omega \quad \ell = M + \omega,$$

where e is the eccentricity, ω is the argument of periapsis and M is the mean anomaly, and, a denotes the semi-major axis. For simplicity, primes are omitted. Primes are used to denote new variables introduced through the canonical transformations built to obtain the analytical solution.

4. Numerical Results

In this section, the analytical solution is applied in a preliminary analysis of some interplanetary missions considering time-fixed transfers from Earth to Venus ($\rho = 0.7270$), and, from Earth to Mars ($\rho = 1.5236$). In this analysis, the following assumptions are considered:

1. The orbits of the planets are circular;
2. The orbits of the planets lie in the plane of the ecliptic;
3. The flight of the space vehicle takes place in the plane of the ecliptic;
4. Only the heliocentric phase is considered; that is, the attraction of planets on the spacecraft is neglected.

The results provided by the analytical theory are compared to numerical results obtained by solving the two-point boundary value through a neighboring extremals algorithm based on the state transition matrix. This algorithm has been chosen as the exact solution for each maneuver, in view of the accuracy obtained in fulfillment of the terminal constraints. A comparison with a linear theory is also presented. The values of the consumption variable J are presented in Table 1. The time evolution of

eccentricity and semi-major axis are presented in Figures 1 to 2 for transfers with time of flight, $t_f - t_0 = 25$ time units. The results are expressed in canonical units.

Table 1. Consumption variable J .

$\rho = r_f / r_0$	$t_f - t_0$	$J_{Analytical}$	$J_{neighboring}$	J_{Linear}
0.7270	25.0	5.9706×10^{-4}	5.9852×10^{-4}	5.7883×10^{-4}
	50.0	2.9864×10^{-4}	2.9901×10^{-4}	2.8941×10^{-4}
	75.0	1.9910×10^{-4}	1.9926×10^{-4}	1.9294×10^{-4}
	100.0	1.4933×10^{-4}	1.4940×10^{-4}	1.4470×10^{-4}
	125.0	1.1947×10^{-4}	1.1949×10^{-4}	1.1576×10^{-4}
1.5236	25.0	7.2912×10^{-4}	7.2468×10^{-4}	6.8706×10^{-4}
	50.0	3.6143×10^{-4}	3.6093×10^{-4}	3.4272×10^{-4}
	75.0	2.4058×10^{-4}	2.4044×10^{-4}	2.2793×10^{-4}
	100.0	1.8034×10^{-4}	1.8028×10^{-4}	1.7067×10^{-4}
	125.0	1.4424×10^{-4}	1.4421×10^{-4}	1.3646×10^{-4}

From the results presented in Table 1 and Figures 1 – 2, major comments are as follows:

The analytical solution gives a very good approximation to the values of the consumption variable J for all transfers. The absolute relative differences vary between 0.00% and 0.65%. The larger differences occur for transfers with lesser time of flight. The time evolution of eccentricity and semi-major axis shows good agreement between analytical and numerical results.

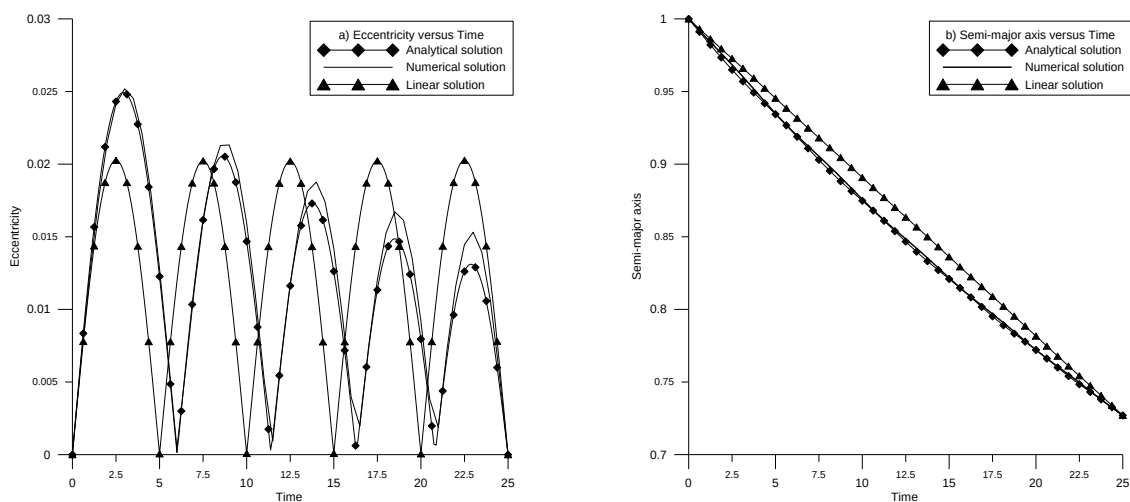


Figure 1. Eccentricity and semi-major axis for $\rho = 0.7270$ and $t_f - t_0 = 25.0$.

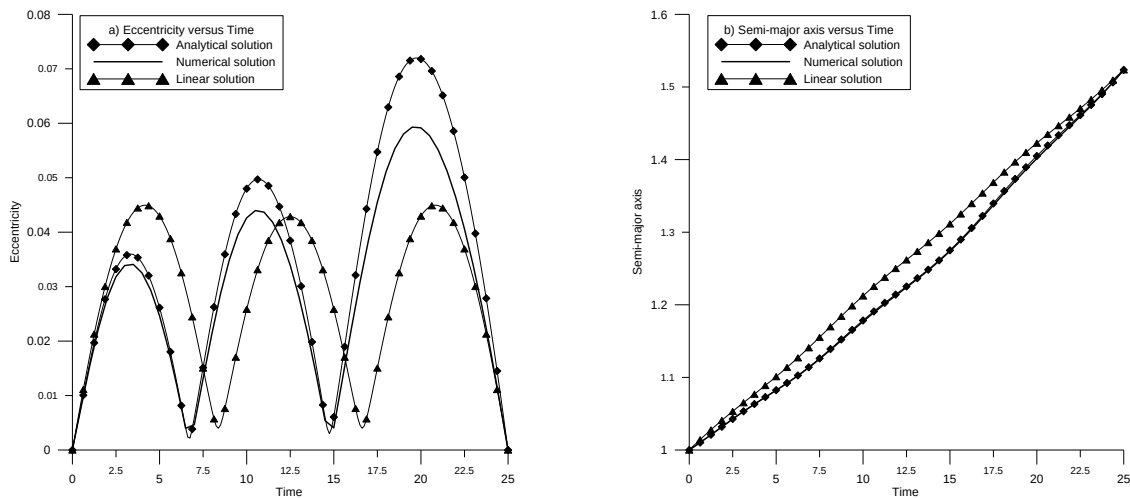


Figure 2. Eccentricity and semi-major axis for $\rho=1.5236$ and $t_f - t_0 = 25.0$.

5. Conclusion

In this paper, a first order analytical theory for optimal time-fixed low-thrust limited-power transfers is described and applied to a preliminary analysis of some interplanetary missions from Earth to Venus, and, Earth to Mars. Numerical results show good agreement between analytical and numerical results.

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7. References

- [1] Bonnard, B, Caillau J B, Dujol, R 2006 Averaging and optimal control of elliptic Keplerian orbits with low propulsion. *Systems Control Lett.* 55, 755 – 760
- [2] Camino, O, Alonso, M, Blake, R et al 2005 SMART-1: Europe's lunar mission paving the way for new cost effective ground operations (RCSGSO) in *Proceedings of the 6th International Symposium on Reducing the Costs of Spacecraft Ground Systems and Operations (RCSGSO)*, 14-17 June **2005**, ESA/ESOC, Darmstadt, Germany
- [3] Da Silva Fernandes, S, Carvalho, F C, Moraes, R V 2015 Optimal low-thrust transfers between coplanar orbits with small eccentricities. *Computational and Applied Mathematics*, DOI: 10.1007/s40314-015-0249-9
- [4] Da Silva Fernandes, S, Carvalho, F C 2008 A first-order analytical theory for optimal low-thrust limited-power transfers between arbitrary elliptical coplanar orbits. *Mathematical Problems in Engineering*, vol. **2008**, 30 pp. (Article ID 525930)
- [5] Geffroy S, Epenoy, R 1997 Optimal low-thrust transfers with constraints – generalization of averaging techniques. *Acta Astronaut.* 41 (3), 133 – 149
- [6] Huang, W 2012 Solving coplanar power-limited orbit transfer problem by primer vector approximation method. *International Journal of Aerospace Engineering*, vol. **2012**, 9 pp., (Article ID 480320)
- [7] Kiforenko, B N 2005 Optimal low-thrust orbital transfers in a central gravity field. *Int. Appl. Mech.* 41 (11), 1211 – 1238
- [8] Marec, J. P. *Optimal Space Trajectories*. Elsevier, New York, USA, 1979
- [9] Pontryagin, L S, Boltyanskii, V G, Gamkrelidze, R V et al 1962 *The Mathematical Theory of Optimal Control Process*. JohnWiley & Sons, New York, USA

- [10] Quarta, A A, Mengali, G 2013 Trajectory approximation for low-performance electric sail with constant thrust angle. *J. Guid. Control Dyn.* 36 (3), 884 – 887
- [11] Racca, G D, Marini, A, Stagnaro, L et al 2002 SMART-1 mission description and developments status. *Planet. Space Sci.* 50, 1323 – 1337
- [12] Rayman, M D, Varghese, P, Lehman, D et al 2000 Results from the Deep Space 1 technology validation mission. *Acta Astronaut.* 47 (2–9), 475 – 487