# Size Effects in Elastic Thin Plates

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**Abstract**. The couple stress theory is applied to describe the size effect in elastic thin plates. An extended equation is developed which is available for the bending problem of the plate whose thickness is close to the material length. A comparison with micropolar theory of plates is made.

## 1. Introduction

In elasticity, $\omega_{ij}=\left(u_{j,i}-u_{i,j}\right)/2$ and $\theta_k=e_{kij}\omega_{ij}/2$ are called, respectively, the rotation tensor and rotation vector. However, the two rotations do not play role in the classical elasticity actually. Mindlin [1] discussed the effect of the rotation gradient on deformation and indicated that the rotation gradient gives rise to bending and twisting of an element. Then, a concept of couple stress was introduced, which acts on the element and produce the bending and twisting. A linear relationship between the couple stress and the curvature is assumed. The proportional coefficient is called the modulus of curvature. Thus, a material constant is introduced into the equations of elasticity, which is a length equal to the square root of the ratio of the modulus of curvature and the modulus of shear. Since the presence of the material length as an intrinsic parameter in the equations of elasticity one can analyze the size effects in deformation problem.

In the classical theory of thin plates, the effect of the rotation gradient on deformation is also neglected. The contribution of the couple stresses on the moment resultant is not taken into account. Of course, the classical equation for thin plates can not be use to describe the size effects.

In the present paper an extended theory for thin plates is developed. In the theory, the bending and twisting of element are allowable, the curvatures of the bending and twisting are considered to be associated with couple stresses. Total moment resultants consist of two parts, one of which is the resultant due to stresses, and the other is that due to couple stresses. In extended equations the material length is explicitly included in the bending stiffness of plates as a size effect term. The presence of the size effect term indicates that the bending stiffness of plates significantly increases and is larger than the classical theory of plates suggests as the thickness of plates is close to the material length. This coincides with the observation in lots of experiments. As a special case, when the thickness of plates is greatly larger than the material length, the extended equation is reduced to the classical equation of plates.

## 2. Curvature and Couple Stress

The gradient of the rotations $\theta_x$ and $\theta_y$ results in the curvatures, which are measured by the curvatures tensor $\underset{\sim}{\chi}$ in the form

$$\chi_{xx}=\frac{\partial^2 w}{\partial x\partial y},\quad \chi_{xy}=\frac{\partial^2 w}{\partial y^2},\quad \chi_{yx}=-\frac{\partial^2 w}{\partial x^2},\quad \chi_{yy}=-\frac{\partial^2 w}{\partial x\partial y} \tag{1}$$

The quantity energetically conjugated with curvature $\underset{\sim}{\chi}$ is couple-stress $\underset{\sim}{\mu}$. The isotropic constitutive equations between $\underset{\sim}{\mu}$ and $\underset{\sim}{\chi}$ are given by Eringen in [3] and Nowacki in [4] as

$$\mu_{ij}=\alpha\delta_{ij}\chi_{kk}+(\beta+\gamma)\chi_{ij}+(\beta-\gamma)\chi_{ji} \tag{2}$$

where the constants $\beta$ and $\gamma$ are the curvature modulus. Substituting (1) into (2) yields

$$\begin{aligned}&\mu_{xx}=2\beta\frac{\partial^2 w}{\partial x\partial y},\quad \mu_{xy}=(\beta+\gamma)\frac{\partial^2 w}{\partial y^2}-(\beta-\gamma)\frac{\partial^2 w}{\partial x^2}\\ &\mu_{yx}=-(\beta+\gamma)\frac{\partial^2 w}{\partial x^2}+(\beta-\gamma)\frac{\partial^2 w}{\partial y^2},\quad \mu_{yy}=-2\beta\frac{\partial^2 w}{\partial x\partial y}\end{aligned} \tag{3}$$

## 3. Moment Resultant

The moment resultants produced by stresses are:

$$\begin{aligned}&M_x^{\sigma}=\int_{-h/2}^{h/2}\sigma_x z dz=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),\quad M_y^{\sigma}=\int_{-h/2}^{h/2}\sigma_y z dz=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\\ &M_{xy}^{\sigma}=M_{yx}^{\sigma}=\int_{-h/2}^{h/2}\tau_S z dz=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\end{aligned} \tag{4}$$

where

$$\tau_S=\frac{1}{2}(\tau_{xy}+\tau_{yx})=G\varepsilon_{xy}=-\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x\partial y}\ \text{ and }\ D=\frac{Eh^3}{12(1-\nu^2)} \tag{5}$$

in which $D$ is the bending rigidity of plates, $G$ is the modulus of shear, $E$ is Young's modulus, $\nu$ is Poisson's ratio, respectively.

Additionally, the moment resultants produced by couple stresses are:

$$\begin{aligned}&M_x^{\mu}=\int_{-h/2}^{h/2}\mu_{yx}dz=h\left[-(\beta+\gamma)\frac{\partial^2 w}{\partial x^2}+(\beta-\gamma)\frac{\partial^2 w}{\partial y^2}\right]\\ &M_y^{\mu}=\int_{-h/2}^{h/2}\mu_{xy}dz=h\left[(\beta+\gamma)\frac{\partial^2 w}{\partial y^2}-(\beta-\gamma)\frac{\partial^2 w}{\partial x^2}\right]\\ &M_{xy}^{\mu}=\int_{-h/2}^{h/2}\mu_{xx}dz=2\beta h\frac{\partial^2 w}{\partial x\partial y},\quad M_{yx}^{\mu}=\int_{-h/2}^{h/2}\mu_{yy}dz=-2\beta h\frac{\partial^2 w}{\partial x\partial y}\end{aligned} \tag{6}$$

Then the total moment resultants are

$$M_x=M_x^{\sigma}+M_x^{\mu},\quad M_y=M_y^{\sigma}-M_y^{\mu},\quad M_{xy}=M_{xy}^{\sigma}-M_{xy}^{\mu},\quad M_{yx}=M_{yx}^{\sigma}+M_{yx}^{\mu} \tag{7}$$

## 4. Strain energy and equilibrium equation

The strain energy of a plate can be written as

$$U = U^{\sigma} + U^{\mu} \tag{8}$$

where

$$U^{\sigma} = \frac{1}{2}\iiint \left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + 2\tau_S \varepsilon_{xy}\right) dxdydz \tag{9}$$

and

$$U^{\mu} = \frac{1}{2}\iiint \left(\mu_{xx}\chi_{xx} + \mu_{yy}\chi_{yy} + \mu_{xy}\chi_{xy} + \mu_{yx}\chi_{yx}\right) dxdydz \tag{10}$$

By defining

$$\{\kappa\} = \left[-\frac{\partial^2 w}{\partial x^2}, -\frac{\partial^2 w}{\partial y^2}, -\frac{\partial^2 w}{\partial x \partial y}, -\frac{\partial^2 w}{\partial x \partial y}\right]^T \tag{11}$$

the strain energy (9) can then be written in the matrix form:

$$U^{\sigma} = \frac{1}{2}\iiint \{\varepsilon\}^T \{\sigma\}\, dxdydz = \frac{1}{2}\iint \{\kappa\}^T \left[D^{\sigma}\right]\{\kappa\}\, dxdy \tag{12}$$

where

$$\left[D^{\sigma}\right] = \frac{Eh^3}{12\left(1-\nu^2\right)}\begin{bmatrix} 1 & \nu & 0 & 0 \\ \nu & 1 & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 1-\nu \end{bmatrix} \tag{13}$$

By defining

$$\{\chi\} = \left[\chi_{yx}, \chi_{xy}, \chi_{xx}, \chi_{yy}\right]^T, \{\mu\} = \left[\mu_{yx}, \mu_{xy}, \mu_{xx}, \mu_{yy}\right]^T \tag{14}$$

the strain energy (10) can then be written in the matrix form:

$$U^{\mu} = \frac{1}{2}\iiint \left[\{\chi\}^T \{\mu\}\right] dxdydz = \frac{1}{2}\iint \{\kappa\}^T \left[D^{\mu}\right]\{\kappa\}\, dxdy \tag{15}$$

where

$$\left[D^{\mu}\right] = \frac{2h\beta}{1-\nu}\begin{bmatrix} 1 & \nu & 0 & 0 \\ \nu & 1 & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 1-\nu \end{bmatrix} \tag{16}$$

Here, the relation $\frac{\gamma-\beta}{\beta+\gamma} = \nu$ is used. Finally, the total stiffness matrix is

$$[D] = \left[D^{\sigma}\right] + \left[D^{\mu}\right] = \left[\frac{Eh^3}{12\left(1-\nu^2\right)} + \frac{2h\beta}{1-\nu}\right]\begin{bmatrix} 1 & \nu & 0 & 0 \\ \nu & 1 & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 1-\nu \end{bmatrix} \tag{17}$$

It may be seen that the ratio of the curvature modulus $\beta$ and the shear modulus $G$ has the dimensions of the square of a length such that

$$\beta = Gl^2 \tag{18}$$

where $l$ is called as material length on which the influence of couple stresses depends strongly.

By substituting (18) into (17) the total stiffness matrix finally becomes

$$[D] = D\left[1+12\left(\frac{l}{h}\right)^2\right]\begin{bmatrix} 1 & \nu & 0 & 0 \\ \nu & 1 & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 1-\nu \end{bmatrix} \tag{19}$$

The total strain energy is written as

$$U = \frac{1}{2}\iint \{\kappa\}^T [D]\{\kappa\}\, dxdy \tag{20}$$

It may be seen that the bending stiffness of plates increases from $D$ to $D\left[1+12\left(\frac{l}{h}\right)^2\right]$ if the local curvature of materials is taken into account. The term $12(l/h)^2$ describes the size effect.

Sequentially, the classical equilibrium equation is extended to the new form:

$$D\left[1+12\left(\frac{l}{h}\right)^2\right]\nabla^2\nabla^2 w = p(x,y) \tag{21}$$

It may be seen that the stiffness in the equilibrium equation (21) is identical with the stiffness in the strain energy (19).

## 5. Conclusions

It can be seen that taking couple stresses into account results in a term $12(l/h)^2$ arising in the stiffness of plates which expresses the size effect. Obviously, when the thickness of plates $h$ is greatly larger than the material length $l$, equation (21) reduces to the classical equation of plates. Contrarily, the size effect term $12(l/h)^2$ even becomes a leading term in the stiffness of plates. It is well known that the material length of most metals is of the order of micrometers. This indicates that the proposed equation (21) in the present paper is applicable for the metal plates of micrometer scale.

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**References**

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