Noncommutative geometry, Grand Symmetry and twisted spectral triple

Agostino Devastato

Dipartimento di Fisica, Università di Napoli Federico II

INFN, Sezione di Napoli, Monte S. Angelo, Via Cintia, 80126 Napoli, Italy

E-mail: `agostino.devastato@na.infn.it` ; `astinodevastato@gmail.com`

Abstract. In the noncommutative geometry approach to the standard model we discuss the possibility to derive the extra scalar field sv - initially suggested by particle physicist to stabilize the electroweak vacuum - from a "grand algebra" that contains the usual standard model algebra. We introduce the Connes-Moscovici twisted spectral triples for the Grand Symmetry model, to cure a technical problem, that is the appearance, together with the field sv, of unbounded vectorial terms. The twist makes these terms bounded, and also permits to understand the breaking making the computation of the Higgs mass compatible with the 126 GeV experimental value.

1. Introduction

Noncommutative geometry [1–4] allows to study a large variety of geometrical frameworks from a totally algebraic approach. Particularly, it is very useful in the derivation of models in high energy physics, for example the Yang-Mills gauge theories [5–9]. The mathematical structure of noncommutative geometry is based on three algebraic objects: a C^*-algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ and a generalization of the Dirac operator D . These three elements already naturally appear in some elementary applications of quantum field theory, for instance the harmonic oscillator in which the set of physical observables (position, time, energy) is an associative algebra; spinors, describing the system, belong to an Hilbert space, and the Dirac operator $\partial\!\!\!/ = -i\gamma^\mu\partial_\mu$ which determines dynamic. The set of these three elements $(\mathcal{A},\mathcal{H},D)$ is named *spectral triple.* If the triple is supplemented with two other ingredients, a *chirality operator* Γ and an anti-unitary operator J called *real structure*, then the spectral triple is said a *graded-real spectral triple* $(\mathcal{A},\mathcal{H},D;J,\Gamma)$.

In the current state, gauge theories require a noncommutative geometry structure that is understood to be an *almost commutative* geometry, i.e. the product of continuous geometry, representing space-time, times an internal algebra of finite dimensional matrices. Not many models can be described in this theory because there are a set of seven mathematical constraints

to be respected. For example, the *order-zero condition*

$$\left[a, Jb^*J^{-1}\right] = 0 \ , \forall a, b \in \mathcal{A} \tag{1.1}$$

ensuring the same action of the algebra on particles and antiparticles; or the *first-order condition*

$$\left[\left[D, a\right], Jb^*J^{-1}\right] = 0 \ , \forall a, b \in \mathcal{A} \tag{1.2}$$

which ensures to obtain a fluctuated Dirac operator of order one in the derivatives. In this geometric framework the spectral action principle [10] enables the description of the full standard model of high energy physics, including the Higgs mechanism, putting it on the same footing geometrical footing as general relativity and making it possible the unification with gravity. Moreover, noncommutative geometry allows to deal with another classical problem of the standard model: the three gauge coupling constants run with energy, and at energies comprised between $10^{13} - 10^{17} GeV$ they are very close, but, in view of present data, they fail to meet at a single point; a first possible way to overcome this problem is to enlarge the Hilbert space, adding new fermions [11]. However it is also possible to deduce, from the spectral action expansion, higher dimensional terms in the Lagrangian, improving the unification, [12] without touching the Hilbert space.

2. The Grand Symmetry

Another useful aspect of noncommutative geometry concerns the prediction of the Higgs mass. This latter, at unification scale, is a function of the other parameters of the theory, especially the Yukawa coupling of fermions y_f and the value of the unified gauge couplings $g = g_3 = g_2 = \frac{5}{3}g_1$. Assuming there is no new physics between the electroweak and the unification scales, i.e. the *big desert hypothesis*, the flow of the Higgs mass under the renormalization group yields a prediction around $170GeV$, [7]. It is not possible in the noncommutative geometry approach to the standard model as well as in the usual Weinberg-Salam electroweak theory, to predict a Higgs mass near to its experimental value, $m_H \simeq 126GeV$ [13] without incurring problems of instability. There is, in fact, an instability in the electroweak vacuum which is meta-stable rather than stable (see [14] for the most recent update). This inconsistency strongly suggests that something may be going on. In particular, particle physicists have shown how it is possible to cure this instability, assuming the existence of a new scalar field - usually denoted σ - suitably coupled to the Higgs. In [15] the noncommutative geometry model has been enlarged obtaining the field σ by turning the Majorana coupling constant y_R , in the finite dimensional part of the Dirac operator, into a field:

$$y_R \to y_R\sigma(x) \tag{2.1}$$

However, by definition the bosonic fields in noncommutative geometry are generated by *inner fluctuations* of the Dirac operator, defining the 1-form connection A,

$$A := \sum_i a_i \left[D, b_i\right] \ , \text{with} \ a_i, b_i \in \mathcal{A} \tag{2.2}$$

and, unfortunately, this is not the case for the field σ because of the first-order condition 1.2 on the Majorana part of the Dirac operator, forcing the related one-form to be zero:

$$\left[[D_M, a], Jb^* J^{-1} \right] = 0 \implies [D_M, a] = 0 \implies A_\sigma = 0 . \tag{2.3}$$

A possible way to generate spontaneously σ, is to take advantage of the fermion doubling in the Hilbert space $\mathcal{H}$ of the standard model [16], introducing the Grand Symmetry [17], a model based on a larger symmetry than the usual one and mixing gauge and spin degrees of freedom. The spectral triple of the Grand Symmetry is the same of the standard model apart from the algebra. It is the product of the spectral triple $(C^\infty(M), L^2(M, S), \slashed{\partial}, \gamma^5, \mathcal{J})$ of a compact Riemannian manifold M $-$ where $L^2(M, S)$ is the Hilbert space of square integrable spinors over M, $\slashed{\partial}$ is the usual Dirac operator, γ^5 the product of the four euclidean gamma matrices and $\mathcal{J}$ the usual charge conjugation operator $-$ by a finite dimensional spectral triple:

$$\mathcal{H}_F = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^c = \mathbb{C}^{32}$$

$$D_F = D_0 + D_M = \begin{pmatrix} 0 & \mathcal{M} & \mathcal{M}_R & 0 \\ \mathcal{M}^\dagger & 0 & 0 & 0 \\ \mathcal{M}_R^\dagger & 0 & 0 & \mathcal{M}^* \\ 0 & 0 & \mathcal{M}^T & 0 \end{pmatrix}$$

$$\gamma_F = \mathrm{diag}(\mathbb{I}_8, -\mathbb{I}_8, -\mathbb{I}_8, \mathbb{I}_8)$$

$$J_F = \begin{pmatrix} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix} cc. \tag{2.4}$$

the finite Hilbert space $\mathcal{H}_F$ is the usual one of the standard model for one particles family; the matrix $\mathcal{M}$ contains the quarks, leptons and neutrinos Dirac masses with CKM mixing; while the matrix $\mathcal{M}_R$ contains the Majorana neutrinos mass y_R and forms the Majorana part of the Dirac operator, namely D_M.

The choice of the finite algebra $\mathcal{A}_F$ will be more involved. Under natural assumptions on the representation of the algebra, it is shown in [17] that the algebra in the spectral triple of the standard model should be a sub-algebra of $C^\infty(M) \otimes \mathcal{A}_F$ with

$$\mathcal{A}_F = \mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C}) , \text{ with } a \in \mathbb{N}. \tag{2.5}$$

The algebra of the standard model,

$$\mathcal{A}_{sm} := \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C})$$

is obtained from $\mathcal{A}_F$ for $a = 2$, by using in addition to 1.1 and 1.2, another axiom of the theory, namely the grading condition, i.e. every element of the algebra has to commute with the chirality operator, $[a, \Gamma] = 0$.

Differentely, the Grand Symmetry model starts with the "grand algebra" ($a = 4$ in 2.5),

$$\mathcal{A}_G = \mathbb{M}_4(\mathbb{H}) \oplus \mathbb{M}_8(\mathbb{C}) \tag{2.6}$$

and one can generate the field σ by inner fuctuation, satisfying the first-order condition imposed by the majorana part D_M of Dirac operator D. In fact, by using again the grading and the

first-order conditions one has the following reductions:

$$\mathcal{A}_G \xrightarrow{\text{grading}} \mathcal{A}'_G = [\mathbb{M}_2(\mathbb{H})_L \oplus \mathbb{M}_2(\mathbb{H})_R] \oplus [\mathbb{M}_4(\mathbb{C})_r \oplus \mathbb{M}_4(\mathbb{C})_l] \tag{2.7}$$

$$\mathcal{A}'_G \xrightarrow{\text{1st order}} \mathcal{A}''_G = (\mathbb{H}_L \oplus \mathbb{H}'_L \oplus \mathbb{C}_R \oplus \mathbb{C}'_R) \oplus (\mathbb{C}_l \oplus \mathbb{M}_3(\mathbb{C})_l \oplus \mathbb{C}_r \oplus \mathbb{M}_3(\mathbb{C})_r)$$

with three of the four complex algebras identified $\mathbb{C}_R = \mathbb{C}_r = \mathbb{C}_l$. The σ field will be given by the difference of two elements of the remaining complex algebras $\mathbb{C}_r$ and $\mathbb{C}'_R$:

$$\sigma \sim y_R(c_R - c'_R), \text{ with } c_R \in \mathbb{C}_R \text{ and } c'_R \in \mathbb{C}'_R \tag{2.8}$$

The final reduction to the standard model, that is $\mathbb{C}_R = \mathbb{C}'_R$, $\mathbb{H}_L = \mathbb{H}'_L$ and $\mathbb{M}_3(\mathbb{C})_l = \mathbb{M}_3(\mathbb{C})_r$ is obtained by the first-order condition on the free Dirac operator:

$$\slashed{D} := \slashed{\partial} \otimes \mathbb{I}_F. \tag{2.9}$$

3. Twisted Grand Symmetry

If, on one hand, the Grand Symmetry has the advantage to generate the field σ in the same way than the other gauge fields, on the other hand it contains two weak points. The first limit is that the breaking to the standard model algebra $\mathcal{A}_{sm}$ is obtained by a mathematical condition, i.e. the first-order condition imposed on the free Dirac operator:

$$\text{1-st order condition on } \slashed{D}: \quad \mathcal{A}''_G \to \mathcal{A}_{sm}. \tag{3.1}$$

Of course, it would be preferable if the reduction to the standard model was obtained by a dynamical breaking rather than a mathematical reduction.

The second and more important question concerns the unboundeness of the one-forms connections. Unfortunately, before the last breaking 3.1 not only is the first-order condition not satisfied, but the commutator

$$[\slashed{D}, a] \text{ , with } a \in C^\infty(M) \otimes \mathcal{A}_G \tag{3.2}$$

is never bounded. This is problematic both for physics, because the connection 1-form containing the gauge bosons is unbounded, and from a mathematical point of view, because the construction of a Fredholm module over $\mathcal{A}$ and Hochschild character cocycle depends on the boundedness of the commutator [18].

In [19] both the problems have been solved by using a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D; \rho)$, introduced by Connes and Moscovici in [20] to incorporate type III examples, such as those arising from the transverse geometry of codimension one foliations. Rather than requiring the boundedness of the commutator, one asks that there exists a automorphism ρ of $\mathcal{A}$ such that the twisted commutator

$$[\slashed{D}, a]_\rho := \slashed{D}a - \rho(a)\slashed{D} \tag{3.3}$$

is bounded for any $a \in \mathcal{A}_G$. In fact, one can show that for a suitable choice of a subalgebra $\mathcal{B} \subset C^\infty(M) \otimes \mathcal{A}_G$, a twisted fluctuation of $D = \slashed{D} + D_M$, satisfying the first order condition, generates a field σ - very similiar to the one of [17]. Together with this new scalar field one has also the generation of an additional bounded vector field X_μ, whose potential will lead a

breaking mechanism to the standard model.

Explicitly, $\mathcal{B}$ is the sub-algebra $\mathbb{H}^2 \oplus \mathbb{C}^2 \oplus \mathbb{M}_3(\mathbb{C})$ of $\mathcal{A}_G$. Labelling the two copies of the quaternions and complex algebras by the left/right spinorial indices l, r and the left/right internal indices L, R, that is

$$\mathcal{B} = \mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{C}_R^l \oplus \mathbb{C}_R^r \oplus \mathbb{M}_3(\mathbb{C}) \tag{3.4}$$

the automorphism ρ consists in the exchange of the left/right spinorial indices:

$$\rho : (\, q_L^l, \, q_L^r, \, c_R^l, \, c_R^r, \, m) \rightarrow (q_L^r, \, q_L^l, \, c_R^r, \, c_R^l, \, m) \tag{3.5}$$

where $m \in M_3(\mathbb{C})$ while the c's and q's are complex numbers and quaternions belonging to their respective copy of $\mathbb{C}$ and $\mathbb{H}$.

Furthermore, the breaking to the standard model is now spontaneous, as conjectured in [17]. Namely the reduction of the grand algebra $\mathcal{A}_G$ to $\mathcal{A}_{sm}$ is obtained dynamically, as a minimum of the potential coming from the spectral action. The scalar field σ then plays a role similar as the Higgs field in the electroweak symmetry breaking. Precisely, since the standard model algebra $\mathcal{A}_{sm}$ is the subalgebra of $\mathcal{B}$ invariant under the twist, one can naturally introduce as physical degrees of freedom the fields

$$\Delta(X_\mu) := X_\mu - \rho(X_\mu) \, , \, \, \Delta(\sigma) := (\sigma - \rho(\sigma)) \, D_M \tag{3.6}$$

to measure how far the grand symmetry is from the SM. In (3.6) X_μ and σ are respectively the twisted fluctuations of $\slashed{D}$ and D_M. The new potential, coming from the spectral action expansion, has the form

$$V = V_X + V_\sigma + V_{X\sigma} \tag{3.7}$$

where V_X depends only by the fields $\Delta(X_\mu)$, V_σ depends only by the field $\Delta(\sigma)$ and $V_{X\sigma}$ is an interaction term containing both the fields. In [19, sect. 5.6] it is shown how the minimum of this potential is obtained for

$$\Delta(X_\mu) = 0 \tag{3.8}$$
$$\Delta(\sigma) = 0$$

which means the invariance of X_μ and σ under the twist ρ. This condition is verified for the biggest invariant sub-algebra of $\mathcal{B}$ i.e. $\mathcal{A}_{sm}$.

4. Conclusions

Let us get the conclusions: the main idea of the *twisted Grand Symmetry* model is that the scalar field σ is related to the dynamical breaking of the grand symmetry to the standard model. This idea was already formulated in [17], but it was not fully implemented because, without the twist, the fluctuation of the free Dirac operator by the grand algebra $\mathcal{A}_G$ yielded an operator whose square was a non-minimal Laplacian. Almost simultaneously, a similar idea was implemented in [21], where the bigger symmetry did not come from a bigger algebra, but it followed from relaxing the first order condition. In the twisted case, here presented, we have shown a reduction mechanism to the standard model very similiar to that of [21]. Therefore, it would be interesting to understand the relationship between the twisted fluctuations and the inner fluctuation without the first order condition. Moreover, the twist ρ is remarkably simple,

and its mathematical structure should be studied more in details, in particular how it should be incorporated in the other axioms of noncommutative geometry, for example the orientability condition where the commutator with the Dirac operator has a very important role. Also, the physical meaning of the twist is intriguing: the un-twisted version of $\mathcal{B}$ forces the action of the algebra to be equal on the right and left components of spinors. In this sense the breaking of the grand symmetry to the standard model could be interpreted as a sort of "primordial" chiral symmetry breaking.

Acknowledgments

[1] Connes A, *Noncommutative Geometry*, Academic Press, 1984.
[2] Landi G, *An Introduction to Noncommutative Spaces and their Geometries*, *Springer Lecture Notes in Physics 51*, Springer Verlag (Berlin Heidelberg) 1997. arXiv:hep-th/9701078.
[3] Gracia-Bondia J M, Varilly J C, Figueroa H, *Elements of Noncommutative Geometry*, Birkhauser, 2000.
[4] Connes A, Marcolli M, "Noncommutative Geometry, Quantum Fields and Motives", AMS 2007;
[5] Connes A and Lott J, Particle Models And Noncommutative Geometry (expanded Version), Nucl. Phys. Proc. Suppl. **18B** (1991) 29.
[6] Schucker T, Forces from Connes' geometry, Lect. Notes Phys. **659** (2005) 285 *Preprint* [hep-th/0111236].
[7] Chamseddine A H, Connes A and Marcolli M, Gravity and the standard model with neutrino mixing, Adv. Theor. Math. Phys. **11** (2007) 991 *Preprint* [arXiv:hep-th/0610241].
[8] van den Dungen K and van Suijlekom W D, Particle Physics from Almost Commutative Spacetimes, *Preprint* arXiv:1204.0328 [hep-th].
[9] Jureit J H, Krajewski T, Schucker T and Stephan C A, Seesaw and noncommutative geometry, Phys. Lett. B **654** (2007) 127 *Preprint* [arXiv:0801.3731 [hep-th]].
[10] Chamseddine A H and Connes A, The spectral action principle, Commun. Math. Phys. **186**, 731 (1997) *Preprint* [arXiv:hep-th/9606001].
[11] Andrianov A A, Espriu D, Kurkov M A and Lizzi F, Universal Landau Pole, Phys. Rev. Lett. **111** (2013) 011601 *Preprint* [arXiv:1302.4321 [hep-th]].
[12] Devastato A, Lizzi F, Flores C V, and Vassilevich D, Unification of coupling constants, dimension 6 operators and the spectral action, International Journal of Modern Physics A, 0, 0, 1550033.
[13] Beringer et al. (Particle Data Group), Phys. Rev. D86, 010001 (2012).
[14] Buttazzo D, Degrassi G, Giardino P P, Giudice G F, Sala F, and Salvio A, Investigating the near-criticality of the Higgs boson, *Preprint* arXiv:1307.3536 [hep-ph].
[15] Chamseddine A H and Connes A, Resilience of the Spectral Standard Model, *Preprint* arXiv:1208.1030 [hep-ph].
[16] Lizzi F, Mangano G, Miele G and Sparano G, Fermion Hilbert space and fermion doubling in the noncommutative geometry approach to gauge theories, Phys. Rev. D **55** (1997) 6357 *Preprint* [hep-th/9610035].
[17] Devastato A, Lizzi F and Martinetti P, Grand Symmetry, Spectral Action, and the Higgs mass, *Preprint* arXiv:1304.0415 [hep-th].
[18] Connes A and Moscovici H, Cyclic cohomology and Hopf symmetry, *Preprint* math/0002125 [math-oa].
[19] Devastato A and Martinetti P, Twisted spectral triple for the Standard Model and spontaneous breaking of the Grand Symmetry, *Preprint* arXiv:1411.1320 [hep-th].
[20] Connes A and Moscovici H, Type III and spectral triples, Traces in number theory, geometry and quantum fields, Aspects Math. **E38** (2008), no. Friedt. Vieweg, Wiesbaden, 57–71.
[21] Chamseddine, A H, Connes A and van Suijlekom W D, Inner Fluctuations in Noncommutative Geometry without the first order condition, *Preprint* arXiv:1304.7583 [math-ph].