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Multiple island chains in wave-particle interactions

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Abstract. We analyze the isochronous island chains that appear in the Poincaré sections of near integrable twist systems. When the system presents just one resonant perturbation with a winding number, the number of chains is constant and it is completely determined by the perturbation. However, for systems that are perturbed by an infinite number of resonant perturbations with the same winding number, the number of isochronous chains depends on the superposition of the perturbations and it is a function of the parameters. Considering a system that describes wave-particle interaction, we show that the number of island chains increases without limit when the wave period or wave number are increased.

1. Introduction

The Poincaré-Birkhoff Fixed Point Theorem states that the resonances of a near integrable system present an even number of periodic points [1-6]. Half of these points are stable and half of them are unstable. The stable points are surrounded by resonant islands and the set of islands that appear around a single periodic trajectory forms a chain [1-4, 6, 7]. When the resonance presents more than one stable periodic trajectory, the island chains surrounding these trajectories are called isochronous [5, 8].

The winding number, also known as rotation number, is not a monotonic function for nontwist systems [5, 6, 9]. In this case, it is common to observe more than one isochronous chain with different values for the action variables in the Poincaré sections, as can be seen in Refs. [6-10]. For twist systems, the winding number is a monotonic function of the action variables [5, 6, 9], which makes it impossible for the system to present isochronous chains in different regions of the Poincaré sections. Although the Poincaré-Birkhoff Fixed Point Theorem does not make any claim on the number of isochronous chains [1-3], we generally observe in the literature twist systems that present only one chain [2]. However, Refs. [11-13] show that it is possible for twist systems to present more than one isochronous chain with the same value for the action variables.

In this paper, we consider two near integrable twist systems that may present more than one isochronous chain in their Poincaré sections. We begin by analyzing a Hamiltonian that describes a nonlinear two-oscillator system, as discussed by Walker and Ford in Ref. [11]. This Hamiltonian presents two perturbative terms with different winding numbers. By changing the perturbative terms, we observe that the number of isochronous chains is altered, although the winding numbers remain the same. On the other hand, the number of chains does not vary with the parameters of the system.

In Ref. [13], we studied a Hamiltonian describing wave-particle interaction. When the wave is given as a series of periodic pulses, the system presents an infinite number of resonant perturbations with the same winding number, and each of these perturbations may generate islands in the same region of phase space. This superposition makes the number of island chains vary as a function of the wave parameters. For sufficiently high values of the parameters, all the resonances of the system present more than one isochronous chain.

2. Two-oscillator system

A (m,n) primary resonance is characterized by the winding number $\Omega = n/m$, where m and n are two positive relative primes. For a two dimensional system described by the action-angle variables $(\boldsymbol{J},\boldsymbol{\varphi})$, the Poincaré-Birkhoff Fixed Point Theorem states that a (m,n) resonance presents L isochronous chains with m islands each in the Poincaré section $J_1 \times \varphi_1$, and L isochronous chains with n islands each in the Poincaré section $J_2 \times \varphi_2$. However, the theorem does not make any claim on the value of L.

We point out that the winding number is not sufficient to determine all the features of a resonance because there is an infinite number of perturbations characterized by the same winding number, i.e., all the $(lm,\ ln)$ perturbations, with l a not null integer, are characterized by $\Omega = n/m$. Therefore, to determine the number of isochronous chains in the Poincaré sections, it is necessary to know which $(lm,\ ln)$ perturbations act on the system.

In Ref [11], Walker and Ford analyzed a Hamiltonian describing a nonlinear two-oscillator system:

$$\begin{aligned} H &= H_0(J_1,J_2) + H_1(J_1,J_2,\varphi_1,\varphi_2), \\ H &= J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2 + \alpha J_1J_2\cos(r_1\varphi_1 - s_1\varphi_2) + \beta J_1 J_2^{3/2}\cos(r_2\varphi_1 - s_2\varphi_2), \end{aligned} \tag{1}$$

where r_i and s_i are not null integers with the same signal [13] that can be written as $(r_i,s_i) = (l_i m_i,\ l_i n_i)$. Walker and Ford used the polar coordinates

$$q_j = \sqrt{2J_j}\cos\varphi_j \quad \text{and} \quad p_j = -\sqrt{2J_j}\sin\varphi_j$$

to represent the Poincaré sections of the system, and they considered $(r_1,s_1) = (2,2)$ and $(r_2,s_2) = (2,3)$. They verified that the (1,1) resonance generated by the (2,2) perturbation presents 2 island chains in the Poincaré section $p_2 \times q_2$, while the (2,3) resonance generated by the (2,3) perturbation presents just one chain.

In figure 1, we show three Poincaré sections built for the same values of the parameters but considering different resonant perturbations. In each panel of figure 1, the (m_2,n_2) = (2,3) resonance (inner resonant islands) is generated by a different perturbation. In panel (a), $(r_2,s_2) = (2,3)$ and the $(m_2,n_2) = (2,3)$ resonance presents one isochronous chain with three islands, i.e., the three inner islands in the Poincaré section are generated by a single initial condition. In panel (b), we consider $(r_2,s_2) = (4,6)$, and the $(m_2,n_2) = (2,3)$ resonance exhibits two chains with three islands each. It means that we need two different initial conditions to generate the six inner islands. Moreover, we

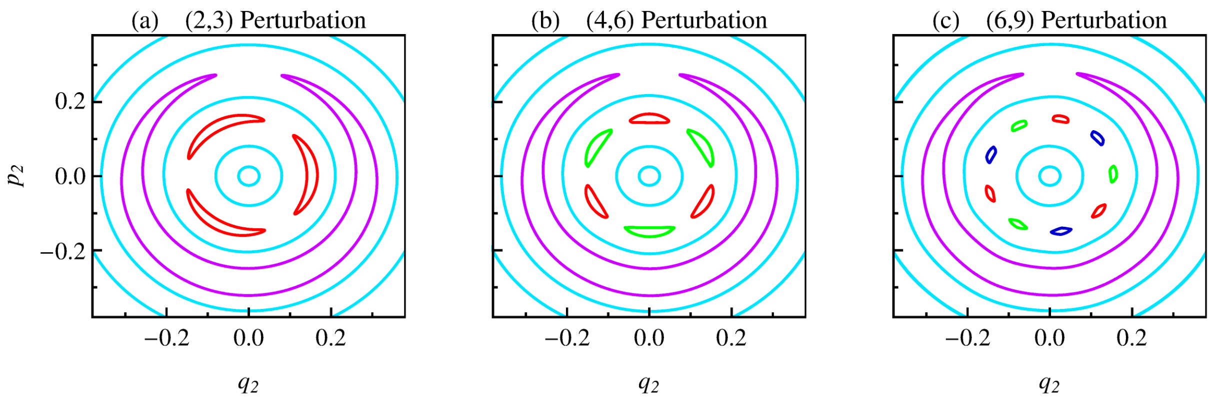

Figure 1. Poincaré section $p_2 \times q_2$ showing the (m_2,n_2) = (2,3) resonance (inner islands) generated by different resonant perturbations: (a) $(r_2,s_2) = (2,3)$; (b) $(r_2,s_2) = (4,6)$; and (c) $(r_2,s_2) = (6,9)$.

observe that the islands of the two chains alternate in the Poincaré section. In panel (c), the $(r_2, s_2) = (6,9)$ perturbation generates three island chains with three islands each.

In all the panels of figure 1, $(r_1, s_1) = (1,1)$ and this perturbation generates the outer resonant island in the pictures. For $(r_1, s_1) = (1,1)$, the $(m_1, n_1) = (1,1)$ resonance presents only one isochronous chain with one island. When we change the resonant perturbation generating the $(m_1, n_1) = (1,1)$ resonance, the number of chains varies similarly to the behavior observed for the $(m_2, n_2) = (2,3)$ resonance.

While the winding number $\Omega = n_i / m_i$ defines the number of islands per chain (as stated by the Poincaré-Birkhoff theorem), figure 1 shows that the number L_i of chains is defined by $|l_i|$. Therefore, to determine the number of isochronous chains it is necessary to know not only the winding number characterizing the resonance, but also which $(l_i m_i,\, l_i n_i)$ resonant perturbation acts on the system to generate the islands.

3. Wave-particle interaction

In Ref. [13], we analyzed the islands of a system describing wave-particle interaction. One particle moves under the action of a uniform magnetic field and a stationary electrostatic wave given as a series of periodic pulses. The dimensionless Hamiltonian that describes the dynamics transverse to the magnetic field is given by [14, 15]

$$H(I,\theta,t) = \sqrt{1+2I} + \frac{\varepsilon}{2}\cos(k\sqrt{2I}\sin\theta)\sum_{l=-\infty}^{+\infty}\delta(t-lT), \tag{2}$$

where (I,θ) are action-angle variables, T is the wave period, k is the wave number and $\varepsilon/2$ is the wave amplitude. The Hamiltonian (2) can be expanded in a Fourier-Bessel series as [13]

$$H = \sqrt{1+2I} + \frac{\varepsilon}{2T}\sum_{r=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty} J_r(k\sqrt{2I})\cos\left(r\theta - \frac{2\pi st}{T}\right), \tag{3}$$

where $J_r(k\sqrt{2I})$ are Bessel functions of order r.

In the resonant islands of the system, the wave transfers energy to the particle and it is regularly accelerated [14-16]. References [14, 16] calculate the parameters interval for which the acceleration can be maximum. However, to achieve the condition of maximum acceleration, it is also necessary to properly adjust the initial conditions of the particle to make it follow the best trajectory in phase space. For this purpose, one should know the position of the resonances and the number of isochronous chains in phase space.

Reference [13] shows that Hamiltonian (3) presents an infinite number of resonant perturbations with the same winding number. Actually, all the (r,s) perturbations that can be written as $(r,s) = (l_i m,\, l_i n)$, with m and n fixed and different values of l_i, are characterized by the winding number $\Omega = n/m$. For example, the perturbations (1,1), (2,2), (-1,-1), (-3,-3), etc., they all have $\Omega = 1/1$.

All the resonant terms that present the same winding number may generate islands in the same region of phase space. This superposition of terms acting on the same region makes the number of isochronous chains vary as a function of the wave period and wave number [13], as can be seen in figure 2. This figure shows the phase space near the (2,3) resonance for different values of the wave number. In panel (a), $k=1$ and the (2,3) resonance presents just one island chain, i.e., the two islands are generated by the same initial condition. Panel (b) shows two isochronous chains with two islands each for $k=3$. The islands of one chain are centered at $\theta = 0,\ \pi$, while the islands of the other chain are centered at $\theta = \pi/2,\ 3\pi/2$. The islands of the two distinct chains can also be recognized by their different size and shape. Similarly to figure 1, the islands of the different chains in figure 2(b) alternate in phase space.

Figure 3 shows the parameter spaces for the (2,3) and (5,2) resonances. The parameter spaces represent the number of isochronous chains as a function of T and k, as indicated in the pictures. For the (2,3) resonance, the number of chains increases monotonically with the parameters of the waves. For the (m,n) resonances with $m>4$, the number of chains is not a monotonic function of the

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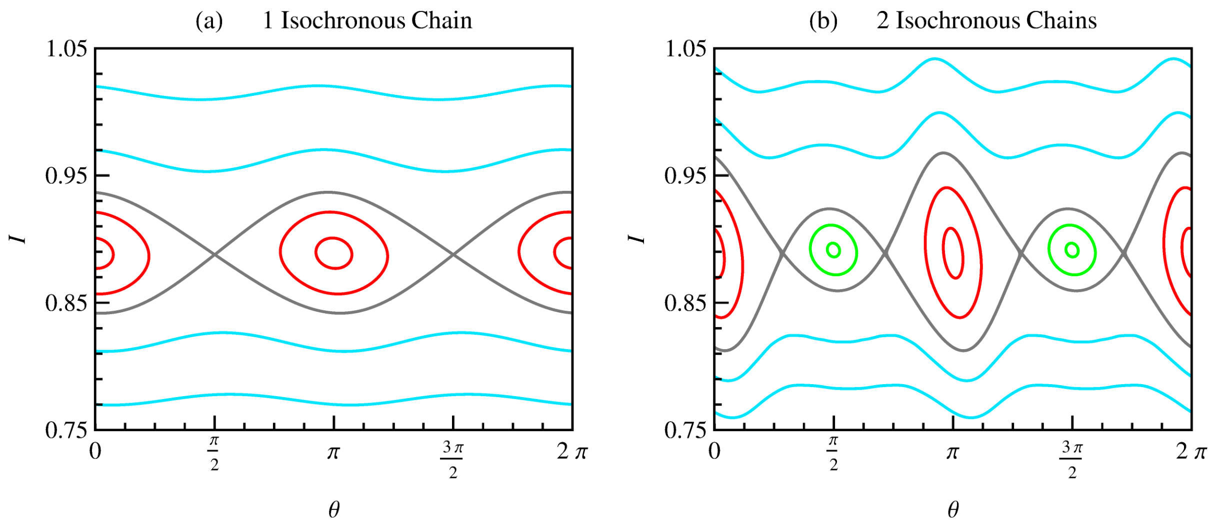

Figure 2. Phase space near the (2,3) resonance for $T = 5\pi$, $\varepsilon = 0.01$, and (a) $k = 1$; (b) $k = 3$.

parameters, as can be seen in figure 3(b) for the (5,2) resonance. As we increase the value of m , the pattern becomes more complicated.

In Ref. [13], we discussed how the number of isochronous chains are related to the resonant perturbations in Hamiltonian (3). For a (m, n) resonance with $m \le 4$, the L chains observed in phase space are generated by the $(r, s) = (l_i m,\ l_i n)$ perturbations for which $|l_i| \le L$. It happens because the amplitudes of the $(l_i m,\ l_i n)$ perturbations with $|l_i| > L$ are generally much smaller and they do not contribute to the generation of islands.

When $m > 4$, the coupling of the $(l_i m,\ l_i n)$ resonant terms becomes more complicated and the previous statement is not true anymore [13]. For the (5,2) resonance in figure 3(b), for example, we observe two isochronous chains when $T \to 4\pi/5$ or $k \to 0$. These two chains are generated by the $(r, s) = (5l_i,\ 2l_i)$ perturbations with $|l_i| \le 2$. Increasing the values of T or k , figure 3(b) presents a very thin region corresponding to four island chains, which are generated by the perturbations with $|l_i| \le 4$. If we continue to increase the values of T or k , the (5,2) resonance exhibits a second region in the parameter space that corresponds to two chains. However, this second region is generated by the coupling of the $(5l_i,\ 2l_i)$ perturbations for which $|l_i| \le 4$ and not $|l_i| \le 2$ as observed for the first region representing two chains.

Although the L chains may be generated by the $(l_i m,\ l_i n)$ perturbations with $|l_i| > L$ when $m > 4$, the number of resonant terms that actually contribute to the generation of islands in phase space is always finite [13]. We also showed in Ref. [13] that the borders separating two regions with a different number of chains in the parameter space respect the same kind of power law for all the resonances. This power law is related to the position of the resonance in phase space with respect to the action variable.

Due to a symmetry in Hamiltonian (3), all the resonances of the system present an even number of islands in phase space [13]. Therefore, all the (m, n) resonances with m odd present an even number of isochronous chains, as can be seen in figure 3(b). On the other hand, when the number m of islands per chain is even, the number of chains may be even or odd, as shown in figure 3(a). This symmetry in (3) implies that most of the resonances of the system always presents more than one chain in phase space. Furthermore, when the values of the wave period and wave number are sufficiently high, all the resonances have more than one isochronous chain. Such behavior is not commonly observed in the literature that generally presents twist systems with single chains [2].

4. Conclusions

The Poincaré-Birkhoff Fixed Point Theorem deals with the winding number that characterizes a

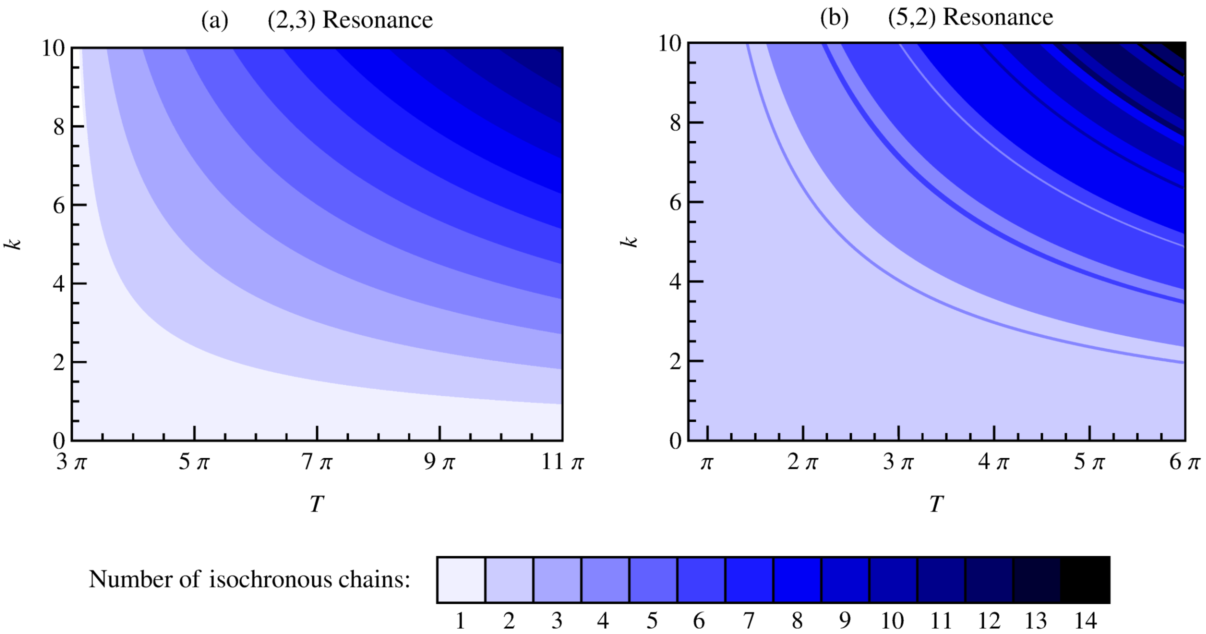

Figure 3. Number of isochronous chains as a function of T and k for the (a) (2,3); and (b) (5,2) resonances.

resonance. In this paper, we showed that the winding number is not enough to determine all the features of a resonance because an infinite number of resonant perturbations have the same winding number. In particular, the number of isochronous chains present in phase space depends on the perturbations that act on the system and their superposition.

When the system has just one resonant perturbation with the same winding number, the number of chains is completely determined by this perturbation and it is a constant for the system. For systems that present an infinite number of perturbations with the same winding number, all these perturbations may generate islands in the same region of phase space. Such a superposition alters the number of isochronous chains according to the amplitudes of the perturbations and the coupling between them. For a system describing wave-particle interaction as discussed in this paper, we observed that the number of chains increases without limit as we increase the wave period or wave number. For sufficiently high values of these parameters, all the resonances of the system exhibit more than one chain in phase space.

We point out that the results presented in this paper are not a particular feature of the analyzed systems. They are valid for near integrable twist systems that are perturbed by either one or infinite resonant terms with the same winding number. For systems that are perturbed by a finite number of resonant terms with the same winding number, we also observe that the number of chains varies as a function of the parameters of the system. However, in this case, the number of chains is limited to a finite set and it does not increase without bounds.

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