4th International Conference on Mathematical Modeling in Physical Sciences (IC-MSquare2015) IOP Publishing
Journal of Physics: Conference Series **633** (2015) 012121 doi:10.1088/1742-6596/633/1/012121

Emergence of periodic behaviours from randomness

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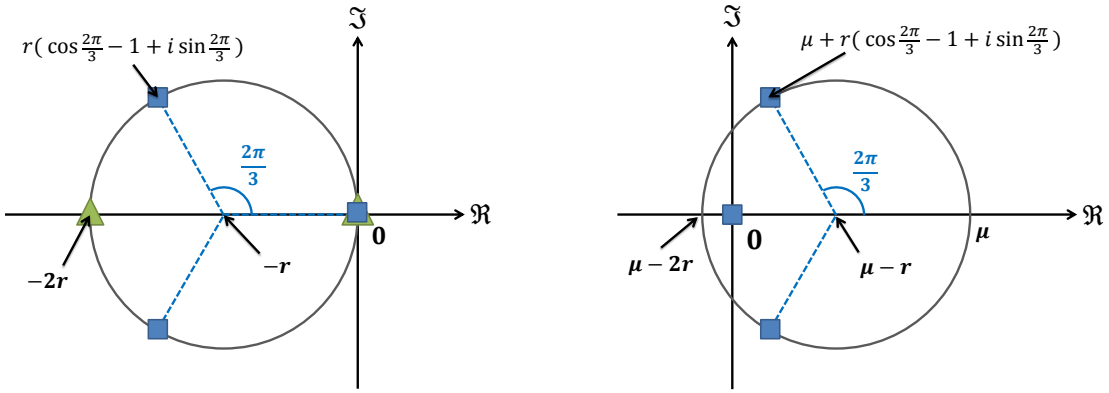
Abstract. This paper discusses how periodic behaviours can arise in discrete systems where the underlying dynamics are purely random. We consider non-interacting particles moving randomly on a network of nodes forming a closed loop. The population dynamics describing the number of particles at a node is a stochastic birth-death process, augmented by particles migrating randomly to adjacent nodes. This can result in the emergence of periodic behaviours occurring because of the interaction between the dynamics of the particles and the spatial structure through which they move. The conditions for this requires the network to comprise of three or more nodes and the migration to have a preferred direction. Moreover there are three classes of equilibria for the populations at nodes that depend on the relative values of the migration and birth rates.

1. Introduction

Periodic effects can be observed in collections of discrete objects, be they fire-flies signalling to attract mates [1], synapses firing in the brain [2] or photons emerging from a cavity [3]. The identification and origin of wave-like properties becomes difficult to interpret without the conceptual aid of a field. In this paper we demonstrate how periodic behaviours may emerge in such systems when the individual components' dynamics are random. In section 2 we describe how a system of non-interacting particles moving spontaneously on a network of nodes forming a loop can display oscillatory behaviours provided two conditions are satisfied – that the network comprise at least three nodes and that there be a preferred direction for the particles spontaneous movement. The oscillations so produced are damped and lead to the particles being uniformly distributed around the network. However, in section 3 we show that the disturbance can be made to persist if the particles' dynamics are augmented by a birth-death process operating at each of the nodes [4]. Then three distinct dynamic equilibria emerge for the distribution of particles on the network whose adoption is dependent on the relative values of the birth and migration rates – an incoherent uniform distribution, a coherent propagating wave-packet and a collapse into a non-propagating state located at a single node.

2. Model

Consider a population of non-interacting particles that are constrained to move around a network of N nodes forming a closed loop. Such a network can be viewed as a line with periodic boundary conditions. We impose that the whole system cannot hold any information other than its current state. This implies that each particle moves through random migration to an adjacent node at constant rate r, the time between these spontaneous events being independent and exponentially distributed. These dynamics and the network's structure can be represented through an $N \times N$



(a) (b)

Figure 1. (a) Argand diagram of the eigenvalues without a birth and death process. Green triangles and blue squares denote the eigenvalues for when $N = 2, 3$ respectively. (b) Corresponding eigenvalues with the birth-death process.

rate-matrix $\mathbf{Q}$ with elements Q_{ij} representing the rate at which a particle jumps from state i to state j and diagonal entries Q_{ii} defined such that the rows sum to zero

$$Q_{ij} = \begin{cases} -r & \text{if } i = j, \\ r & \text{if } j \equiv i+1 \pmod N, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Representing the distribution of particles at time t by the N-dimensional row vector $\mathbf{n}(t)$ we can write down a linear differential equation for the expectation of this distribution $\langle \mathbf{n}(t) \rangle$

$$\frac{d}{dt}\langle \mathbf{n}(t) \rangle = \langle \mathbf{n}(t) \rangle \mathbf{Q} \quad \Longrightarrow \quad \langle \mathbf{n}(t) \rangle = \mathbf{n}(0) \exp(t\mathbf{Q}) . \tag{2}$$

The behaviour of the system is therefore dependent on the eigenvalues ω_k of the rate-matrix

$$\omega_k = r\left(e^{i\frac{2\pi}{N}(k-1)} - 1\right), \qquad k = \{1, 2, \ldots, N\}. \tag{3}$$

The eigenvalue ω_1 is zero and corresponds to the equilibrium distribution of the particles, which is uniform. For $k > 1$, the eigenvalues are complex provided that $N > 2$. The imaginary part of ω_k gives rise to oscillations, but because the real part of ω_k is negative these are damped. Thus the system exhibits damped oscillations towards a uniform equilibrium. The complex spectrum of eigenvalues can be represented graphically as in Figure 1(a) which shows these for a network comprising 2 and 3 nodes. Eigenvalues lie on a circle of radius r with centre located on the real axis at $-r$. If $N = 2$, then the eigenvalues $(0, -2r)$ are real and no oscillatory behaviour can occur. If $N = 3$, two eigenvalues form a complex conjugate pair and can exhibit oscillations, damped due to the negative real part. If the particles were permitted to migrate in both directions around the loop the imaginary part of ω_k would tend towards the real axis, eventually becoming zero when the probabilities of moving in both directions are equal. Hence for oscillations to occur there must be a preferred direction to the migration.

3. Birth and death processes

The damped oscillations can be sustained by augmenting the migration dynamics with a birth-death process acting at each node. The births occur at rate μ for each particle and thus at a

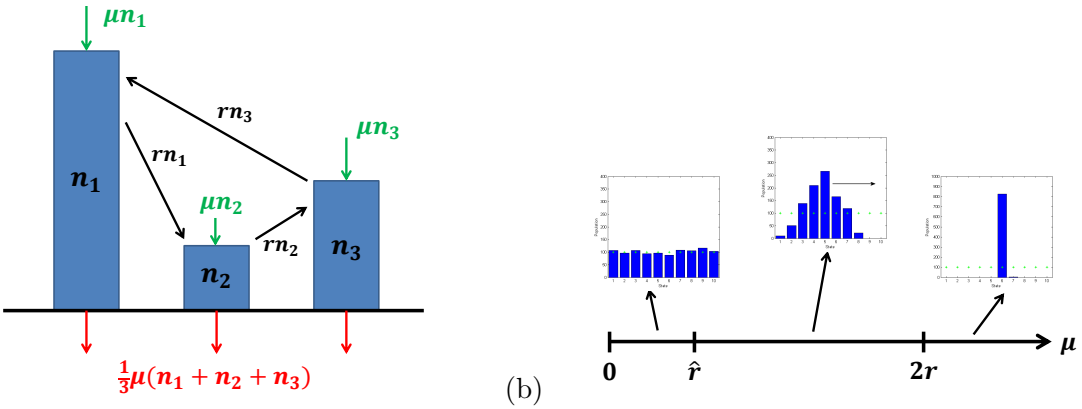


Figure 2. (a) The dynamics for $N = 3$. (b) The rate regimes and corresponding equilibrium distributions.

combined rate μn_i for a node containing a population of n_i particles. In contrast the combined death rate occurring at one of the $N^+(\leq N)$ nodes hosting particles is $\mu n/N^+$, proportional to the *total* instantaneous population size n on the entire network, with the effect that information about the global state of the network is introduced to a node and not just local information via interactions with itself and its nearest neighbours. Thus although the births at a node occur independently of other nodes, the deaths do not. This non-locality leads to the emergence of coherent behaviours. A schematic of these dynamics is illustrated in Figure 2(a) for a three node network. When there exist particles at every node ($N^+ = N$) the dynamics given by equation (2) are modified according to

$$\frac{d}{dt}\langle \mathbf{n}(t)\rangle = \langle \mathbf{n}(t)\rangle \left(\mathbf{Q} + \mu\mathbf{I} - \mu\frac{1}{N}\mathbf{1}_{N\times N}\right) \equiv \langle \mathbf{n}(t)\rangle \mathbf{A} \quad \Longrightarrow \quad \langle \mathbf{n}(t)\rangle = \mathbf{n}(0)\exp(t\mathbf{A})\,. \tag{4}$$

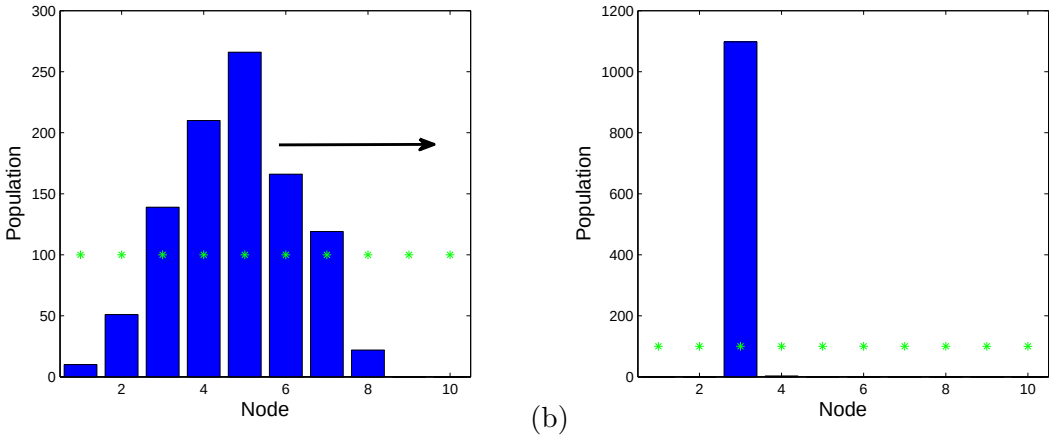
where $\mathbf{I}$ is an $N \times N$ identity matrix and $\mathbf{1}_{N\times N}$ is a matrix with all elements unity. Comparing with equation (2) we have two additional terms on the right-hand side corresponding to the effect of births and deaths respectively. Equation (4) is adjusted when the population at some of the nodes is zero ($N^+ < N$) to account for the changing death rates. These adjustments ensure the populations cannot become negative.
The eigenvalues λ_k of $\mathbf{A}$ can be expressed in terms of the eigenvalues (3) of the rate-matrix

$$\lambda_1 = \omega_1 = 0, \quad \lambda_k = \omega_k + \mu, \qquad k = \{2, 3, \ldots, N\}. \tag{5}$$

All non-zero eigenvalues have been shifted by the birth rate and now lie on a circle with centre $\mu - r$, see Figure 1(b). This shift dramatically changes how the system behaves compared to the incoherent uniform state that results when $\mu = 0$. A diagram of the values of μ corresponding to three different classes of behaviour is given by Figure 2(b). By increasing the birth and death rates we find that a bifurcation occurs when the eigenvalue λ_2 and its conjugate λ_N cross the imaginary axis and the uniform state becomes unstable. This happens when

$$\mu = \hat{r} \equiv r - r\cos\left(\frac{2\pi}{N}\right). \tag{6}$$

Increasing the number of nodes in the network causes $\hat{r}$ to decrease towards zero as N^{-2}, meaning the change is more likely to occur in larger sized networks. Eigenvalues with positive real part allow oscillations to emerge from small perturbations from the stationary distribution, growing



(a) (b)

Figure 3. Simulation results of an initial population of 1000 particles in a network of $N = 10$ nodes in a loop. Both cases started in the stationary distribution with 100 particles at each node, indicated by green asterisks. Corresponding parameter values are (a) $\mu = 0.3r$ and (b) $\mu = 3r$.

in amplitude until the growth is halted by the restriction of non-negative populations. The resulting equilibrium behaviour of the system is a wave of particles that propagates around the loop at a constant speed determined by the imaginary parts of the eigenvalues. Simulation results demonstrating this behaviour are shown in Figure 3(a).

As the rate μ is increased further, the effect of the boundaries becomes more prominent causing the wave-packet to propagate at a slower speed. This effect continues until $\mu = 2r$ at which point a second bifurcation occurs and the equilibrium behaviour of system is characterised by a collapse into a non-propagating state locked at a single node, say node 1. Simulation results are shown in Figure 3(b). The second bifurcation point is explained by considering what happens to the small number of particles that migrate to the neighbouring node 2. The equation for the expected population $\langle n_2 \rangle$ at this node is then

$$\frac{d}{dt}\langle n_2 \rangle = r\langle n_1 \rangle - r\langle n_2 \rangle + \mu\langle n_2 \rangle - \frac{1}{2}\mu\left(\langle n_1 \rangle + \langle n_2 \rangle\right) = \left(r - \frac{1}{2}\mu\right)\left(\langle n_1 \rangle - \langle n_2 \rangle\right). \tag{7}$$

When $\mu > 2r$ the right-hand side is negative and the population at node 2 is driven downwards, the rate of birth and deaths dominate the rate of the migrations.

4. Conclusion

We have described how periodic behaviours are capable of emerging from discrete systems with purely random dynamics, specifically particles migrating around a network can exhibit wave-like properties when a birth-death process is present at each node. Key to the emergent oscillations was arresting the attenuation with a non-local birth-death process at each node. In order for wave-like behaviour to emerge the birth-death rates are required to be large enough to amplify oscillations but small enough to allow particles to migrate.

References

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