Beyond the Standard Model with noncommutative geometry, strolling towards quantum gravity

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Abstract. Noncommutative geometry in its many incarnations appears at the crossroad of many researches in theoretical and mathematical physics: from models of quantum spacetime (with or without breaking of Lorentz symmetry) to loop gravity and string theory, from early considerations on UV-divergencies in quantum field theory to recent models of gauge theories on noncommutative spacetime, from Connes description of the standard model of elementary particles to recent Pati-Salam like extensions. We list several of these applications, emphasizing also the original point of view brought by noncommutative geometry on the nature of time. This text serves as an introduction to the volume of proceedings of the parallel session "Noncommutative geometry and quantum gravity", as a part of the conference "Conceptual and technical challenges in quantum gravity" organized at the University of Rome *La Sapienza* in September 2014.

1. Introduction

Starting with early considerations of Bronstein, Mead, Wheeler, Pauli, Heisenberg and others (see [1] for a nice historical view on the subject), there exists a vast literature dealing with the reasons why putting together general relativity with quantum mechanics asks for a revision of the classical concept of spacetime. A common feature is the emergence of noncommutativity in the description of spacetime itself, as opposed to quantum mechanics alone where noncommutativity lives on the phase space. The appearance of spacetime noncommutativity has a wide range of motivations, from pure mathematics to phenomenological arguments. We shall not try to be exhaustive here, and simply point out some directions of research that have been through interesting developments in the last years. We shall make the distinction between *noncommutative spacetimes* intended as spaces whose coordinates no longer commute, and *spectral geometries* intended as a space whose algebra of functions is non-necessarily commutative.

Noncommutative spacetimes, as recalled in section 2, are obtained as a deformation of a usual space by trading the (commutative) coordinate functions x^μ, x^ν of a manifold with coordinate operators q^μ, q^ν satisfying non-trivial commutation relations. Besides the seminal quantum spacetime model of Doplicher, Fredenhagen, Roberts [2] treated in L. Tomassini talk, such spaces are present in many - if not all - approaches to quantum gravity, including loop quantum gravity, string theory (see P. Aschieri and R. Szabo text), as well as in more phenomenology-oriented models like doubly special relativity, as illustrated by S. Bianco in his presentation.

Noncommutative spacetimes also emerged very early as a possible solution to ultraviolet divergencies in quantum field theory, especially in the work of Snyder [3]. Quantum field and gauge theories on noncommutative spacetimes have thus developed as a theory on their own, independently of any consideration on quantum gravity. Recent advances are presented in the texts of T. Juric, S. Meljanac, A, Samasarov one the one side, and J.-C. Wallet & A. Géré for noncommutative gauge theories [4] on the other.

Nevertheless, revisiting the work of Snyder in the light of nowadays quantum-gravity problematic offers an intriguing point of view, which is presented in V. Astuti paper.

Spectral geometries [5] is the subject of section 3. It consists in a generalization of Gelfand duality between locally compact spaces and C^*-algebras, so that to encompass all the aspects of Riemannian geometry [6] beyond topology. It furnishes a geometrical interpretation of the Lagrangian of the standard model of elementary particles [7, 8], as well as some possibilities to go beyond [9, 10]. Recent progress on that matter are reported in the contribution of A. Devastato. Finally F. Besnard. discusses the extension of this approach to the pseudo-Riemannian case.

2. Noncommutative spacetimes

2.1. Poincaré covariant spacetime

Spacetime as a pseudo-Riemannian manifold loses sense at Planck scale

$$\lambda_P = \sqrt{\frac{G\hbar}{c^3}} \simeq 1.6 \times 10^{-33} cm.$$

This is because an arbitrary accurate localization process requires to concentrate an arbitrary amount of energy in a small volume, yielding the creation of a black hole. To maintain an operational meaning to the measurement process, one should impose some non-zero minimal uncertainties in the simultaneous measurement of spacetime coordinates. This can be realized by promoting the coordinates functions x^μ to operators q^μ satisfying the non-commutative relation

$$[q_\mu, q_\nu] = Q_{\mu\nu}. \tag{1}$$

The behavior of (1) under a Poincaré transformation marks the difference between two classes of models, that both have given birth to an extended literature and many discussions, sometimes quite vivid. For simplicity, let us assume that the commutator of two coordinates is a central element (although some models of non-central commutators have been investigated, see in Tomassini's paper), that is

$$[q_\mu, q_\nu] = i\lambda_P^2 \theta_{\mu\nu} \mathbb{I} \tag{2}$$

where $\mathbb{I}$ is the identity operator in the Hilbert space on which the q^μ are represented and $\Theta = \{\theta_{\mu\nu}\}$ is an antisymmetric matrix. Obviously (2) is not invariant under the action of the Poincaré group

$$q_\mu \mapsto q'_\mu \doteq \Lambda_\mu^\alpha q_\alpha + a_\mu \mathbb{I} \qquad \Lambda \in SO(3,1), a \in \mathbb{R}^3 \tag{3}$$

since

$$\begin{aligned} [q'_\mu, q'_\nu] &= [a_\mu, a_\nu]\mathbb{I} + q_\alpha([a_\mu \mathbb{I}, \Lambda_\nu^\alpha \mathbb{I}] - [a_\nu \mathbb{I}, \Lambda_\mu^\alpha \mathbb{I}]) + \Lambda_\mu^\alpha [q_\alpha, q_\beta] \Lambda_\nu^\beta && (4)\\ &= i\lambda_P^2\, \Lambda_\mu^\alpha \Lambda_\nu^\beta \theta_{\alpha\beta} \mathbb{I} \neq [q'_\mu, q'_\nu]. && (5) \end{aligned}$$

One may ask that Θ transforms under the conjugate action of the Poincaré group, yielding the *Poincaré covariant* model of Doplicher, Fredenhagen, Roberts. The Planck length, viewed as the norm of the tensor $\lambda_P \Theta$ is Poincaré invariant, and there is no modification of the dispersion relation $E^2 = p^2c^2 + m^2c^4$. Applications of this model to cosmology are presented in **Luca Tomassini**'s *Noncommutative Friedmann spacetimes from Penrose-like inequalities.*

Journal of Physics: Conference Series **634** (2015) 012001 doi:10.1088/1742-6596/634/1/012001

2.2. Deformed-Poincaré invariant spacetime

Alternatively one may require the *Poincaré invariance* of the relation (2) by imposing

$$[q'_\mu, q'_\nu] = i\lambda_P^2\theta_{\mu\nu}. \tag{6}$$

This means that the symmetry group of the quantum space is no longer the Poincaré group, but a quantum group deformation of it (the so called θ-Poincaré quantum group), characterized by a non-trivial commutation relation for translations

$$[a_\mu, a_\nu] = i\theta_{\mu\nu} - i\theta^{\alpha\beta}\Lambda^\mu_\alpha\Lambda^\nu_\beta. \tag{7}$$

Under this symmetry, λ_P is again Lorentz invariant but there is now a modification of the dispersion relation

$$E^2 = p^2c^2 + m^2c^4 + f(m, p, E). \tag{8}$$

The possible experimental signature of (8) have been intensively explored, also for the Lie algebra- like noncommutativity (κ-Poincaré deformation)

$$[q_\mu, q_\nu] = \kappa\,\epsilon^\rho_{\mu\nu}q_\rho. \tag{9}$$

Such quantum-group deformations of Poincaré symmetries provide a useful tool to implement the idea of *Doubly Special Relativity*, that is the implementation of the Planck length λ_P as an invariant scale, in a similar manner as the speed of light c [11]. This has been interpreted later as a geometry where the space of momenta is curved (*Relative Locality*). Recent developments on that matter are treated in **Stefano Bianco** *Phenomenology from the DSR-deformed relativistic symmetries of 3D quantum gravity via the relative-locality framework.*

2.3. Quantum field theory on noncommutative space

Quantum field theories on noncommutative spacetime were put at the front of the scene by (open) string theory, when it was observed that D-brane world volume acquire a noncommutative deformation in the background of a non-zero B-field. This, together with new developments on closed string and nonassociative algebras, is recalled in **Paolo Aschieri & Richard J. Szabo** *Triproducts, nonassociative strar products and geometry of R-flux string compactifications.*

However one should not forget that quantum field and gauge theories on noncommutative spacetimes have been originally introduced independently of quantum gravity, as a tool to avoid ultraviolet divergencies. This was the original idea of Snyder, that has been somehow subsumed by renormalization. Nevertheless the idea that noncommutative spaces offer a beautiful ground to understand better quantum field and gauge theories, especially their renormalization properties, has been intensively investigated in the last decade. A scalar field model on a generalized κ-Minkowski space is presented in **Tajron Juric, Stjepan Meljanac, and Andjelo Samsarov** *Light-like κ-deformation and scalar field theory via Drinfeld twist.* The recent advances in gauge theory on noncommutative spacetimes [12] are reported in **Jean-Christophe Wallet** *Spectral theorem in noncommutative field theories: Jacobi dynamics.*

Finally, to go back to quantum gravity, let us also mention that Snyder's ideas re-thought from a quantum gravity perspective yields intriguing results. Recently, it has been used to question how much of the noncommutativity of the coordinates actually survive the description of physical systems. This is the object of **Valerio Astuti** *Covariant quantum mechanics applied to noncommutative geometry.*

3. Spectral geometry

On may question the physical meaning of the noncommutative coordinate q^μ in (1). As elements of the (abstract) polynomial algebra they generate, the spectrum of the q^μ's is not real, which makes their interpretation as physical observable problematic. A solution is to represent the q^μ's on some Hilbert space, as this is done in quantum mechanics. To do so, it is convenient to view the q_μ's as affiliated to the algebra of compact operators, as pointed out in the Doplicher-Fredenhagen-Roberts. But this indicates that the algebra generated by the coordinates may not be the most accurate tools to describe a quantum space, an algebra suitably associated to the q^μ's can do a better job. With this idea in minds, one has no reason to restrict one's attention to noncommutative deformations of commutative coordinates: there are much more noncommutative algebras to play with ! This idea is enforced by Gelfand duality between commutative C^*-algebras and locally compact spaces, which suggests that a natural definition of a *noncommutative geometry* is an object such that its "algebra of functions" (and not only its coordinates) is noncommutative.

3.1. The standard model and beyond

Connes' theory of spectral triples $(\mathcal{A}, \mathcal{H}, D)$ extends Gelfand duality, beyond topology, so that to encompass all the geometrical aspects of Riemannian geometry.

A spectral triple consists in a involutive algebra $\mathcal{A}$, a faithful representation on $\mathcal{H}$, a densely defined operator D on $\mathcal{H}$ such that $[D, a]$ is bounded and $a[D - \lambda\mathbb{I}]^{-1}$ is compact for any $a \in \mathcal{A}$ and $\lambda \notin$ Sp D. Imposing a set of further conditions, one defines *real* spectral triples, whose paradigmatic example is

$$(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \partial\!\!\!/ = -i\gamma^\mu\partial_\mu), \tag{10}$$

that is the algebra of smooth functions on a closed spin manifold $\mathcal{M}$, acting on the Hilbert space of square integrable spinors, with $D = \partial\!\!\!/$ the Dirac operator. Conversely, one has the following reconstruction theorem [6]: given a real spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A}$ unital commutative, then there exists a compact oriented Riemannian spin manifold $\mathcal{M}$ such that $\mathcal{A} = C^\infty(\mathcal{M})$. These conditions still makes sense for non-commutative $\mathcal{A}$, so that one defines a *noncommutative geometry* as a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is non necessarily commutative. To summarize:

$$\begin{array}{rcl} \text{commutative spectral triple} & \to & \text{noncommutative spectral triple} \\ \updownarrow & & \downarrow \\ \text{Riemannian geometry} & & \text{non-commutative geometry} \end{array}$$

Spectral triple turn out to be a powerful tool to describe the standard model of elementary particles from a purely geometric point of view. The starting point is to view spacetime no more as a manifold, as in general relativity, but as the product of a manifold by a matrix geometry. This allows to incorporate in the geometry the degrees of freedom of the standard model. More precisely, one considers the *almost-commutative* algebra

$$C^\infty(\mathcal{M}) \otimes \mathcal{A}_F, \tag{11}$$

where $\mathcal{A}_F$ is a finite dimensional algebra that carries the gauge group of the standard model (which is retrieved as the group of unitaries of $\mathcal{A}_F$). It acts on the space of fermions, that is $\mathbb{C}^{96}$ (two colored quarks, one neutrino and one electron make 8, that multiplies 2 for the chirality, another 2 for antiparticles and 3 for the number of generations). There is a finite dimensional Dirac operator D_F, that is a 96×96 matrix whose entries are the Yukawa coupling of the fermions and the mixing angles of quarks and neutrinos. A general formula for product of spectral triples yields the generalized Dirac operator $D := \partial\!\!\!/ \otimes \mathbb{I}_{96} + \gamma^5 \otimes D_F$. Bosonic fields

Journal of Physics: Conference Series **634** (2015) 012001 doi:10.1088/1742-6596/634/1/012001

are generated by the so-called fluctuations of the metric, that is the substitution of D with the covariant Dirac operator

$$D_A := D + A + JAJ^{-1} \tag{12}$$

where A is a selfadjoint element of the set of generalized 1-forms $\Omega^1_D := \{a^i[D, b_i]\}$. The asymptotic expansion $\Lambda \to \infty$ of the *spectral action* $\mathrm{Tr}f(\frac{D}{\Lambda})$ [13] where Λ is a cutoff parameter and f a smooth approximation of the characteristic function of the interval $[0, 1]$ yields the bosonic Lagrangian of the standard model (including the Higgs) minimally coupled to Einstein-Hilbert action (in Euclidean signature).

In other terms the standard model appears as a purely gravitational theory, but on a (slightly) noncommutative space. As a bonus, the Higgs field comes out as the component of a connection in the noncommutative part of the geometry.

Practically, the spectral action provides relations between the parameters of the theory at a putative energy of unification. In particular the mass term of the Higgs appears as a function of the input of the models, namely the Yukawa couplings of fermions. Assuming the big-desert hypothesis, the running of this mass under the flow of the renormalization group yields a prediction for the mass of the Higgs of 170 GeV, a value ruled out by Tevatron in 2008.

Since then the Higgs-Brout-Englert boson has been discovered with a mass around 125 GeV. This mass is problematic, or at least intriguing, because it lies just below the threshold of stability, meaning that electroweak vacuum is a metastable state rather than a stable one. One solution to stabilize the electroweak vacuum is to postulate there exists another scalar field, called σ, suitably coupled to the Higgs. Chamseddine and Connes have noticed in [14] that taking into account this new scalar field in the spectral action, by promoting the Yukawa coupling of the right neutrino (which is one of the constant component of the matrix D_F) to a field,

$$k_R \to k_R\sigma, \tag{13}$$

then one obtains the correct coupling to the Higgs as well as a way to pull back the mass of the Higgs from 170 to 126GeV. In [10, 15], as reported in **Agostino Devastato** *Noncommutative geometry, grand symmetry and twisted spectral triple*, its is shown how the substitution (13) can be obtained as a fluctuation of the Dirac operator, but in a slightly modified version inspired by the notion of *twisted spectral triple* introduced previously by Connes and Moscovici. The field σ thus appears as a Higgs-like field associated to a spontaneous symmetry breaking to the standard model of a "grand symmetry" model where the spin degrees of freedom ($C^\infty(\mathcal{M})$ acting on the space of spinors) are mixed with the internal degrees of freedom ($\mathcal{A}_F$ acting on the space of particles).

3.2. Noncommutative space versus noncommutative space-time

Spectral triple provides a generalization of Riemannian geometry to the noncommutative setting, but there is no reconstruction theorem for pseudo-Riemannnian manifolds. Once computed the spectral action, one makes a Wick rotation $t \to it$, as done for instance in the path integral approach to quantum gravity. However one might like to make sense of Minkovskian signature from the beginning. In *Two roads to noncommutative causality*, **Fabien Besnard** presents a state of the art, including his own recent results, on various attempts to incorporate a causal structure in spectral triples.

Let us end this discussion by our own contribution, stressing the algebraic approach on how to put time into the game. This is the idea, sometimes advertised by Connes as "the heart of noncommutative geometry", that time involution is intrinsically contained within the noncommutativity of the algebra. Namely, given a von Neumann algebra $\mathcal{A}$ acting on an Hilbert space $\mathcal{H}$ together with a vector Ω in $\mathcal{H}$ cyclic and separating for the action of $\mathcal{A}$, one defines by

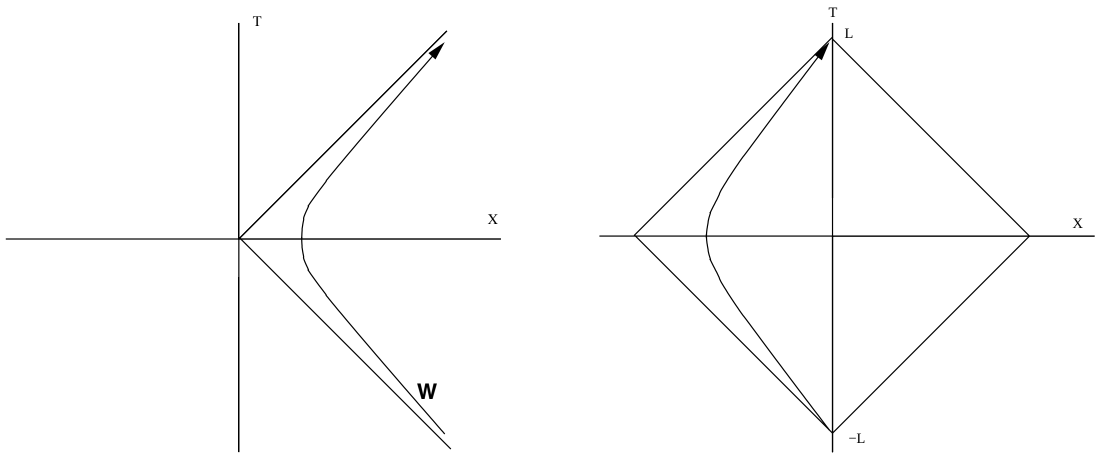

Figure 1. An orbit of the vacuum modular group for the algebra of local observables localized in the Rindler wedge (left) and in a double-cone of Minkowski spacetime (right).

Tomita-Takesaki modular theory a 1-parameter group of automorphism $\sigma_t^\Omega \in \mathrm{Aut}(\mathcal{A})$. Connes has shown that the group obtained from another state Ω' differs from the former only by unitaries, that is

$$\sigma_t^{\Omega'} = U_t^{\Omega'\Omega} \sigma^\Omega (U_t^{\Omega'\Omega})^* \quad \forall t \in \mathbb{R} \tag{14}$$

where the unitary intertwining is given by Connes cocycle $U_t^{\Omega'\Omega}$. Hence there is a canonical group of outer automorphism σ_t canonically associated to the von Neumann algebra $\mathcal{A}$, where

$$\mathrm{Out}(\mathcal{A}(\mathcal{O})) = \mathrm{Aut}(\mathcal{A}(\mathcal{O}))/\mathrm{Inn}(\mathcal{A}(\mathcal{O})).$$

The physical interpretation of this group as a time evolution is enforced by the fact that σ_Ω^t satisfies with respect to Ω the same properties as do in statistical physics the time evolution $e^{iHt} \cdot e^{-iHt}$ with respect to a Gibbs equilibrium state, namely the KMS condition.

An interesting framework to test the interpretation of the modular group as a real "physical" time is algebraic quantum field theory, where the von Neumann algebra is the algebra $\mathcal{A}(\mathcal{O})$ of local observables associated to an open region $\mathcal{O}$ of Minkowski spacetime. In particular, for $\mathcal{O}$ the Rindler Wedge, it is well know that the modular group of the vacuum vector is generated by the boosts. Hence it has a geometrical action whose orbits are the trajectories of uniformly accelerated observers. The KMS condition is interpreted as the vacuum being a thermal equilibrium state for this observer with a temperature proportional to the acceleration (see [16] for a recent critical view on this interpretation, though). A similar analysis holds for double-cone regions of Minkowski spacetime and yields a correction to the Unruh temperature for an observer with a finite life-time [17], see fig. 1.

Interestingly, a similar analysis for a double-cone in a bidimensional conformal field theory permits to compute explicitly the action of Connes cocycle. The field in the double-cone is determined by its components $\psi(x_1), \psi(x_2)$ on each interval on the boundary defining the double-cone (fig. 2). There is a state whose modular group has pure geometrical action $x_1(t), x_2(t)$ (but the orbit is not the trajectory of an observer with constant acceleration) while the modular group for the vacuum mixes this geometric action with a mixing of the components

$$\sqrt{\frac{dx_i}{d\zeta}}\, \sigma_t(\psi(x_i)) = \sum_{k=1,2} O_{ik}(t) \sqrt{\frac{dx_k}{d\zeta}}\, \psi(x_k(t))$$

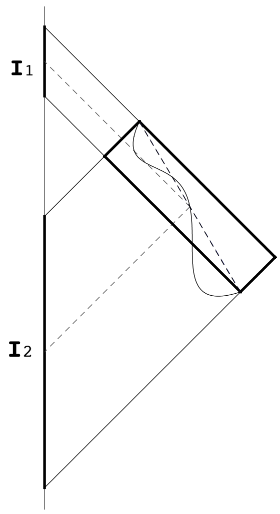

Figure 2. A modular orbit in $2D$-conformal theory with boundary.

where ζ is a suitable parametrization of the orbit. In this sense, the action of the unitary cocyle is non-geometric, and amounts to mix the components of the conformal field on the two intervals. For a further interpretation of this, see [18] and [20].

Besides quantum field theory, the hope is that this way of extracting a time flow from an algebra of observables and a state may be relevant in quantum gravity [19].

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