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A Green-Naghdi approach for thermo-electroelasticity

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Abstract. The constitutive relations of piezoelectric ceramics are essentially nonlinear since the so-called piezoelectric moduli depend on the induced strains. Pioneering papers in these topics dealt mainly with the isothermal case. In view of applications, however, thermal effects have to be taken into account in connection with thermo-electric behaviors. Here we briefly compare continuum theories for nonlinear thermoelettroelasticity. In particular we describe an extension of Green-Naghdi thermoelasticity theory for an electrically polarizable and finitely deformable heat conducting elastic continuumn, which interacts with the electric field. In this theory, unlike other, thermal waves propagate at a finite speed.

1. Introduction

The materials exhibiting couplings between elastic, electric, magnetic and thermal fields have attracted great attention in the last decades, in connection with increasing wide use in sensing and actuation. The interest of many researchers turned to mathematical theories of such materials, in order to give certainty to experimental results and applications. Many applications have their mathematical formulation within a linear framework, and the theoretical study began from this context. Foundamental is Nowacki's paper [1], where a uniqueness theorem for the solutions of the initial boundary value problems is proved in linear thermopiezoelectricity referred to a natural state, i.e., with no biasing (or initial) fields. Hence Nowacki [2] also deduced the generalized Hamilton principle and a theorem of reciprocity of work.

For nonlinear continuous media, in order to find the thermodynamic restrictions on the constitutive relations for an electrically polarizable and finitely deformable heat conducting elastic continuum which interacts with the electric field, some authors (e.g. Tiersten [4]) use a theory based on the Clausius-Duhem entropy inequality, following the standard Coleman-Noll procedure [3]. Such theories, which we call here theories of type (A), predict an infinite speed of heat propagation. To avoid such physical contradiction, several papers were written to present continuum theories capable of predicting thermal waves propagating at finite speeds in various media, referred as *second sound*; such theories are often referred as generalized thermoelasticity.

In addition, Rybalko [9] in 2004 showed by experiments that a second-sound (thermal) wave is accompanied by the appearance of electric induction. This fact demonstrates the need for second sound theories of an electrically polarizable and finitely deformable heat conducting elastic continuum which interacts with the electric field.

A group of such theories, referred here as theories of type (B), assumes the Clausius-Duhem inequality together with a generalized Cattaneo equation [5].

Another theoretical approach of continuum thermodynamics is proposed by Green-Naghdi theories [6], [7]. It is based on an integral thermodynamic equality rather than an entropy inequality and uses the notion of thermal displacement α associated with empirical temperature T. Such theories have attracted considerable interest and have been applied in many physical situations, where heat propagation is coupled e.g. with elasticity (see [8]). We refer to them as theories of the type (C).

In [13] there is an extension of the thermodynamic theories [6], [7] to an electrically polarizable and finitely deformable heat conducting elastic continuumn. Here we briefly present it and show the thermodynamic restrictions on response mappings within each type $(A)-(C)$ of theory.

2. Notations and preliminary definitions

Let $\mathcal{E}$ denote a three-dimensional Euclidean point space. We consider a body B whose particles are identified with the positions $\boldsymbol{X} \in \mathcal{E}$ they occupy in a given reference configuration $\boldsymbol{B}$. The material filling B is characterized by a given process class $I\!\!P(B)$ of B as a set of ordered $10-$tuples of functions on $\boldsymbol{B} \times I\!\!R$

$$p = \Big(\boldsymbol{x}(.),\, \theta(.),\, \varphi(.),\, \varepsilon(.),\, \eta(.),\, \boldsymbol{\tau}(.),\, \boldsymbol{P}(.),\, \boldsymbol{q}(.),\, \boldsymbol{b}(.),\, r(.)\Big) \in I\!\!P(B) \tag{1}$$

defined with respect to $\boldsymbol{B}$, satisfying the balance laws of linear momentum, moment of momentum, energy, an entropy inequality or equality and the field equations of electrostatics, where

- $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X},\, t)$ is the *motion*, $\boldsymbol{v} = \dot{\boldsymbol{x}}(\boldsymbol{X},\, t)$ is the *velocity*,
- $\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$ is the *deformaton gradient*,
- $\theta = \theta(\boldsymbol{X},\, t) > 0$ is the *absolute temperature*, $\boldsymbol{g} = \frac{\partial \theta}{\partial \boldsymbol{x}}$ is the *temperature gradient*,
- $\varphi = \varphi(\boldsymbol{X},\, t)$ is the *electric potential*,
- $\boldsymbol{E}^M = -\nabla_{\boldsymbol{x}} \varphi$ is the *(Maxwellian) spatial electric vector*
- $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{X},\, t)$ is the *polarization vector*, $\boldsymbol{\pi} = \boldsymbol{P}/\rho$,
- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}^M + \boldsymbol{P}$ is the *electric displacement* ($\varepsilon_0 =$ *vacuum electric permittivity*),
- $\varepsilon = \varepsilon(\boldsymbol{X},\, t)$ is the specific *internal energy* per unit mass,
- $\eta = \eta(\boldsymbol{X},\, t)$ is the specific *entropy* per unit mass,
- $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{X},\, t)$ $\Big(\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{X},\, t)\Big)$ is the Cauchy stress tensor,
- $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{X},\, t)$ is the spatial *heat flux vector*,
- $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{X},\, t)$ is the external specific *body force* per unit mass,
- $r = r(\boldsymbol{X},\, t)$ is the *radiating heating* per unit mass

The *free energy function* is then defined by

$$\psi = \varepsilon - \theta\eta - \boldsymbol{E}^M \cdot \boldsymbol{\pi} \tag{2}$$

3. Theories of type (A)

3.1. Local balance laws in spatial form

Under suitable assumptions of regularity the usual integral forms of the balance laws of linear momentum, moment of momentum, energy, the field equations of electrostatics, and the entropy inequality are equivalent to the spatial field equations

$$\rho\dot{\boldsymbol{v}} = div\boldsymbol{\tau} + \boldsymbol{P} \cdot \nabla_x \boldsymbol{E}^M + \rho\boldsymbol{b}\,, \tag{3}$$

$$\rho\dot{\varepsilon} = \boldsymbol{\tau} \cdot \nabla\boldsymbol{v} - div\boldsymbol{q} + \boldsymbol{E}^M \cdot \rho\dot{\boldsymbol{\pi}} + \rho r\,, \tag{4}$$

$$\boldsymbol{E}^M = -\nabla_{\boldsymbol{x}}\varphi\,, \qquad div\boldsymbol{D} = \boldsymbol{0}\,, \tag{5}$$

$$\rho\dot{\eta} \geq \rho(r/\theta) - div(\boldsymbol{q}/\theta)\,. \tag{6}$$

3.2. Constitutive Assumptions in Spatial Form

Let $\mathcal{D}$ be an open, simply connected domain consisting of $4-$tuples $(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{g})$, and assume that if $(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{g}) \in \mathcal{D}$, then $(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{0}) \in \mathcal{D}$.

For every $p \in I\!P(B)$ the specific free energy $\psi(\boldsymbol{X},\, t)$, the specific entropy $\eta(\boldsymbol{X},\, t)$, the Cauchy stress tensor $\boldsymbol{T}(\boldsymbol{X},\, t)$, the specific polarization vector $\boldsymbol{P}(\boldsymbol{X},\, t)$, and the heat flux $\boldsymbol{q}(\boldsymbol{X},\, t)$ are given by continuously differentiable functions on $\mathcal{D}$ such that

$$\psi = \overline{\psi}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{g})\,, \tag{7}$$

$$\eta = \overline{\eta}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{g})\,, \tag{8}$$

$$\boldsymbol{\tau} = \overline{\boldsymbol{\tau}}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{g})\,, \tag{9}$$

$$\boldsymbol{P} = \overline{\boldsymbol{P}}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{g})\,, \tag{10}$$

$$\boldsymbol{q} = \overline{\boldsymbol{q}}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{g})\,. \tag{11}$$

Note that the dependence upon $\boldsymbol{X}$ is not written only for brevity; when the body is not materially homogeneous it becomes active.

3.3. Dissipation Principle

For any given motion, temperature field and electric potential field, the process p constructed from the constitutive equations (7)-(11) belongs to the process class (1) of B. Therefore the constitutive functions (7)-(11) are compatible with the second law of thermodynamics in the sense that they satisfy the dissipation inequality (6).

3.4. Constitutive restrictions implied by the entropy inequality

The following proposition holds (e.g. see [12]).

The Dissipation Principle is satisfied if and only if the conditions $(i-ii)$ below hold:

(i) *The free energy response function $\overline{\psi}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{g})$ is independent of $\boldsymbol{g}$ and determines the response functions for entropy, first Piola-Kirchhoff stress, and polarization vector through the relations*

$$\overline{\eta}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M) = -\partial_\theta \overline{\psi}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M) \tag{12}$$

$$\overline{\boldsymbol{\tau}}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M) = \rho \boldsymbol{F} \partial_{\boldsymbol{F}} \overline{\psi}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M) \tag{13}$$

$$\overline{\boldsymbol{\pi}}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M) = -\partial_{\boldsymbol{E}^M} \overline{\psi}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M) \tag{14}$$

(ii) *The reduced dissipation inequality (Fourier inequality)*

$$\boldsymbol{q} \cdot \boldsymbol{g} \,\leq\, 0 \tag{15}$$

is satisfied along any process.

4. Theories of type (B)

4.1. Local balance laws in spatial form

They are just (3)-(6) in Section 3.1.

4.2. Spatial Constitutive Assumptions

Let $\mathcal{D}$ be an open and simply connected domain consisting of $5-$tuples $(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q},\, \boldsymbol{g})$ and assume that *if* $(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q},\, \boldsymbol{g}) \in \mathcal{D}$, *then* $(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{0},\, \boldsymbol{0}) \in \mathcal{D}$.

Below we use the time derivative for the heat flux vector

$$\overset{\circ}{\boldsymbol{q}} = \dot{\boldsymbol{q}} - \boldsymbol{L}\boldsymbol{q} + (tr\boldsymbol{L})\boldsymbol{q}\,, \qquad \boldsymbol{L} = grad\boldsymbol{v}\,. \tag{16}$$

The equality $\dot{\boldsymbol{Q}} = J\boldsymbol{F}^{-1}\, \overset{\circ}{\boldsymbol{q}}$, where $\boldsymbol{Q}$ is the material heat flux vector, shows that the spatial counterpart of the material derivative $\dot{\boldsymbol{Q}}$ is represented by $\overset{\circ}{\boldsymbol{q}}$ rather than by $\dot{\boldsymbol{q}}$.

For every $p \in I\!P(B)$ the specific free energy $\psi(\boldsymbol{X},\, t)$, the specific entropy $\eta(\boldsymbol{X},\, t)$, the Cauchy stress tensor $\boldsymbol{\tau}(\boldsymbol{X},\, t)$, the polarization vector $\boldsymbol{P}(\boldsymbol{X},\, t)$, and the time rate of the heat flux $\overset{\circ}{\boldsymbol{q}}\,(\boldsymbol{X},\, t)$ are given by continuously differentiable functions on $\mathcal{D}$ such that

$$\psi = \overline{\psi}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q},\, \boldsymbol{g})\,, \tag{17}$$

$$\eta = \overline{\eta}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q},\, \boldsymbol{g})\,, \tag{18}$$

$$\boldsymbol{\tau} = \overline{\boldsymbol{\tau}}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q},\, \boldsymbol{g})\,, \tag{19}$$

$$\boldsymbol{P} = \overline{\boldsymbol{P}}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q},\, \boldsymbol{g})\,, \tag{20}$$

$$\overset{\circ}{\boldsymbol{q}} = \boldsymbol{h}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q},\, \boldsymbol{g})\,. \tag{21}$$

Further, the tensors $\partial_{\boldsymbol{q}}\boldsymbol{h}(.)$ and $\partial_{\boldsymbol{g}}\boldsymbol{h}(.)$ are non-singular.

4.3. Dissipation Principle

Just as in Section 3.3.

4.4. Constitutive restrictions implied by the entropy inequality

The following proposition holds (see [5]). *The Dissipation Principle is satisfied if and only if the conditions* $(i - ii)$ *below hold:*

(i) *The free energy response function* $\overline{\psi}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q},\, \boldsymbol{g})$ *is independent of the temperature gradient* $\boldsymbol{g}$ *and determines the entropy, the Cauchy stress tensor, and the polarization vector through the relations*

$$\overline{\eta}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q}) = -\frac{\partial\overline{\psi}}{\partial\theta}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q})\,, \tag{22}$$

$$\overline{\boldsymbol{\tau}}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q}) = \rho\boldsymbol{F}\frac{\partial\overline{\psi}}{\partial\boldsymbol{F}}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q})\,, \tag{23}$$

$$\overline{\boldsymbol{\pi}}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q}) = -\frac{\partial\overline{\psi}}{\partial\boldsymbol{E}^M}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q})\,; \tag{24}$$

(ii) *The reduced dissipation inequality*

$$\rho\theta\frac{\partial\overline{\psi}}{\partial\boldsymbol{q}}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q}) \cdot \dot{\boldsymbol{q}} + \boldsymbol{q}\cdot\boldsymbol{g} \,\leq\, 0\,, \tag{25}$$

where

$$\dot{\boldsymbol{q}} = \boldsymbol{h}(\boldsymbol{F},\, \theta, \boldsymbol{E}^M,\, \boldsymbol{q},\, \boldsymbol{g}) + [\boldsymbol{L} - (tr\boldsymbol{L})\boldsymbol{I}]\,\boldsymbol{q}\,, \tag{26}$$

is satisfied.

5. Theories of type (C)

Now we present the extension [13] of Green-Naghdi theories [6], [7] to thermoelectroelasticity, which is based on an integral thermodynamic equality rather than an entropy inequality and use the notion of thermal displacement α associated with empirical temperature T. The following relations hold:

$$\alpha = \alpha(\boldsymbol{x}, t) = \int_0^t T(\boldsymbol{x}, \tau)\, d\tau + \alpha_0(\boldsymbol{x}), \qquad t > 0\,, \tag{27}$$

$$T = \dot{\alpha}\,, \qquad \boldsymbol{\beta} = \frac{\partial \alpha}{\partial \boldsymbol{X}}\,, \qquad \boldsymbol{\gamma} = \frac{\partial T}{\partial \boldsymbol{x}}\,, \qquad \boldsymbol{g} = \frac{\partial \theta}{\partial \boldsymbol{x}} = \frac{\partial \theta}{\partial T}\boldsymbol{\gamma}\,. \tag{28}$$

5.1. Local balance laws in spatial form

$$\begin{aligned} \dot{\rho} + \rho\, div\boldsymbol{v} &= 0\,, \\ \rho\dot{\boldsymbol{v}} &= div\boldsymbol{\tau} + \boldsymbol{P}\cdot\nabla_x\boldsymbol{E}^M + \rho\boldsymbol{f}\,, \\ skw\boldsymbol{\tau} + skw\boldsymbol{T}^E &= \boldsymbol{0}\,, \\ \rho\dot{\eta} &= \rho(s+\xi) - div\boldsymbol{p}\,, \\ \rho\dot{e} &= \boldsymbol{\tau}\cdot\nabla\boldsymbol{v} - div\boldsymbol{q} + \boldsymbol{E}^M\cdot\rho\dot{\boldsymbol{\pi}} + \rho r\,, \\ \nabla_x\cdot\boldsymbol{D} = 0\,, \qquad \boldsymbol{E}^M &= -\nabla_{\boldsymbol{x}}\phi\,, \end{aligned} \tag{29}$$

where $s = r/\theta$ is the *external rate of supply of entropy per unit mass* and ξ is the *internal rate of supply of entropy per unit mass*. Eliminating r between equations $(29)_4$, $(29)_5$ and using the free energy (2) yields the *reduced energy equation*

$$\rho(\dot{\psi} + \dot{\theta}\eta) + \boldsymbol{p}\cdot\boldsymbol{g} + \rho\theta\xi - \boldsymbol{\tau}\cdot\nabla\boldsymbol{v} + \dot{\boldsymbol{E}}^M\cdot\boldsymbol{P} = 0\,, \tag{30}$$

where $\boldsymbol{p} = \boldsymbol{q}/\theta$ is the entropy flux vector per unit area.

5.2. Constitutive equations and Dissipation Principle

Constitutive equations are assumed that have the form

$$\psi = \hat{\psi}(T,\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M) \tag{31}$$

$$\eta = \hat{\eta}(T,\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M) \tag{32}$$

$$\theta = \hat{\theta}(T,\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M) \tag{33}$$

$$\xi = \hat{\xi}(T,\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M) \tag{34}$$

$$\boldsymbol{p} = \hat{\boldsymbol{p}}(T,\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M) \tag{35}$$

$$\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}(T,\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M) \tag{36}$$

$$\boldsymbol{P} = \hat{\boldsymbol{P}}(T,\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M) \tag{37}$$

The dissipation inequality

$$\xi \geq 0 \tag{38}$$

holds along any process of $\mathcal{B}$ ([10], [11]).

5.3. Constitutive restrictions implied by the reduced energy equality

The following proposition holds (e.g. see [13]).

Assume constitutive equations of the forms (31)*, with*

$$\boldsymbol{q} = \theta \boldsymbol{p} \,, \tag{39}$$

$$\frac{\partial \hat{\theta}}{\partial T} > 0 \,, \tag{40}$$

and let the internal energy response function be defined by (2)*. Then the validity of the reduced energy equation* (30) *along any smooth enough process implies the following conditions for the response functions*

$$\psi = \hat{\psi}(T,\, \boldsymbol{\beta},\, \boldsymbol{F},\, \boldsymbol{E}^M) \,, \qquad \theta = \hat{\theta}(T) \,, \tag{41}$$

$$\hat{\boldsymbol{\tau}} = \rho \boldsymbol{F} \frac{\partial \hat{\psi}}{\partial \boldsymbol{F}} \,, \qquad \hat{\boldsymbol{P}} = -\rho \frac{\partial \hat{\psi}}{\partial \boldsymbol{E}^M} \,, \qquad \hat{\eta} = -\frac{\partial \hat{\psi}}{\partial \theta} \,. \tag{42}$$

$$\rho \frac{\partial \hat{\psi}}{\partial \boldsymbol{\beta}}(T,\, \boldsymbol{\beta},\, \boldsymbol{F},\, \boldsymbol{E}^M) \cdot \boldsymbol{F}^T \boldsymbol{\gamma} + \rho \hat{\theta}(T) \hat{\xi}(T,\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M) + \hat{\boldsymbol{p}}(T,\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M) \cdot \boldsymbol{g} = 0 \,. \tag{43}$$

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