

# An accurate solution of elastodynamic problems by numerical local Green's functions

F S Loureiro<sup>1,2\*</sup>, J E A Silva<sup>2</sup> and W J Mansur<sup>3</sup>

<sup>1</sup>Department of Thermal and Fluid Sciences, Federal University of São João del-Rei, Brazil

<sup>2</sup>Postgraduate Program in Computational Modeling, Federal University of Juiz de Fora, Brazil

<sup>3</sup>Department of Civil Engineering, COPPE/Federal University of Rio de Janeiro, Brazil

E-mail: felipe.loureiro@ufjf.edu.br

**Abstract.** Green's function based methodologies for elastodynamics in both time and frequency domains, which can be either numerical or analytical, appear in many branches of physics and engineering. Thus, the development of exact expressions for Green's functions is of great importance. Unfortunately, such expressions are known only for relatively few kinds of geometry, medium and boundary conditions. In this way, due to the difficulty in finding exact Green's functions, specially in the time domain, the present paper presents a solution of the transient elastodynamic equations by a time-stepping technique based on the Explicit Green's Approach method written in terms of the Green's and Step response functions, both being computed numerically by the finite element method. The major feature is the computation of these functions separately by the central difference time integration scheme and locally owing to the principle of causality. More precisely, Green's functions are computed only at  $t = \Delta t$  adopting two time substeps while Step response functions are computed directly without substeps. The proposed time-stepping method shows to be quite accurate with distinct numerical properties not presented in the standard central difference scheme as addressed in the numerical example.

## 1. Introduction

The elastodynamic equations appear in a great deal of applications such as geophysics, soil dynamics, structural dynamics, earthquake engineering, etc [1]. The elastodynamic equations for a general problem in the time domain can be written, in index notation, as follows:

$$\rho \ddot{u}_i - \sigma_{ij,j} = b_i \text{ in } \Omega \times (0, t_f] \quad (1)$$

where  $u_i$ ,  $\sigma_{ij,j}$  stand for the displacement and Cauchy stress tensor, respectively and  $\rho$  is the mass density while  $b_i$  is the given body force. In addition to the above equation, Dirichlet and Neumann type boundary conditions as well as initial conditions are given by:

$$u_i = \bar{u}_i \text{ on } \Gamma_{D_i} \times (0, t_f]; \quad \sigma_{ij} n_j = \bar{t}_i \text{ on } \Gamma_{N_i} \times (0, t_f]; \quad u_i = u_{0i}; \quad \dot{u}_i = v_{0i} \text{ in } \Omega \text{ for } t = 0 \quad (2)$$

where  $\bar{u}_i$ ,  $\bar{t}_i$ ,  $u_{0i}$  and  $v_{0i}$  denote prescribed functions with  $n_j$  being the normal vector components.

Here Eqs. (1)-(2), within the context of elastic wave problems limited to the linear case (i.e.,  $\sigma_{ij} = C_{ijkl} u_{k,l}$ ), are solved numerically by means of the Explicit Green's approach (ExGA) method as described in what follows.



## 2. Time domain solution

In the ExGA method, the problem domain  $\Omega$  is partitioned into non-overlapping element domains, i.e.,  $\bar{\Omega} = \bar{\Omega}^h = \cup_e \bar{\Omega}_e$ . Then an approximation of unknown fields (i.e., Green's functions, displacements and velocities), in a similar way to that of the semi-discrete FEM formulation where the space is discretized independently of the time domain, is employed [2, 3]. Finally, the ExGA displacement and velocity time integral expressions written in a step-by-step manner where the time domain is split into  $N$  equal time intervals of length  $\Delta t = t_{k+1} - t_k = t_f/N$  read (convolution integrals over the time range  $[0, \Delta t]$  have been approximated by the trapezoidal rule) [4]:

$$\mathbf{U}^{k+1} = \mathbf{H}(\Delta t) \mathbf{M} \mathbf{U}^k + \mathbf{G}(\Delta t) \bar{\mathbf{U}}^k; \quad \dot{\mathbf{U}}^{k+1} = \dot{\mathbf{H}}(\Delta t) \mathbf{M} \mathbf{U}^k + \dot{\mathbf{G}}(\Delta t) \bar{\mathbf{U}}^k + \mathbf{M}^{-1} \mathbf{F}^{k+1} \Delta t/2 \quad (3)$$

where  $\bar{\mathbf{U}}^k = \mathbf{M} \dot{\mathbf{U}}^k + \mathbf{F}^k \Delta t/2$ . In addition,  $\mathbf{G}(\Delta t)$  and  $\mathbf{H}(\Delta t)$  stand, respectively, for the Green's and the Step response matrices for the discrete system with entries representing their nodal values. Moreover,  $\mathbf{M}$  and  $\mathbf{F}$  denote, respectively, the standard FEM mass matrix (here assumed to be lumped) and load vector.

The continuum equations that govern the Green's functions and also the Step response functions are those of the problem under consideration (Eqs. (1)-(2)) considering homogeneous (null) boundary conditions. Their computations for a given nodal source point, say  $\mathbf{y} = \mathbf{y}_p$ , along an  $\mathbf{e}_l$  direction by the semi-discrete FEM are carried out by the following discrete weak form:

$$\int_{\Omega_s^h} \rho v_i^h \ddot{w}_i^h + v_{i,j}^h C_{ijkl} w_{k,l}^h d\Omega = 0; \quad \int_{\Omega_s^h} \rho (v_i^h w_i^h - v_i^h w_{0i}) d\Omega = 0; \quad \int_{\Omega_s^h} \rho (v_i^h \dot{w}_i^h - v_i^h \dot{w}_{0i}) d\Omega = 0 \quad (4)$$

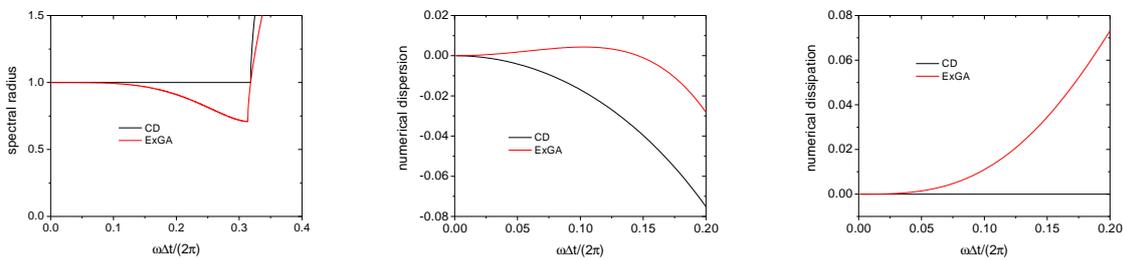
where  $w_i^h$  can be either an approximation for the Green's function ( $w_i^h(\mathbf{x}, t) \equiv G_{il}^h(\mathbf{x}, \mathbf{y}_p, t)$ ) or for the Step response function ( $w_i^h(\mathbf{x}, t) \equiv H_{il}^h(\mathbf{x}, \mathbf{y}_p, t)$ ) components that when evaluated at the nodal points gives the entries of a column of the Green's (or Step response) matrix [3]. Physically, the Green's function can be interpreted as the response of the system due to an initial velocity applied along an  $\mathbf{e}_l$  direction defined as  $\tilde{w}_{0i} = \delta(\mathbf{x} - \mathbf{y}_p) \delta_{il}/\rho$ , whereas the Step response function is the response of the system due to an initial displacement defined as  $w_{0i} = \delta(\mathbf{x} - \mathbf{y}_p) \delta_{il}/\rho$ . Due to the causality, the response is confined to a region defined as  $\Omega_s = \{\mathbf{x}, \mathbf{y} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{y}\| \leq c_d \Delta t\} \subset \Omega \subset \mathbb{R}^2$  called local subdomain that may be very small once the time step length  $\Delta t$  is also small which is the case here. Notice that this upper bound for the local subdomain is defined in terms of the largest wave velocity which is the dilatational (or primary) one ( $c_d$ ); besides, the discrete local subdomain  $\Omega_s^h$  constructed around the nodal source point  $\mathbf{y} = \mathbf{y}_p$  is slightly larger than the subdomain inferred by the principle of causality because of the semi-discretization procedure [3].

The semi-discretization procedure of Eq. (4) gives rise to a system of second-order ordinary differential equation in the time domain. In this sense its solution must be performed adopting, for instance, time integration schemes. In this work, the well-known Central Difference time integration scheme [5] is employed to compute Green's and Step response functions only at time  $t = \Delta t$  and in an independent way. The novel feature of the proposed formulation is as follows:

- Green's functions are computed from  $t = 0$  to  $t = \Delta t/2$  and then from  $t = \Delta t/2$  to  $t = \Delta t$  or, in other words, two time substeps are adopted until reach the desired solution at the end of the first time interval;
- Step response functions are computed directly from  $t = 0$  to  $t = \Delta t$  without substeps.

After computing the Green's and Step response functions locally taking into account all the nodal source points, the Green's and Step response matrices, which are very sparse, are

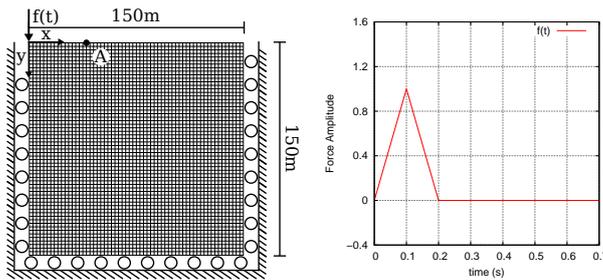
readily constructed, and the time marching process can be initialized by Eq. (3). The final task is to show that such time-stepping technique is convergent yielding meaningful results. To do so, a convergence analysis following the standard literature guideline [5] is performed and the conclusions are summarized in Fig. 1 as a function of the circular frequency. From this figure we can conclude that the proposed ExGA method has some distinct features over the standard Central Difference scheme, namely: (i) the stability constraint obtained from the spectral radius is not decreased and remains the same as that of the CD scheme; (ii) less numerical dispersion; and (iii) numerical dissipation is introduced (spectral radius less than one). It is worthy mentioning that the numerical dissipation in spite of leading to a non-conservative scheme may be beneficial to the solution of some practical problems to dissipate the inaccurate higher frequencies as presented later in the example.



**Figure 1.** Convergence analysis for the ExGA and standard Central Difference (CD) methods.

### 3. Results and discussion

In this section, an example consisting of a homogeneous plane strain two-dimensional soil strip model is considered and the results are analyzed. A sketch of the FE model is depicted in Fig. 2(left); the time-dependent load and the physical properties are in Fig. 2(right) and Table 1, respectively. An element length  $l_e = 2.5$  m and time interval  $\Delta t = 5.5 \times 10^{-3}$  s are adopted.



**Figure 2.** Soil strip FE model (left) and time variation of the load (right).

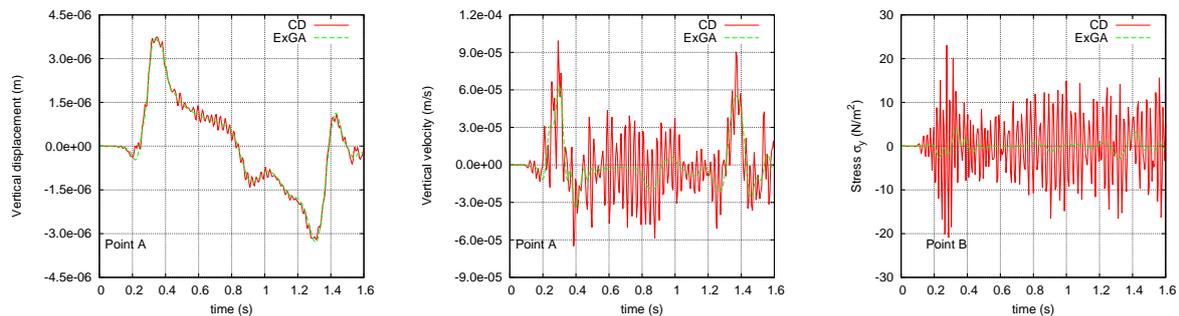
$\nu$	$\rho(\text{kg/m}^3)$	$E(\text{N/m}^2)$
0.25	2000	$2.5 \times 10^8$

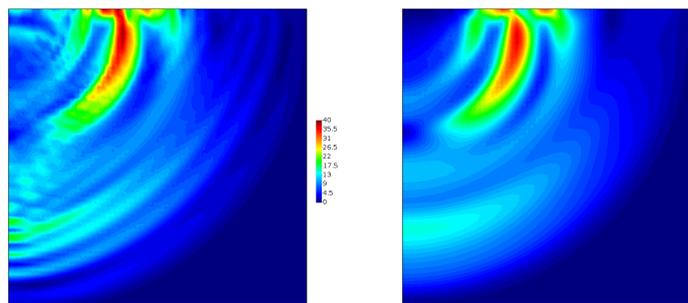
$c_d(\text{m/s})$	$c_s(\text{m/s})$	$c_r(\text{m/s})$
388	224	205.54

**Table 1.** Physical properties of the model.

The time-histories of the vertical displacement and velocity at point A=(40,0) are plotted, respectively, in Fig. 3(left) and Fig. 3(middle), considering both the CD and the ExGA methods. Stress  $\sigma_y$  time-histories at the Gauss integration point B=(41.25,1.125), close to point A, are computed, and the results are presented in Fig. 3(right). Analyzing the figures, it can be observed the presence of high spurious oscillations into the CD results, which can lead to a misinterpretation. On the other hand, due to the capability of filtering spurious oscillations by means of the numerical dissipation, the results concerning the ExGA method are free of oscillations, indicating that accurate and reliable results are achieved. This is better visualized in Fig. 4 where the Von-Mises stress snapshots of the CD and ExGA methods are compared.



**Figure 3.** Comparison of the numerical results: vertical displacement (left), vertical velocity (middle); and stress  $\sigma_y$  (right).



**Figure 4.** Snapshots of the Von-Mises stress at  $t = 0.385$  s: CD(left) and ExGA(right).

#### 4. Conclusions

An improved Explicit Green's Approach (ExGA) method based on concepts of numerical local Green's and Step response functions has been presented in this paper for solving the elastodynamic equations. The use of time substeps to compute only the Green's functions in the ExGA method provided some important numerical properties leading to a very accurate time-stepping scheme. One such property is the numerical dissipation responsible for filtering out undesirable spurious oscillations that may arise in some simulations. It is worth pointing out that the time substeps are applied only once at  $t = \Delta t$  and both the Green's and Step response functions are computed locally due to the causality, resulting at a low computational cost. The numerical example analyzed here has confirmed the effectiveness and robustness of the proposed ExGA method. Further research with the goal of incorporating the so-called fundamental solution (analytical expression) by modifying the standard ExGA formulation [2] is in progress.

#### 5. Acknowledgements

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