Non commutative Friedmann spacetimes from Penrose-like inequalities

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Abstract. In this paper a generalisation of the Doplicher-Fredenhagen-Roberts (DFR) construction of a physically motivated non commutative Minkowski spacetime is generalised to the case of Friedmann flat expanding spacetimes. In particular, uncertainty relations quantitatively expressing limitations in the localisation of quantum observables are described and commutation relations implying them are derived from a general analysis. They reduce in a suitable sense to the DFR ones in the flat limit. Finally, it is shown that representations of these commutation relations do exist.

1. Introduction

The basic geometrical object of classical General Relativity is a four dimensional manifold (spacetime) M equipped with a lorentzian metric, all points of which can be labelled by (local) coordinates x^μ. Given this structure, relativistic quantum matter is introduced by means of quantum fields living on M. However, it has been since a long time recognised that such a picture becomes problematic at small scales, even though the very first motivation has been the need to get rid of the well known ultraviolet divergences (still) plaguing Quantum Field Theory. Indeed, in the very first speculations of Heisenberg, Sneyder and Yang of a possible non commutativity of spacetime there seems to have been no substantial direct physical motivation. It was only some years later, apparently with the work Mead [25], that the idea appeared that General Relativity itself could lead to limitations in the precision of spacetime position measurements. The idea was natural and compelling. To make observations in a given spacetime region we must use (say) radiation with comparable wavelength. In case the localisation region is very small (so that the energy density can be assumed very big) this should eventually lead to the formation of an event horizon, i.e. of a black hole ([3]). Thus, no information could come out of the region itself, making the measurement meaningless. Mead also gave the first quantitative estimate of such a lower bound, showing it to be of the scale of the Planck length, that is the utterly small quantity

$$\lambda_P = \sqrt{\frac{G\hbar}{c^3}} \simeq 1,6.10^{-33}\text{cm},$$

where c and G are respectively the speed of light and the gravitational constant. Still, though Mead's result was derived making use of a kind of Heisenberg's microscope argument, for many years to come none of his findings was interpreted as a signal of the non commutativity of

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spacetime, nor any connection with the previously mentioned problems of Quantum Field Theory was explicitly indicated. Perhaps, among the reasons were on the one side the irrelevance of gravity in scattering experiments and on the other the lack of an explicit non commutative geometry as constructed thanks to the work [1] by Connes, Takesaki and many others.

Finally, the pieces were all taken together for the first time in the pioneering work of Doplicher, Fredenhagen and Roberts [15][2], who advocated the idea that the structure of non commutative spacetime Quantum Fields live on should be derived by (or at least not be in contrast with) established physical principles. They thus started from a clearcut formulation of Mead's result, eventually to take the form of a Principle of gravitational stability for localisation of events (PGSL):

- The gravitational field generated by the concentration of energy required to localise an event in spacetime should not be so strong to hide the event itself to any distant observer - distant compared to the Planck scale.

From it, they were able to derive the following bounds on the uncertainties of the coordinates of events for Minkowski spacetime:

$$c\Delta t\left(\Delta x^1 + \Delta x^2 + \Delta x^3\right) \geq \lambda_P^2, \tag{1}$$

$$\Delta x^1\Delta x^2 + \Delta x^1\Delta x^3 + \Delta x^2\Delta x^3 \geq \lambda_P^2. \tag{2}$$

where the Δx^μ's, $\mu = 0, \dots, 3$ should be interpreted as lengths of the edges of the relevant "localisation box". The quantities Δx^μ, $\mu = 0, \dots, 3$ were then interpreted as the mean standard deviations

$$\Delta x^\mu \doteq \Delta_\omega x^\mu = \sqrt{\omega((x^\mu)^2) - \omega(x^\mu)^2} = \sqrt{(\phi, (x^\mu)^2\phi) - (\phi, x^\mu\phi)^2} \tag{3}$$

of hermitian operators x^μ, $\mu = 0, \dots, 3$, acting on some Hilbert space $\tilde{\mathcal{H}}$ with scalar product $\widetilde{(\cdot,\cdot)}$ (here ω_ϕ is a so called vector state and $\phi \in \tilde{\mathcal{H}}$). These operators had to satisfy commutation relations such that (1) would follow, which were explicitly found. This is of course in complete analogy with Quantum Mechanics, but with the crucial difference that now the same mathematical objects are interpreted in a completely different way. The x^μ are to be regarded to as (global) non commutative coordinates on a non commutative Minkowski spacetime $\mathcal{E}$, which in turn was (as usual in non commutative geometry) identified with the C^*-algebra (see [6]) they generate (see below for more details).

To implement the PGSL, we need general conditions for the (non) formation of horizons on the relevant background. These (when they exist) are usually expressed in terms of some definition of the energy of the localised matter or radiation. In [15] the linear approximation of Einstein's equations was used to derive these conditions and the Heisenberg uncertainty relations where used to estimate the (local) energy. Later, several improvements followed in this respect which did not change the essential picture. First, in [30], where Penrose's Inequality was used to derive a more precise answer to 1) and accordingly stricter inequalities for the localisation uncertainties were derived. It is a remarkable fact that equations (1) could be recovered from them. This method has also the advantage to admit straightforward generalisations to general curved spacetimes. Accordingly, a modification of Penrose's inequality derived in [13, 14] for the case of Friedmann expanding spacetimes was used in [31] to derive uncertainty inequalities in the *flat* case. In both papers, however, the energy transfer to spacetime caused by the localisation process was still estimated in terms of the Heisenberg uncertainty relations. A completely different approach was taken in [18]. In this remarkable paper the ambiguities in the definition of energy were removed by making use of a relativistic scalar quantum field evaluated in a suitable state (and the backreaction fully taken into account using Raychaudhury equation).

[2] see also the interesting reviews [10, 12, 28].

The treatment, however, was limited to the case of spherically symmetric localisation regions and for this reason no uncertainty relations could be recovered. Still, all obtained results are in perfect quantitative agreement with (1).

One more important points have to be stressed concerning the road to Quantum Gravity proposed in [15] (see [19] for a further discussion). While being "semiclassical" in the sense that the existence of spacetime itself is taken as a starting assumption as opposed for example to Loop Quantum Gravity (see for example [8] for a comprehensive introduction), the DFR approach is fully quantum field theoretical in the sense that the above mentioned spacetime is considered as a way to describe the localisation properties of relativistic quantum fields. In other terms, while in non relativistic Quantum Mechanics "position" is position of (say) a single particle and is of course an observable, a relativistic field is nothing more (and less) than a way to attach physical degrees of freedom to a spacetime point. This means that the physical quantities the theory speaks about are precisely those degrees of freedom and spacetime points have to be seen as a mere labels or (if coordinates are introduced) parameters[3]. In the case of Quantum Field Theory on non commutative spacetime (*à la* DFR) these parameters are non commutative. Thus, the term "event" is used in the formulation of the PGSL above in the very same sense of Einstein: an event is *not* a physical entity but rather a place (specified by some parameters called coordinates) where a physical process could take place. Of course, quantum fields do not really make sense when evaluated at a point and it is usually assumed that all quantum observables localised in some spacetime region $\mathcal{O}$ may be constructed from the fields "smeared" with test functions f whose support is contained in $\mathcal{O}$, but this does not at all change the picture. Quantum fields on DFR non commutative Minkowski spacetime were actually constructed from the very beginning in [15] and their properties studied in a long series of papers (see for example [10, 11]).

This paper, in which we present ongoing work with G. Morsella [26], is organised as follows. In Section 2 we give a description of non commutative Minkowski spacetime. We include a brief discussion of quantum fields on it, with the aim to stress that these are the fundamental objects of the DFR approach to Quantum Gravity. In Section 3 we illustrate the use of Penrose-like inequalities to derive uncertainty relations. First we illustrate the method for the case of Minkowski spacetime, then use it for the case of general expanding flat Friedmann spacetimes. Sections 4 and 5 are dedicated to the derivation of commutation relation from which the Friedmann uncertainty relations follow. We will see on general grounds that in order to get the DFR model in the flat limit a new generator has to be added to the coordinates. Finally, in Section 6 we show that representations of our commutation relations indeed exist.

2. The Doplicher-Fredenhagen-Roberts model

2.1. DFR non commutative Minkowski spacetime

As it was mentioned in the introduction, for the case of Minkowski spacetime in [15] a C^*-algebra $\mathcal{E}$ was constructed whose (unbounded) selfadjoint generators x_μ have commutation relations such that the uncertainty inequalities (1) are satisfied in any state ω on $\mathcal{E}$ in their domain (*i.e.* such that $\omega(x_\mu^2) < \infty$). These commutation relations are

$$[x_\mu, x_\nu] = i\lambda_P^2 Q_{\mu\nu}, \qquad x^\mu = (x^\mu)^*, \tag{4}$$

and follow from an abstract analysis based on fairly general assumptions. Of these, the physically most significant is *covariance*, that is existence of an automorphic action of the (full) Poincaré group $\mathcal{P}$ on $\mathcal{E}$ such that

$$\alpha_{(\Lambda,a)}(x_\mu) = \Lambda_\mu{}^\nu x_\nu + a_\mu \mathbb{I}, \qquad \alpha_{(\Lambda,a)}(Q_{\mu\nu}) = \Lambda_\mu{}^{\mu'} \Lambda^{\nu'}{}_\nu Q_{\mu'\nu'}. \tag{5}$$

[3] Their status may be compared with the one of time in non-relativistic Quantum Mechanics.

This is to be contrasted with models of non commutative Minkowski spacetime in which actions of *quantum* versions of the Poincaré group are assumed and is justified by the expectation that global space motions should look the same in the large and in the small, while being the classical ones in the large[4]. Moreover, the so called "Quantum conditions" are required. They read

$$[x_\mu, Q_{\nu\rho}] = 0, \qquad Q^{\mu\nu}Q_{\mu\nu} = 0, \qquad (Q^{\mu\nu}(*Q)_{\mu\nu})^2 = 16\mathbb{I}. \tag{6}$$

where $\mathbb{I}$ is the identity operator, $(*Q)_{\mu\nu} = \epsilon_{\mu\nu\eta\rho}Q^{\eta\rho}$ and $\epsilon_{\mu\nu\eta\rho}$ is the totally antisymmetric tensor. The last two equality in terms of the electric and magnetic components $\boldsymbol{e} = (Q_{01}, Q_{02}, Q_{03})$ and $\boldsymbol{m} = (Q_{12}, Q_{32}, Q_{13})$ of Q read

$$\boldsymbol{e}^2 = \boldsymbol{m}^2 \qquad (\boldsymbol{e}\cdot\boldsymbol{m})^2 = \mathbb{I} \tag{7}$$

To understand what $\mathcal{E}$ and its generators look like, observe that because of their centrality (6) the operators $Q_{\mu\nu}$ have a joint spectrum Σ and because of covariance they reduce in an irreducible representation to an ordinary matrix

$$\sigma_{\mu\nu} = \mathrm{Ad}_{(\Lambda_\sigma)}(S) = (\Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}S_{\mu'\nu'}), \qquad S = (S_{\mu\nu}) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

where $\mathrm{Ad}(\cdot)$ indicates the adjoint action of the Lorentz group $\mathscr{L}$. Correspondingly, we must have $x_\sigma = \Lambda_\sigma \tilde{x}$ with

$$\tilde{x}_0 = p_1,\ \tilde{x}_1 = p_2,\ \tilde{x}_2 = q_1,\ \tilde{x}_3 = q_2, \tag{8}$$

p_i, q_i being a pair of the usual Schrödinger operators. It is thus natural to define $\mathcal{E}$ through a generalised Weyl correspondence, namely as the (unique) C^*-completion of the Banach algebra of continuous functions from Σ to $L^1(\mathbb{R}^4, d^4\alpha)$ and vanishing at infinity equipped with the star-product, involution and norm

$$(f \star g)(\sigma, \alpha) = \int f(\sigma, \alpha')g(\sigma, \alpha - \alpha')e^{\frac{i}{2}\alpha_\mu\sigma^{\mu\nu}\alpha'_\nu}, \tag{9}$$

$$f^*(\sigma, \alpha) = \overline{f(\sigma, -\alpha)}, \qquad ||f|| = \sup_{\sigma\in\Sigma} ||f(\sigma, \cdot)||_1. \tag{10}$$

A fiberwise application of von Neumann's Uniqueness Theorem implies then the existence of a unique (up to unitary equivalence) representation of $\mathcal{E}$ acting on the direct integral (over $\mathscr{L}$) Hilbert space $\mathcal{H} = \int^{\oplus} d\Lambda \left(L^2(\mathbb{R}) \otimes L^2(\mathbb{R})\right) \simeq \int^{\oplus} d\Lambda\, L^2(\mathbb{R}^2)$. This allows to prove that $\mathcal{E}$ is isomorphic to $\mathscr{C}_0(\Sigma, \mathcal{K})$, $\mathcal{K}$ being the compact operators.

According to the principles of non-commutative geometry (pure) states ω on $\mathcal{E}$ should be viewed as "non-commutative points" (see [23, 24] for a discussion on that matter in the DFR context). Although for general ω positivity and the commutation relations prevent pointlike localisation, the interesting class of *maximally localised* states minimising the quantity $\sum_\mu(\Delta(x_\mu))^2$ was introduced in [15]. Essentially, they are ground states of the Hamiltonian $H = 1/2(p_1^2 + q_1^2 + p_2^2 + q_2^2)$ of the two-dimensional harmonic oscillator, *i.e.* translated gaussian states (sometimes called coherent states). A maximally localised state ω is completely characterised by the averages $\omega(x_\mu) = \overline{x}_\mu$, $\overline{x} \in \mathbb{R}^4$. Moreover, for any $\overline{x} \in \mathbb{R}^4$ and $\mu = 0, \ldots, 3$ we have $\Delta_{\omega_{\overline{x}}}(x_\mu) = \lambda_P$: we see that, so to speak, non commutativity makes points thick. The maximally localised states are looked at as the only ones to have geometrical meaning. There are some more reasons to consider maximally localised states. First, we have for them a well defined classical limit for $\lambda_P \to 0$ which in [15] was identified as $M \times \Sigma_0$, with $\Sigma_0 \subset \Sigma$ a certain six-dimensional compact manifold. Second, they naturally arise in the construction of Wick products of quantum fields on our non commutative spacetime $\mathcal{E}$.

[4] But see [29] for a proof that these are always suitable restrictions of the DFR model.

2.2. Quantum Fields

The generalisation of quantum fields to the non commutative setting has attracted much attention but once again we will stick to the approach outlined in [15] for Minkowski spacetime.

By analogy with the definition of an ordinary field on commutative Minkowski spacetime, one considers

$$\phi(x) = \int_{\mathbb{R}^4} dk\, e^{ikx} \otimes \hat{\phi}(k) \tag{11}$$

where $\hat{\phi}(k)$ is just the Fourier transform of the field on commutative spacetime but now e^{ikx} are the generalised Weyl operators associated to the DFR non commutative coordinates. Thus ϕ is now to be interpreted as a map from states on the C^*-algebra of non commutative Minkowski spacetime $\mathscr{E}$ to smeared field operators. More explicitly

$$\omega \to \phi(\omega) = \langle \omega \otimes \mathbb{I}, \phi(x) \rangle = \int_{\mathbb{R}^4} dx\, \phi(x)\psi_\omega(x),$$

where the r.h.s. is a quantum field on the ordinary spacetime, smeared with the test function ψ_ω defined by $\hat{\psi}_\omega(k) = \langle \omega, e^{ikx} \rangle$. In other terms, (smeared) non commutative quantum fields are functions from a quantum spacetime[5] to a C^*-algebra $\mathscr{F}$ (the C^*-algebra generated by ordinary fields) and are described by elements affiliated to $\mathscr{E} \otimes \mathscr{F}$.

We conclude this subsection with two important facts. First, one has

$$[\phi(\omega), \phi(\omega')] = i \int d^4x d^4y\, \Delta(x-y)\psi_\omega(x)\psi_{\omega'}(y),$$

so that knowledge of the classical commutator entails knowledge of its non commutative spacetime counterpart one. This will be the later taken as the starting point for our (proposed) generalisation to non commutative Friedmann spacetimes. Second, the modification of test functions illustrated above will in general entail that the evaluation of products of fields in a state generally involve non-local expressions.

3. Uncertainty relations from Penrose-like inequalities

3.1. Minkowski spacetime

The main idea in [30] to overcome the obvious contradictions involved in the application of the linear approximation of General Relativity to derive no-horizon conditions and thus describe very strong gravitational fields was to make use of Penrose inequality (see [22] for a beautiful introduction to the subject and a comprehensive bibliography). Here, we formulate it in somewhat loose terms.

For asymptotically flat data, horizons form if and only if

$$A < 16\pi \frac{G^2}{c^4} M_{\mathrm{ADM}}^2, \tag{12}$$

where A is the proper area enclosing the collapsing matter of total ADM mass M_{ADM}.

As it is well known the ADM mass is nothing else the total mass, *i.e.* the proper (local) mass M_p together with the (negative) contribution due to the binding gravitational energy. However, since to obtain our uncertainty relations we use the Heisenberg uncertainty relations

$$\Delta x_j \Delta p_j \geq \frac{\hbar}{2},\ i,j = 1,2,3 \qquad\qquad \Delta t \Delta E \geq \frac{\hbar}{2}, \tag{13}$$

[5] Recall that "non commutative points" are states on a C^*-algebra.

we need a formulation of Penrose inequality in terms of a proper local quantity. Could we limit ourselves to the case of spherically symmetric collapsing matter, we would have at our disposal the theorems in [13, 14, 21], where the condition that horizons form is expressed only in terms of proper lengths and masses. This is not the case but luckily we only need a sufficient no horizon condition, as provided by inverting (12) and observing that that $M_{\text{ADM}} \leq M_p$ (from now on we write $M = M_p$). Thus we get:

$$A \geq 16\pi \frac{G^2}{c^4} M^2. \tag{14}$$

First of all, we need to evaluate the proper area A of the localising region. As a working approximation and following [30], we take the one of the background (that is the chosen spacetime *without* the experiment). If we choose cartesian coordinates and indicate by Δx_i, $i = 1, 2, 3$, the sides length of a parallelepiped, in view of the fact that our background is for now by hypothesis minkowskian, we may write

$$A = \frac{\pi}{3}(\Delta x^1 \Delta x^2 + \Delta x^1 \Delta x^3 + \Delta x^2 \Delta x^3), \tag{15}$$

with the numerical constant chosen so that with $\Delta x^1 = \Delta x^2 = \Delta x^3 = 2\Delta R$ we have $A = 4\pi \Delta R^2$. Recalling now that for a single particle we have $E^2 \geq c^2(p_1^2 + p_2^2 + p_3^2)$, we identify momenta with the their quantum counterparts Δp_i, $i = 1, 2, 3$ in (13) and substitute in (14). We get:

$$\Delta A \geq 12\lambda_P^4 \left[\left(\frac{1}{\Delta x^1} \right)^2 + \left(\frac{1}{\Delta x^2} \right)^2 + \left(\frac{1}{\Delta x^3} \right)^2 \right]. \tag{16}$$

where we write from now on $\Delta A = \Delta x^1 \Delta x^2 + \Delta x^1 \Delta x^3 + \Delta x^2 \Delta x^3$. After some manipulation (see [30]) we get the space-space uncertainty relation

$$(\Delta_\omega x^1)^2 (\Delta_\omega x^2)^2 (\Delta_\omega x^3)^2 \geq 12\lambda_P^4 \left(\Delta_\omega x^1 \Delta_\omega x^2 + \Delta_\omega x^1 \Delta_\omega x^3 + \Delta_\omega x^2 \Delta_\omega x^3 \right). \tag{17}$$

whereas using the time-energy Heisenberg relation we obtain

$$(c\Delta_\omega t)^2 \left(\Delta_\omega x^1 \Delta_\omega x^2 + \Delta_\omega x^1 \Delta_\omega x^3 + \Delta_\omega x^2 \Delta_\omega x^3 \right) \geq 12\lambda_P^4. \tag{18}$$

Here, we added the subscript ω to emphasise the dependence on states now that the inequalities have a quantum interpretation. For later reference, we mention that

$$\Delta_\omega x^1 \Delta_\omega x^2 \Delta_\omega x^3 \geq \sqrt[4]{12^3} \lambda_P^3, \tag{19}$$

must then hold [30], so that a lower bound for the volume appears. From it, the existence of a mean maximal mass-energy density for such states can be deduced.

Significantly, the DFR uncertainty relations could be recovered as a weakening[6] of relations (17),(18) by making use of the following simple algebraic inequalities

$$\begin{aligned} (a + b + c)^2 &\geq ab + bc + ac, \\ (ab + bc + ac)^3 &\geq a^2 b^2 c^2. \end{aligned} \tag{20}$$

It is thus reasonable to regard the DFR commutation relations as a reasonable approximation to commutation relations implementing (17),(18). In Subsection 3.2, we will use the same approximation to build a DFR-like quantum Friedmann expanding spacetime.

[6] In particular, the ratio $\Delta A/\Delta V$ is substituted by the smaller quantity $(\Delta A)^{-1/2}$. Thus, localisation regions with zero volume are allowed.

3.2. Uncertainty relations for Friedmann expanding spacetimes

Consider now any Friedmann flat expanding spacetimes M and fix *comoving coordinates* $(t, \mathbf{x})$. Thus takes the form

$$ds^2 = dt^2 - a(t)^2(dx_1^2 + dx_2^2 + dx_3^2), \tag{21}$$

and the Hubble parameter is $H(t) = a'(t)/a(t)$. In a previous paper the following generalisation of Penrose isoperimetric inequality with was formulated:

- Black holes do not form if the (positive) excess of proper mass-energy δE inside a two-surface S of proper area $\Delta\mathscr{A}$ contained in a slice of constant universal time t_0 satisfies the inequality:

$$\sqrt{\Delta\mathscr{A}}\left(\frac{1}{4\sqrt{\pi}} + \frac{H\sqrt{\Delta\mathscr{A}}}{4\pi c}\right) \geq \frac{G}{c^4}\delta E. \tag{22}$$

Since we restrict to expanding cosmologies, we shall assume from now on $H \geq 0$. Notice that Penrose inequality for Minkowski spacetime is recovered when $H = 0$, so that the condition for no black hole formation is *stronger* in this case. We will regard at the second factor in the left hand side of (22) as a correction to the flat case brought about by the expansion.

If we consider a box-like localisation region with comoving edges $\Delta x_1, \Delta x_2, \Delta x_3$, we get

$$\Delta\mathscr{A} = a^2(t)(\Delta x_1\Delta x_2 + \Delta x_1\Delta x_3 + \Delta x_2\Delta x_3) = a^2(t)\Delta A.$$

Calculations completely analogous to those in [31] lead to the following inequalities:

$$a^2(t)\Delta A\left(\frac{1}{4\sqrt{3}} + \frac{H(t)\sqrt{a(t)^2\Delta A}}{12c}\right) \geq \frac{\lambda_P^2}{2}, \tag{23}$$

$$c\Delta t \cdot \sqrt{a^2(t)\Delta A}\,\min_{t\in\Delta t}\left\{\left(\frac{1}{4\sqrt{3}} + \frac{H(t)\sqrt{a(t)^2\Delta A}}{12c}\right)\right\} \geq \frac{\lambda_P^2}{2}. \tag{24}$$

We now solve it with respect to the comoving area ΔA. For the sake of clarity, we state the following

Lemma 3.1. Let $f = (x_0 - c/\sqrt{3}H\lambda_P)^2$, where

$$x_0 = \begin{cases} \frac{2c}{\sqrt{3}H\lambda_P}\cos\left[\frac{1}{3}\arccos\left(-1 + \frac{9\sqrt{3}H^2\lambda_P^2}{c^2}\right)\right], & \text{for } \Delta > 0, \\ -sign(q)\frac{2c}{\sqrt{3}H\lambda_P}\cosh\left[\frac{1}{3}arccosh|1 - \frac{9\sqrt{3}H^2\lambda_P^2}{c^2}|\right], & \text{for } \Delta < 0, \end{cases} \tag{25}$$

is the greatest solution of the cubic equation $x^3 + px + q = 0$ and

$$\Delta = \frac{c^4}{\sqrt{3}H^4\lambda_P^4}(\frac{9\sqrt{3}H^2\lambda_P^2}{c^2} - 2),$$

$$p = -\left(\frac{c}{H\lambda_P}\right)^2, \qquad q = -\frac{4\sqrt{3}c^3}{9H^3\lambda_P^3}\left(1 - \frac{9\sqrt{3}H^2\lambda_P^2}{c^2}\right).$$

Then, inequality (23) is equivalent to

$$\Delta A \geq \lambda_P^2\, a(t)^{-2} f\left(H(t)\right). \tag{26}$$

The function f is strictly positive and analytic as a function of the variable H. Moreover, $f \sim H^{-2/3}$ when $H \to +\infty$ and $f \to 2\sqrt{3}$ when $H \to 0$.

Proof. An obvious application of the formulas for the solutions of cubic equations with respect to the unknown $\tilde{x} = \sqrt{a^2\Delta A/\lambda_P}$. The behaviour of the solution for $H \to +\infty$ and $H \to 0$ can also be recovered from equation (24). □

We observed that the same conditions are stronger for Minkowski spacetime. It is easily seen that they are actually the strongest possible, so that it is reasonable to expect that (suitable) intermediate cases model situations in which the effects of curvature are underestimated but still taken into account. We thus feel free to substitute from (23) into (24) to obtain

$$c\Delta t \cdot \sqrt{a^2\Delta A} \geq \max_{t\in\Delta t}\{a(t)\Delta A\} \geq \lambda_P^2 \max_{t\in\Delta t}\left\{a(t)^{-2} f\left(H(t)\right)\right\}. \tag{27}$$

This, together with (26), will be our starting point for quantisation. In fact, the time average over Δt of the positive functions a, f_1 being smaller than the corresponding maximum and using the first of inequalities (20) to evaluate $\sqrt{\Delta A}$, we consider the following uncertainty relations:

$$\Delta_\omega A \geq \frac{\lambda_P^2}{2}|\omega(a^{-2}f)|, \tag{28}$$

$$c\Delta_\omega t\left(\Delta_\omega x_1 + \Delta_\omega x_2 + \Delta_\omega x_3\right) \geq \frac{\lambda_P^2}{2}|\omega(a^{-1}f)|, \tag{29}$$

where again the subscript ω emphasises the we now regard the inequalities as holding for any state on our non commutative spacetime and for future convenience we rescaled Planck's constant λ_P.

Consider now, at the non commutative level, a coordinate transformation

$$(t, \mathbf{x}) \to (t' = w(t), \mathbf{x}' = \mathbf{x}), \tag{30}$$

where w is any diffeomorphism with $w' > 0$. Since we are working at planckian scales, the approximation

$$\Delta t(t') \simeq \frac{\Delta t}{\Delta t'}\Delta t' = \left(\frac{dt}{dt'}\right)\Delta t' \tag{31}$$

is more than reasonable and we can write

$$c\Delta t'\left(\Delta x_1 + \Delta x_2 + \Delta x_3\right) \geq \frac{\lambda_P^2}{2}\left(\frac{dt}{dt'}\right)^{-1}\left(a^{-1}f\right)\left(t(t')\right).$$

where the function f is the same as before. We thus obtain, with obvious notation,

$$\Delta_\omega A \geq \frac{\lambda_P^2}{2}|\omega(f)|, \tag{32}$$

$$c\Delta_\omega t'\left(\Delta_\omega x_1 + \Delta_\omega x_2 + \Delta_\omega x_3\right) \geq \frac{\lambda_P^2}{2}|\omega\left(\left(\frac{dt}{dt'}\right)^{-1} a^{-1}f\right)|. \tag{33}$$

In the specific case of conformal coordinates

$$(\tau, \mathbf{x}) = \left(\int_{t_0}^{t} a(s)^{-1}ds + t_0, \mathbf{x}\right),$$

this gives

$$\Delta_\omega A \geq \frac{\lambda_P^2}{2}|\omega(a^{-2}f)|, \tag{34}$$

$$c\Delta_\omega \tau\left(\Delta_\omega x_1 + \Delta_\omega x_2 + \Delta_\omega x_3\right) \geq \frac{\lambda_P^2}{2}|\omega(a^{-2}f)|. \tag{35}$$

From now on, we shall work in conformal coordinates both because of the simple symmetry of the corresponding STUR (34) and (35) and in view of the forthcoming applications to quantum field theory.

In the next sections, we provide explicit realisations of (34) and (35) in terms of generators of a non commutative Friedmann C^*-algebra. Then, we will see that an implementation of the above time-diffeomorphism symmetry can be obtained in terms of unitary operators.

4. Quantum Friedmann expanding spacetime

Following the general ideas of the DFR approach to Quantum Field Theory on non commutative spacetimes, we regard the coordinates as noncommutative generators of a C^*-algebra $\mathcal{A}$ describing the relevant non commutative spacetime. We fix conformal coordinates $(x_0 = \tau, \mathbf{x}$. Our commutation relations will be

$$[x_\mu, x_\nu] = i\lambda_P^2 Q_{\mu\nu}, \qquad [Q_{\mu\nu}, \tau] = 0, \qquad [Q_{\mu\nu}, Q_{\lambda\rho}] = 0 \tag{36}$$

where the second and third equalities are motivated by the fact that, being Friedmann spacetimes spatially homogeneous, we do expect the commutators to depend only on the time coordinate. The covariance properties of our operators above will be specified below. For a sensible generalisation of the structures and properties of DFR Minkowski spacetime, it is natural to formulate the following additional requirements:

1) The STUR (34), (35) should be satisfied for a physically relevant class of states.
2) There should be a compatible continuous action of the isometry group $G = SO(3) \ltimes \mathbb{R}^3$ of Friedmann spacetimes.
3) We should in some suitable sense recover the DFR model in the flat limit $a \to 1$.
4) The commutative background spacetime $M = I \times \mathbb{R}^3$ should be recovered (perhaps as a factor) in the limit $\lambda_P \to 0$.

Of course, 1) and 2) will be fully satisfied. Concerning 3), one should not be too strict. As a matter of fact, not even in the commutative case one fully recovers Minkowski space from Friedmann spacetimes in the zero density, flat limit, and as far as we know there is no standard procedure circumvent this problem. One way to understand this is to think about the different isometry groups of these spacetimes, the one of Friedmann spacetime being of course smaller than Minkowski's (see the end of this section for a further discussion). We will not address 4) in this paper.

Remark 4.1. From now on we shall always assume we work in some representation π on some (separable) Hilbert space $\mathcal{H}$ in which the operators $\pi(x_0) = \pi(\tau), \pi(\mathbf{x}), \pi\left(Q_{\mu\nu}\right)$ are selfadjoint and $[\pi(x_\mu), \pi(x_\nu)]$ have selfadjoint closures $i\lambda_P^2\pi\left(Q_{\mu\nu}(\tau)\right)$[7]. Moreover, we assume the existence of a common invariant dense domain D for the operators $\pi(\mathbf{x}), \pi\left(Q_{\mu\nu}\right)$ and $\alpha(\tau)$, where α is any complex valued smooth function on $\mathrm{sp}(\tau)$. In the following, we shall drop the symbol π unless necessary.

Requirement 2) above now becomes: the non commutative coordinates transform "classically" under the given unitary action of $SO(3) \ltimes \mathbb{R}^3$. That is, with $R \in SO(3)$ and $R_{\mu\nu}$ having the obvious meaning,

$$U(R)\mathbf{x}U(R)^{-1} = R\mathbf{x}, \quad U(R)Q_{\mu\nu}U(R)^{-1} = R_{\mu\rho}R_{\nu\sigma}Q^{\rho\sigma}.$$

[7] Throughout this operators will be said to commute when they strongly commute, that is have commuting spectral resolutions.

Moreover, for $\mathbf{a} \in \mathbb{R}^3$, we must have

$$U(\mathbf{a})\mathbf{x}U(\mathbf{a})^{-1} = \mathbf{x} + \mathbf{a}\mathbb{I}, \qquad U(\mathbf{a})Q_{\mu\nu}U(\mathbf{a})^{-1} = Q_{\mu\nu}.$$

The generator τ must of course be invariant under both classes of transformations. Then, for $Q(\tau)$ to be a 2-tensor under the action of $SO(3)$, it is necessary and sufficient that its electric and magnetic component $\boldsymbol{e} = (Q_{01}, Q_{02}, Q_{03})$ and $\boldsymbol{m} = (Q_{12}, Q_{32}, Q_{13})$ of Q transform as vectors.

As in the case of the DFR model, our non commutative Friedmann spacetimes will be (to a large extent) characterised by the invariant quantities which can be constructed out of the Q's. Thus we impose the following *Friedmann Quantum Conditions*:

$$\boldsymbol{e}^2 = h_1(\tau)^2\mathbb{I}, \quad \boldsymbol{m}^2 = h_2(\tau)^2\mathbb{I}, \quad \boldsymbol{e}\cdot\boldsymbol{m} = h_3(\tau)\mathbb{I}, \tag{37}$$

where the dot indicates the usual euclidean scalar product and the functions $h_i\colon \mathrm{sp}(\tau) \to \mathbb{R}$ are assumed to be smooth with $h_1, h_2 > 0$.

We will determine these function through the uncertainty relations (34), (35).

By Remark 4.1 the operators $h_1(\tau), h_2(\tau)$ have inverses $h_1(\tau)^{-1}, h_2(\tau)^{-1}$ leaving D invariant and by (36) they commute with the operators $Q_{\mu\nu}$. Hence we can consider the operators $\tilde{\boldsymbol{e}} = h_1(\tau)^{-1}\boldsymbol{e}, \tilde{\boldsymbol{m}} = h_2(\tau)^{-1}\boldsymbol{m}$. They leave D invariant and commute with each other by (36). Moreover, by the Friedmann Quantum Conditions they are *bounded*. An important role will be played by the joint spectrum Z of $\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{m}}$. Writing $Z \ni \zeta = (e, m)$, we see that by covariance Z is a G-space with respect to the action of the group $SO(3)$ given by $\zeta = (e, m) \to \zeta' = (Re, Rm)$. Thus Z is included in the set of pairs of real three-vectors e, m satisfying

$$e^2 = 1, \quad m^2 = 1, \tag{38}$$

and hence, being closed, it is compact.

We now make one more assumption, which will have very important consequences.

- The operators $\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{m}}$ are central, that is $[\tilde{\boldsymbol{e}}, x_i] = [\tilde{\boldsymbol{m}}, x_i] = 0$, with $i = 1, 2, 3$.

It follows that the commutation relation (36) may now be written, with obvious notation,

$$[\tau, \boldsymbol{x}] = i\lambda_P^2 h_1(\tau)\tilde{\boldsymbol{e}}, \qquad [\boldsymbol{x}, \boldsymbol{x}] = i\lambda_P^2 h_2(\tau)\tilde{\boldsymbol{m}}. \tag{39}$$

Now, notice that if we define the (strong) commutant of a densely defined operator X on $\mathcal{H}$ as

$$(X)'_s := \{B \in \mathcal{B}(\mathcal{H}) \,:\, BX \subset XB\},$$

we immediately see that the (strong) commutativity $\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{m}}$ with the selfadjoint operators $\mathbf{x}$ implies that $\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{m}} \in (x_i)'_s$, $i = 1, 2, 3$ and similarly replacing x_i with τ.

Proposition 4.2. Let the operators $\tau, x_i, \boldsymbol{e}, \boldsymbol{m}$, with $i = 1, 2, 3$, be as above. Then the uncertainty relations

$$c\Delta_\omega\tau \sum_{j=1}^{3} \Delta_\omega x_j \geq \frac{\lambda_P^2}{2}|\omega(h_1(\tau))|, \tag{40}$$

$$\sum_{1\leq j<k\leq 3} \Delta_\omega x_j \Delta_\omega x_k \geq \frac{\lambda_P^2}{2}|\omega(h_2(\tau))|, \tag{41}$$

hold for all vector states $\omega = \omega_\psi$, $\psi \in D$.

Proof. Let $\mathcal{M}$ be the abelian von Neumann algebra generated by $\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{m}}$. By [2, Thm. 2 of part II.6], one has a direct integral decomposition

$$\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(\zeta) d\mu(\zeta),$$

such that $\mathcal{M}$ is the algebra of (bounded) diagonal operators. Since any one of $x_\mu = (\tau, x_i)$ is a closed operator, its strong commutant is weakly closed [7, Lemma 7.2.8], and therefore $\mathcal{M} \subset (x_\mu)'_s$. This implies, by [7, Prop. 12.1.7], that X also decomposes along the direct integral

$$x_\mu = \int_Z^{\oplus} X(\zeta) d\mu(\zeta), \tag{42}$$

with selfadjoint $x_\mu(\zeta)$ on $\mathcal{H}(\zeta)$ such that $D(x_\mu) = \{\psi \in \mathcal{H} : \psi(\zeta) \in D(x_\mu(\zeta)) \text{ for a.e. } \zeta \in Z\}$. Therefore, with $\omega_\zeta := \omega_{\psi(\zeta)}$, $\psi \in D$, an argument similar to that in [15, Prop. 3.3] shows that

$$\Delta_\omega \tau \Delta_\omega x_j \geq \int_Z \Delta_{\omega_\zeta} \tau(\zeta) \Delta_{\omega_\zeta} x_j(\zeta) d\mu(\zeta)$$

and similarly for $\Delta_\omega x_j \Delta_\omega x_k$. Moreover

$$\int_Z \sum_{k=1}^{3} \Delta_{\omega_\zeta}(\tau) \Delta_{\omega_\zeta}(x_i) d\mu(\zeta) \geq \int_Z \sqrt{\sum_{k=1}^{3} \Delta_{\omega_\zeta}(\tau)^2 \Delta_{\omega_\zeta}(x_i)^2 d\mu(\zeta)} \geq$$
$$\geq \frac{1}{2} \int_Z \big\| \omega_\zeta\big([\tau(\zeta), \boldsymbol{x}(\zeta)]\big) \big\| d\mu(\zeta)$$

Similarly,

$$\int_Z \sum_{1 \leq j < k \leq 3} \Delta_{\omega_\zeta}(x_i) \Delta_{\omega_\zeta}(x_k) d\mu(\zeta) \geq \int_Z \sqrt{\sum_{1 \leq j < k \leq 3} \Delta_{\omega_\zeta}(x_i)^2 \Delta_{\omega_\zeta}(x_k)^2} d\mu(\zeta) \geq$$
$$= \frac{1}{2} \int_Z \sqrt{\sum_{1 \leq j < k \leq 3} \big| \omega_\zeta\big([x_j(\zeta), x_k(\zeta)]\big) \big|^2} d\mu_\omega(\zeta).$$

It is also clear that $\mathcal{M}$ commutes with the spectral projections of τ, which easily implies $\mathcal{M} \subset (F(\tau))'_s$, i.e., the decomposability of $h_1(\tau)$ too. Then, from the obvious relation

$$\int_Z^{\oplus} [\tau(\zeta), x_i(\zeta)] \psi(\zeta)\, d\mu(\zeta) = [\tau, x_j]\psi = i h_1(\tau) e_i \psi$$
$$= \int_Z^{\oplus} i h_1(\tau)(\zeta) \tilde{e}_i(\zeta) \psi(\zeta)\, d\mu(\zeta),$$

valid for $\psi \in D$, there follows, for a.e. $\zeta \in Z$,

$$\big\| \omega_\zeta\big([\tau(\zeta), \boldsymbol{x}(\zeta)]\big) \big\| = \big\| \omega_\zeta\big(h_1(\tau)(\zeta)\big) \tilde{\boldsymbol{e}}(\zeta) \big\| = \big| \omega_\zeta\big(h_1(\tau)(\zeta)\big) \big|,$$

having used the fact that $\tilde{e}_i$ is diagonal and $\|\tilde{\boldsymbol{e}}(\zeta)\| = 1$. By a similar reasoning one also has

$$\sum_{1 \leq j < k \leq 3} \big| \omega_\zeta\big([x_i(\zeta), x_j(\zeta)]\big) \big|^2 = \big\| \omega_\zeta\big(h_2(\tau)(\zeta)\big) \tilde{\boldsymbol{m}}(\zeta) \big\|^2 = \big| \omega_\zeta\big(h_2(\tau)(\zeta)\big) \big|^2.$$

Finally, we recall that by assumption h_1, h_2 are positive functions. $\square$

Thanks to the previous result, we see that the uncertainty relations (34), (35) are satisfied if we set $h_1 = h_2 = a^{-2} f(H) \doteq F$ (notice that F is regular and stricly positive). We also observe that the result holds for *any* a, H.

The above assumptions on the commutators $Q_{\mu\nu}$ have interesting consequences, but to show this we will need the following technical result (see [26]).

Proposition 4.3. Let τ, x, E be selfadjoint operators on a Hilbert space $\mathcal{H}$, $I \subset \mathbb{R}$ an interval containing the spectrum of τ, $F : I \to \mathbb{R}$ a Borel function, and assume that E strongly commutes with τ, x and that there is a dense subspace $D \subset \mathcal{H}$ contained in $D([\tau, x]) \cap D(F(\tau)E)$ such that

$$[\tau, x]\psi = iF(\tau)E\psi, \qquad \psi \in D.$$

Assume moreover that $\alpha : I \to \mathbb{R}$ is a C^1 function and that the dense subspace D is invariant for $x, \tau, (\tau \pm i)^{-1}$ and is contained in $D(\alpha(\tau)) \cap D(\alpha F(\tau)E) \cap D(\alpha' F(\tau)E)$. Then there holds, for all $\psi \in D$,

$$[\alpha(\tau), x]\psi = i\alpha'(\tau)F(\tau)E\psi.$$

Now we are ready to prove the following

Proposition 4.4. Let the covariant generators $\tau, \mathbf{x}, \boldsymbol{e}, \boldsymbol{m}$ satisfy (39) and the assumptions of Remark 4.1. Then $\tilde{\boldsymbol{e}} \cdot \tilde{\boldsymbol{m}} = 0$.

Proof. From equations (39) and the assumed centrality of $\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{m}}$, we see that Proposition 4.3 applies. Thus

$$[\boldsymbol{e}, x_i] = i\lambda_P^2 F'(\tau)F(\tau)\tilde{\boldsymbol{e}}\tilde{e}_i, \qquad [\boldsymbol{m}, x_i] = i\lambda_P^2 F'(\tau)F(\tau)\tilde{\boldsymbol{m}}\tilde{e}_i.$$

It follows that the only non trivial Jacobi identity for the x_μ is satisfied if

$$F(\tau)F'(\tau)\tilde{\boldsymbol{e}} \cdot \tilde{\boldsymbol{m}} = 0. \tag{43}$$

The operator on the left hand side being a joint functional calculus with a regular function, one concludes that such function must be identically vanishing on the joint spectrum of the operators. The result thus follows from $F, F' \neq 0$. □

In view of Requirement 3) above, we regard this as a *no go* result. In fact, taking into account that $F \to 1$ in the flat limit, we see that the DFR model cannot be recovered. In particular, while we expect that the first of the DFR Quantum Conditions (6) can be met, this is clearly impossible for the second. Instead, we see we recover a restriction to $SO(3)$ symmetry (we neglect translations) of the dilation covariant DFR-like model studied in [27], with a restriction to the corresponding Z (which now happens to be as in (58)) of the joint spectrum of the analogs of the operators $\boldsymbol{e}, \boldsymbol{m}$. Should we enlarge our symmetry group to the full Lorentz group we would be forced to do so with Z too, thus losing the minimal length λ_P. This problem will be addressed in the next section.

5. A quantum FRW with (almost) DFR Minkowskian limit

In order to comply with requirement 3) in Section 4, in the spirit of [16] we add a further auxiliary generator S to the (conformal) coordinates τ, x_i, $i = 1, 2, 3$, and assume commutation relations of the form

$$[\tau, x_i] = i\alpha(\tau)\tilde{e}_i, \qquad [x_i, x_j] = i\varepsilon_{ijk}(\beta(\tau)\tilde{m}_{1k} + \gamma(\tau)\tilde{m}_{2k}S) \tag{44}$$

$$[\tau, S] = 0, \qquad [x_i, S] = iC_i. \tag{45}$$

with $\tilde{e}_i, \tilde{m}_{ai}, C_i,\ i = 1,2,3,\ a = 1,2$, central elements, and α, β, γ given regular functions. As in Section 4, we will assume that the assumption of Remark 4.1, obviously extended to the operators $\boldsymbol{C}, S$ are satisfied. In particular, the operators $\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{m}}_1, \tilde{\boldsymbol{m}}_2$ are assumed to be central.

As before, the arbitrariness in the functions α, β, γ is restricted by the necessity of satisfying the Jacobi identities and Proposition 4.3 guarantees that the formal manipulations are indeed allowed. They imply

$$\beta'(\tau)\alpha(\tau)\tilde{\boldsymbol{m}}_1 \cdot \tilde{\boldsymbol{e}} - \gamma(\tau)\tilde{\boldsymbol{m}}_2 \cdot \boldsymbol{C} + \gamma'(\tau)\alpha(\tau)(\tilde{\boldsymbol{m}}_2 \cdot \tilde{\boldsymbol{e}})S = 0.$$

Once again (see the proof of Proposition 4.4) the operator on the left hand side are a joint functional calculus with a regular function, so that such function must be identically vanishing and therefore one gets the equations

$$\beta'(\tau)\alpha(\tau)\tilde{\boldsymbol{m}}_1 \cdot \tilde{\boldsymbol{e}} = \gamma(\tau)\tilde{\boldsymbol{m}}_2 \cdot \boldsymbol{C}, \qquad \gamma'(\tau)\alpha(\tau)(\tilde{\boldsymbol{m}}_2 \cdot \tilde{\boldsymbol{e}}) = 0.$$

Discarding the solution $\gamma = \text{const}$ of the second equation, this implies the conditions

$$\beta'(\tau)\alpha(\tau) = k\gamma(\tau), \qquad \tilde{\boldsymbol{m}}_2 \cdot \boldsymbol{C} = k\tilde{\boldsymbol{m}}_1 \cdot \tilde{\boldsymbol{e}}, \qquad \tilde{\boldsymbol{m}}_2 \cdot \tilde{\boldsymbol{e}} = 0, \tag{46}$$

for a suitable constant $k \in \mathbb{R}$. Without going further in the analysis of the consequences of (46), we consider the following particular solution

$$\begin{gathered} \alpha(\tau) = \beta(\tau) = F(\tau), \quad \gamma(\tau) = -F'(\tau)F(\tau), \\ \tilde{\boldsymbol{e}} = \tilde{\boldsymbol{m}}_1, \quad \tilde{\boldsymbol{m}} := \tilde{\boldsymbol{m}}_2 = -\boldsymbol{C}, \quad \tilde{\boldsymbol{e}}^2 = \tilde{\boldsymbol{m}}^2 = 1, \quad \tilde{\boldsymbol{e}} \cdot \tilde{\boldsymbol{m}} = 0, \end{gathered} \tag{47}$$

with F the right hand side of the STUR (26). This choice is motivated by a result completely analogous to Proposition 4.2, that is one can prove the inequalities

$$c\Delta_\omega \tau \sum_{j=1}^{3} \Delta_\omega x_j \geq \frac{\lambda_P^2}{2} |\omega(F(\tau))|, \tag{48}$$

$$\sum_{1 \leq j < k \leq 3} \Delta_\omega x_j \Delta_\omega x_k \geq \frac{\lambda_P^2}{2} \left(\left|\omega\big(F(\tau)\big)\right|^2 + \left|\omega\big(F(\tau)F'(\tau)S\big)\right|^2 \right) \geq \lambda_p^2 \left|\omega\big(F(\tau)\big)\right|^2. \tag{49}$$

Moreover, it will be shown below that there are representations in which one can choose states with arbitrary localisation properties such that $\omega\big(F(\tau)F'(\tau)S\big) = 0$. We also point out the fact that on Minkowski spacetime F is a constant, and therefore (44) and (48), (49) reduce to the corresponding relations for the DFR model restricted to $Z^{(1)} = \{\zeta = (e,m) : e = m, e^2 = 1\} \cong \mathbb{S}^2$, a single orbit under the action of $SO(3)$. Again $(e,m) \mapsto (Re, Rm)$, $R \in SO(3)$.

All in all, the non zero commutation relations are thus, with obvious notation,

$$[\tau, \boldsymbol{x}] = iF(\tau)\tilde{\boldsymbol{e}}, \quad [\boldsymbol{x}, \boldsymbol{x}] = iF(\tau)(\tilde{\boldsymbol{e}} - F'(\tau)\tilde{\boldsymbol{m}}S) \tag{50}$$

$$[\tau, S] = 0, \quad [\boldsymbol{x}, S] = -i\tilde{\boldsymbol{m}}. \tag{51}$$

They can be rewritten in a particularly convenient form, as we now show. First, define the new central quantity $\tilde{\boldsymbol{n}} = \tilde{\boldsymbol{e}} \times \tilde{\boldsymbol{m}}$ (the symbol '$\times$' indicates here the exterior product). Then, an easy calculation shows that the scalar (with respect to the action of space rotations) quantities $\tilde{\boldsymbol{e}} \cdot \boldsymbol{x}, \tilde{\boldsymbol{m}} \cdot \boldsymbol{x}, \tilde{\boldsymbol{n}} \cdot \boldsymbol{x}$ satisfy

$$\begin{gathered} [\tau, \tilde{\boldsymbol{e}} \cdot \boldsymbol{x}] = iF(\tau), \quad [\tau, \tilde{\boldsymbol{m}} \cdot \boldsymbol{x}] = [\tau, \tilde{\boldsymbol{n}} \cdot \boldsymbol{x}] = [\tilde{\boldsymbol{e}} \cdot \boldsymbol{x}, \tilde{\boldsymbol{m}} \cdot \boldsymbol{x}] = 0, \\ [\tilde{\boldsymbol{e}} \cdot \boldsymbol{x}, \tilde{\boldsymbol{n}} \cdot \boldsymbol{x}] = iF(\tau)F'(\tau)S, \quad [\tilde{\boldsymbol{m}} \cdot \boldsymbol{x}, \tilde{\boldsymbol{n}} \cdot \boldsymbol{x}] = iF(\tau), \\ [\tilde{\boldsymbol{e}} \cdot \boldsymbol{x}, S] = [\tilde{\boldsymbol{n}} \cdot \boldsymbol{x}, S] = 0, \quad [\tilde{\boldsymbol{m}} \cdot \boldsymbol{x}, S] = -i\mathbb{I}, \\ [\tilde{\boldsymbol{e}} \cdot \boldsymbol{x}, F(\tau)S] = -iF(\tau)F'(\tau)S, \end{gathered}$$

from which we see that

$$[\tilde{\boldsymbol{e}}\cdot\boldsymbol{x}, \tilde{\boldsymbol{n}}\cdot\boldsymbol{x}+F(\tau)S] = [\tilde{\boldsymbol{m}}\cdot\boldsymbol{x}, \tilde{\boldsymbol{n}}\cdot\boldsymbol{x}+F(\tau)S] = [t, \tilde{\boldsymbol{n}}\cdot\boldsymbol{x}+F(\tau)S] = 0. \tag{52}$$

Setting $y = \tilde{\boldsymbol{n}}\cdot\boldsymbol{x}+F(\tau)S$, we summarise our findings:

$$[\tau, \tilde{\boldsymbol{e}}\cdot\boldsymbol{x}] = iF(\tau), \quad [\tilde{\boldsymbol{m}}\cdot\boldsymbol{x}, S] = -i\mathbb{I}, \tag{53}$$

all other commutators being zero. Equations (50), (51) thus decouple and this will be very useful when explicitly constructing concrete realisations. We now turn to discuss this problem.

6. Existence of representations

We will consider first relations (53) and then show how to get (50), (51). It should be clear that a simpler version of the construction below applies for Section 4.

The idea is that a solution is obtained by taking $\mathcal{H} = L^2(\mathbb{R}^2)$ (with the Lebesgue measure $d\xi_1 d\xi_2$) and defining

$$\tau := \Phi(\xi_1), \quad \tilde{\boldsymbol{e}}\cdot\boldsymbol{x} := i\frac{\partial}{\partial\xi_1}, \tag{54}$$

$$\tilde{\boldsymbol{m}}\cdot\boldsymbol{x} := \xi_2, \quad y := c\,\mathbb{I}, \quad S := i\frac{\partial}{\partial\xi_2}, \tag{55}$$

with $c \in \mathbb{R}$ and where a function of (ξ_1, ξ_2) is of course understood as the corresponding multiplication operator. The function Φ is given by the equations (see (53))

$$-(\xi - \underline{\xi}) = \int_{\Phi(\underline{\xi})}^{\Phi(\xi)} \frac{ds}{F(s)} = \tilde{G}(\Phi(\xi)) \;\rightarrow\; \Phi(\xi) = \tilde{G}^{-1}(-\xi), \tag{56}$$

because $\tilde{G}$ is invertible as a function of Φ (this follows from strict positivity of the function F). Finally, we point out that we will later obtain an invariant domain so that all the assumptions in Remark 4.1 from $D = C_c^\infty(\mathbb{R}^2)$. While irreducibility is obvious, the Remark below suggests that in general uniqueness of the realisation under the given assumptions is far from granted. However, we believe that this should be considered a problem only in case the different model have different *physical consequences* and we believe (but for now cannot prove) this is not the case.

Remark 6.1. As a matter of fact, the above construction of the operator τ does not work for arbitrary (increasing) expansion parameters $a(t(\tau))$ (see (21) and below). The point is that the function $\Phi(\xi)$ above as given by (56) might well diverge for *finite* values of ξ and thus be defined only in some open interval. The situation is best illustrated if we take, in the limit of small values of the universal time t, $a(t) \sim t^\gamma$ with $\gamma > 1$. It follows that

$$\tau \sim \int_{t_0}^{t} a(s)^{-1} ds \sim \frac{1}{1-\gamma}\left(t^{1-\gamma} - t_0^{1-\gamma}\right), \qquad \Longrightarrow \qquad F(t(\tau)) \sim \left((1-\gamma)\tau + t_0^{(1-\gamma)}\right)^{-\frac{2+3\gamma}{3(1-\gamma)}},$$

taking into account that $H \sim t^{-1}$ and $F \sim H^{-2/3}$ (see Lemma 3.1). A simple calculation then shows that

$$\Phi(\xi_1) = \frac{5(1-\gamma)}{3}\left(-\xi + \underline{\xi} + \frac{3(1-\gamma)}{5}\Phi(\underline{\xi})^{\frac{5}{3(1-\gamma)}}\right)^{\frac{3(1-\gamma)}{5}},$$

which makes the divergence at finite ξ apparent.

Fortunately, this problem may be circumvented as follows. In case the interval of definition of Φ is bounded, say $(\underline{\xi}_1, \underline{\xi}_2)$, then we simply extend the function Φ by periodicity. The Hilbert space will again be $L^2(\mathbb{R})$ and the derivative the usual Schrödinger operator, while the analog of D above will be given by smooth functions of compact support vanishing sufficiently fast with all their derivatives at the points $\underline{\xi}_1 \pm n(\underline{\xi}_2 - \underline{\xi}_1)$, with $n \in \mathbb{N}$. In case it is semibounded from above, say$(-\infty, \underline{\xi})$, we extend the function Φ to all of $\mathbb{R}$ by a reflection, that is we take $\tilde{\Phi} = -\Phi\left(-(\xi + 2\underline{\xi})\right)$ and redefine Φ in the obvious way. All goes as before, but D will be the smooth functions vanishing sufficiently fast with all their derivatives at the point $\underline{\xi}$. The case of an interval semibounded from below can be treated similarly.

So to say, this construction amounts to patching several copies of our Friedman universe one beside the other, a fact which should have no physical consequences as far as they do not "talk" to each other.

We now note that, given a realisation through operators $\tau, \boldsymbol{x}, S, \tilde{\boldsymbol{e}}, \tilde{\boldsymbol{m}}$, and an element $(\boldsymbol{a}, R) \in \mathbb{R}^3 \rtimes SO(3) =: G$ of the isometry group of Friedmann spacetime, a new realisation on the same Hilbert space is obtained by

$$\tau^{(\boldsymbol{a},R)} := \tau, \quad \boldsymbol{x}^{(\boldsymbol{a},R)} := R\boldsymbol{x} + \boldsymbol{a}\mathbb{I}. \quad S^{(\boldsymbol{a},R)} := S,$$
$$\tilde{\boldsymbol{e}}^{(\boldsymbol{a},R)} := R\tilde{\boldsymbol{e}}, \quad \tilde{\boldsymbol{m}}^{(\boldsymbol{a},R)} := R\tilde{\boldsymbol{m}}.$$

Therefore, we will say that a realisation is $\mathbb{R}^3 \rtimes SO(3)$-covariant if there is a unitary strongly continuous representation U of $\mathbb{R}^3 \rtimes SO(3)$ on $\mathcal{H}$ such that

$$U(\boldsymbol{a}, R) X U(\boldsymbol{a}, R)^* = X^{(\boldsymbol{a},R)},$$

for $X = \tau, \boldsymbol{x}, S, \tilde{\boldsymbol{e}}, \tilde{\boldsymbol{m}}$. Moreover, we call a realisation irreducible if any bounded operator $B \in \mathcal{B}(\mathcal{H})$ which commutes with $\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{m}}$ and such that

$$B \in (S)'_s, \quad B \in (\tau)'_s, \quad B \in (\boldsymbol{x})'_s, \tag{57}$$

is a multiple of the identity. Therefore as expected in an irreducible realisation $\tilde{\boldsymbol{e}} = e\mathbb{I}$, $\tilde{\boldsymbol{m}} = m\mathbb{I}$ with (e, m) belonging to the manifold

$$Z^S := \{(e, m) \in \mathbb{R}^6 \,:\, e^2 = m^2 = 1,\, e \cdot m = 0\}. \tag{58}$$

Being Z^S a single orbit under the action $(e, m) \mapsto (R\boldsymbol{e}, R\boldsymbol{m})$, $R \in SO(3)$, we can assume that

$$e = e_0 := (0, 1, 0), \qquad m = m_0 := (0, 0, 1).$$

Similarly to [15], a covariant realisation is then obtained by a direct integral construction:

$$X := \int_G^{\oplus} \tilde{X}^{(\boldsymbol{a},R)} \, d\boldsymbol{a} dR, \tag{59}$$

where $X = \tau, \boldsymbol{x}, S, e_0\mathbb{I}, m_0\mathbb{I}$, and where of course $U(\boldsymbol{a}, R) = \mathbb{I} \otimes \lambda(\boldsymbol{a}, R)$ on $\int_G^{\oplus} L^2(\mathbb{R}^2) \, d\boldsymbol{a} dR \cong L^2(\mathbb{R}^2) \otimes L^2(G)$ and λ is the left regular representation of G. The construction from D of the corresponding domain $D^{(\boldsymbol{a},R)}$ such that the assumptions of Remark 4.1 are verified is now obvious.

Before concluding this work, we would like to point out that, following the lines of [31], it is possible to provide a completely different construction of the operators X above. In this case the main requirement is that $\mathrm{sp}(\tau)$ should coincide with the range of conformal time in the classical limit and then we build τ simply as the obvious multiplication operator on $L^2(\mathrm{sp}(\tau))$. Then, we set $\widetilde{\beta e} \cdot x = [F(\tau), i\partial/\partial\tau]_+$. The problem here is that in general this operator is only maximally symmetric, a fact that forces us to use a certain "doubling construction" to obtain a selfadjoint operator (see [31]).

As a matter of fact, both the procedures outlined in this section lead to difficulties in the definition of a C^*-algebra corresponding to our non commutative Friedman spacetime. The way out will be presented in [26].

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