Large deviation estimates for a Non-Markovian Lévy generator of big order

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Abstract. We give large deviation estimates for a non-markovian convolution semi-group with a non-local generator of Lévy type of big order and with the standard normalisation of semi-classical analysis. No stochastic process is associated to this semi-group.

1. Introduction

There are much more semi-groups than semi-groups which are represented by stochastic processes. On the other hand, there are a lot of formulas in stochastic analysis which are natural. The theory of pseudodifferential operators [1–3] allow to understand a lot of partial differential equations, including parabolic equations. On the other hand we have imported in the theory of non-markovian semi-groups a lot of tools of stochastic analysis [4–17]. Stochastic analysis formulas are valid for the whole process. Their interpretation for non-markovian semi-groups work only for the semi-group.

In [16] and [17], we have done with the classical normalization of semi-classical analysis [18] Wentzel-Freidlin estimates [19] for four order differential operators. Here we extend the method of [16] to the case of an integro-differential operator of big order which generates a non-markovian convolution semi-group. Normalisation are of Maslov type [18].

2. Statement of the theorems

Let $C_b^\infty(R)$ the set of smooth functions on R with bounded derivatives at each order endowed with its natural topology. $C_b(R)$ is the space of bounded contiunous functions endowed with the uniform norm. L^2 is the space of square integrable norms for the Lebesgue measure. This is an Hilbert space endowed with its natural scalar product $<,>$.

Let h be a smooth positive function on R with compact support such that $h(y) = h(-y)$ and such that $h(y) = 1$ is equal to 1 on $[-\beta, \beta]$.

Let be $\alpha \in [0, 1]$. We introduce the Levy generator acting on $C_b^\infty(R)$:

$$Lf(x) = (-1)^{d+1} \int_R (f(x+y) - f(x) - \sum_{i=1}^{d} \frac{y^{2i}}{2i!} f^{(2i)}(x)) \frac{h(y)}{|y|^{2d+\alpha}} dy \tag{1}$$

$\frac{h(y)}{|y|^{2d+\alpha}} dy$ is called the Lévy measure.

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Theorem 1:: *L is symmetric positive on L^2. It has therefore a natural essentially-self adjoint extension which generates a semi-group of contraction P_t* on L^2.

We consider the Hamiltonian

$$H(\xi) = \int_R (\exp[\xi y] - 1 - \sum_{i=1}^{d} \frac{(\xi y)^{2i}}{2i!}) \frac{h(y)}{|y|^{2d+\alpha}} dy \tag{2}$$

Theorem 2: $H(\xi)$ i *is a smooth convex function equals to* 1 *in* 0.

Associate to it, we consider its Legendre transform:

$$L(p) = \sup_{\xi \in R} (\xi p - H(\xi)) \tag{3}$$

If ϕ is a finite energy function in R, we consider the action functional

$$S(\phi) = \int_0^1 L(\frac{d\phi}{dt}) dt \tag{4}$$

Let us recall some basis of the pseudodifferential calculus. $\hat{f}$ is the Fourier transform of f. Let L_1 be an operator acting on $C_b^\infty(R^d)$ by

$$L_1 f(x) = \int_R a(x, \xi) \hat{f}(\xi) \exp[2\sqrt{-1}\pi \xi x] d\xi \tag{5}$$

We say that $a(.,.)$ is its symbol. If

$$|\frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial \xi^m} a(x, \xi)| \leq C|\xi|^{r-m} \tag{6}$$

and if for $|\xi| > C_0$

$$|a(x, \xi)| \geq C|\xi|^r \tag{7}$$

we say that L_1 is an elliptic operator of order r. Let us recall that our thesis underline the relationship between pseudodifferential operators and Poisson processes [20].

Theorem 3:*L is an elliptic pseudodifferential operator.*

By elliptic theory, it generates a semi-group on $C_b(R)$.

According the theory of semi-classical analysis [19], we consider the symbol L_1^ϵ associated to the symbol $\epsilon^{-1} a(x, \epsilon\xi)$. This leads to the operator

$$L^\epsilon f(x) = (-1)^{d+1} \frac{1}{\epsilon} \int_R (f(x + \epsilon y) - f(x) - \sum_{i=1}^{d} \frac{(\epsilon y)^{2i}}{2i!} f^{(2i)}(x)) \frac{h(y)}{|y|^{2d+\alpha}} dy \tag{8}$$

By elliptic theory L^ϵ generates a semi-group on L^2 and even on $C_b(R)$ P_t^ϵ. We consider its absolute value $|P_t^\epsilon|$. We have

Theorem 4 (Wentzel-Freidlin estimates): *Let O be the complement in R of the interval $[x - \delta, x + \delta]$. We have when $\epsilon \to 0$*

$$\overline{Lim} \epsilon Log |P_1^\epsilon|[1_O](x) \leq - \inf_{\phi(1) \in O} S(\phi) \tag{9}$$

if $d + 1$ is even.

The proof is very similar to the proof of [16], the only difference being in the algebraic treatment of Davies method [21].

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3. Proofs of Theorem 1, Theorem 2 and Theorem 3

Proof of theorem 1: Let us show that L is symmetric. Let f anf g be smooth with compact supports:

$$(-1)^{d+1} < Lf, g > \; = \int_{R^2} g(x)((f(x+y) - f(x) - \sum_{i=1}^{d} \frac{y^{2i}}{2i!} f^{(2i)}(x)) \frac{h(y)}{|y|^{2d+\alpha}}) dxdy \tag{10}$$

The symmetry holds by integrating by parts and since $h(-y) = h(y)$.

Let us show that L is positive. We have if $y > 0$

$$f(x+y) - f(x) = \sum_{i=1}^{2d-1} \frac{y^i}{i!} f^{(i)}(x) + \int_{0<s_1<..<s_{2d}<y} f^{(2d)}(x+s_1) ds_1..ds_{2d} \tag{11}$$

Due to the parity of h, we have only to look at

$$\int_{R\times R^+} f(x) \int_{0<s_1<..<s_{2d}<y} f^{(2d)}(x+s_1) ds_1..ds_{2d} \frac{h(y)}{|y|^{2d+\alpha}} dxdy \tag{12}$$

By integrating by parts, it is equal to:

$$(-1)^d \int_{R\times R^+} f^{(d)}(x) \int_{0<s_1<..<s_{2d}<y} f^{(d)}(x+s_1) ds_1..ds_{2d} \frac{h(y)}{|y|^{2d+\alpha}} dxdy \tag{13}$$

By Cauchy -Schwartz inequality,

$$\int_R f^{(d)}(x) f^{(d)}(x+s_1) dx \le \int_R (f^{(d)}(x))^2 dx \tag{14}$$

Therefore

$$(-1)^{d+1} \int_{R\times R^+} f(x) \int_{0<s_1<..<s_{2d}<y} (f^{(2d)}(x+s_1) - f^{(2d)}(x)) ds_1..ds_{2d} \frac{h(y)}{|y|^{2d+\alpha}} dxdy \ge 0 \tag{15}$$

This shows the result. The fact that the operator has a natural self-adjoint extension which is essentially self-adjoint holds by standard results.$\diamondsuit$

Proof of theorem 2: $H(\xi)$ is smooth. We have clearly

$$H^{(1)}(\xi) = \int_R y(\exp[\xi y] - \sum_{i=1}^{d} \frac{(\xi y)^{2i-1}}{(2i-1)!}) \frac{h(y)}{|y|^{2d+\alpha}} dy \tag{16}$$

$$H^{(2)}(\xi) = \int_R y^2(\exp[\xi y] - 1 - \sum_{i=1}^{d-1} \frac{(\xi y)^{2i}}{2i!}) \frac{h(y)}{|y|^{2d+\alpha}} dy \tag{17}$$

Due to the fact that $h(y) = h(-y)$, the result holds from the fact by induction on l that

$$\exp[y\xi] + \exp[-\xi y] - \sum_{i=0}^{l} \frac{\xi^{2i} y^{2i}}{2i!} \tag{18}$$

is positive convex in ξ.$\diamondsuit$

Proof of theorem 3: Let us compute the symbol of L.

$$f(x) = C\int_R \hat{f}(\xi)\exp[\sqrt{-1}x\xi]d\xi \tag{19}$$

Therefore

$$Lf(x) + \int_R \frac{h(y)}{|y|^{2d+\alpha}}dy\int_R(\exp[\sqrt{-1}(x+y)\xi] - \sum_{i=0}^{2d}\frac{(\sqrt{-1}y\xi)^{2i}}{2i!}\exp[\sqrt{-1}x\xi])\hat{f}(\xi)d\xi =$$

$$\int_R \hat{f}(\xi)\exp[\sqrt{-1}x\xi]d\xi\int_R(\exp[\sqrt{-1}\xi y] - 1 - \sum_{i=1}^{d}\frac{(\sqrt{-1}\xi y)^{2i}}{2i!})\frac{h(y)}{|y|^{2d+\alpha}}dy \tag{20}$$

Therefore the symbol is given by

$$a(\xi) = H(\sqrt{-1}\xi) \tag{21}$$

By puting $y\xi = z$ if $\xi > 0$ we get that

$$a(\xi) = \xi^{(2d-1+\alpha)}\int_R h(\frac{z}{\xi})(cos[z] - \sum_{i=0}^{2d}\frac{(-1)^i z^{2i}}{(2i)!})dz \tag{22}$$

In (22), we consider a smooth h_1 function which is equal to zero near 0 and which is equal to 1 in a neighborhood of the infinity and which takes its values in $[0,1]$ and we write

$$a(\xi) = a_1(\xi) + a_2(\xi) \tag{23}$$

$$a_1(\xi) = \xi^{(2d-1+\alpha)}\int_R h_1(\frac{z}{\xi})h(\frac{z}{\xi})(cos[z] - \sum_{i=0}^{2d}\frac{(-1)^i z^{2i}}{(2i)!})dz \tag{24}$$

By integrating by parts succesively, $|a_1(\xi)| \leq C\xi^{-n}$ for all n. On the other if the support of $1 - h_1$ is small enough, we have

$$|a_2(\xi)| \geq C\xi^{(2d-1+\alpha)} \tag{25}$$

for some positive C. The result arise by symmetry for $\xi < 0$. $\diamondsuit$

4. Proof of the Wentzel-Freidlin estimates

Let us begin by some elementary remarks. We remark that

$$\hat{Lf} = H(\sqrt{-1}.)\hat{f} \tag{26}$$

such that

$$\hat{P_t f} = \exp[-tH(\sqrt{-1}.)]\hat{f} \tag{27}$$

These elementary remarks (which are true a lot of convolution semi-groups) will allow us to adapt the proof of [16].

Lemma 5: *For all $\delta > 0$, all C there exist t_δ such that if $t < t_\delta$*

$$|P_t^\epsilon|(|1_{[x-\delta,x+\delta]^c})(x) \leq \exp[-\frac{C}{\epsilon}] \tag{28}$$

Proof: We consider the semi-group

$$\exp[-\frac{x\xi}{\epsilon}]P_t^\epsilon[\exp[\frac{x'\xi}{\epsilon}]f(x')](x) \tag{29}$$

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The symbol of its generator is

$$F_\xi^\epsilon(\xi') = \frac{1}{\epsilon} H(\epsilon\sqrt{-1}\xi' + \xi) \tag{30}$$

This is the symbol of an elliptic operator which is positive if $|\xi'|$ is big. It generates therefore a semi-group on $C_b(R)$ $Q_t^{\epsilon,\xi}$. We get the expansion

$$F_\xi^\epsilon(\xi') = \frac{H(\xi)}{\epsilon} + H^{(1)}(\xi)\sqrt{-1}\xi' + \epsilon \int_{0<s_1<s_2<1} (\xi')^2 H^{(2)}(\epsilon s_1\sqrt{-1}\xi' + \xi) ds_1 ds_2 + \frac{H(\xi)}{\epsilon} + H^{(1)}\sqrt{-1}\xi' + R_\xi^\epsilon(\xi') \tag{31}$$

Therefore We get

$$\hat{Q_t^{\epsilon,\xi}} f = \exp[-\frac{tH(\xi)}{\epsilon}] \exp[-t(+H^{(1)}\sqrt{-1}\xi' + R_\xi^\epsilon(\xi'))]\hat{f} \tag{32}$$

The uniform norm of $\exp[-t(+H^{(1)}\sqrt{-1}\xi' + R_\xi^\epsilon(\xi'))$ is bounded and the uniform norm of its derivative is bounded by $\exp[C|\xi|]/\epsilon$. Therefore the norm on $C_b(R)$ of $Q_t^{\epsilon,\xi}$ is bounded by $\exp[-\frac{CtH(\xi)}{\epsilon}]\exp[C|\xi|]$. Therefore

$$|P_t^\epsilon|(|1_{[x-\delta,x+\delta]^c})(x) \leq \exp[-\frac{CtH(\xi)}{\epsilon}] \exp[\frac{\delta\xi}{\epsilon}] \exp[C\xi] \tag{33}$$

But $H(\xi) \geq C|\xi|$ if $|\xi| > K(C)$ for all C.$\diamondsuit$

Remark: This inequality where the classical Davies gauge transform plays a fundamental role [21] replace the role of exponential martingales of [19].

When we have proved this lemma, the estimates follow closely the lines of [16] and [19].

We cut the time interval $[0, 1]$ is small intervals of length $[t_i, t_{i+1}]$. By the semi group property we use that

$$|P_1^\epsilon|[1_{[x-\delta,x+\delta]^c}](x) \leq |P_{t_1}^\epsilon|..|P_{1-t_n}^\epsilon|[1_{[x-\delta,x+\delta]^c}](x) \tag{34}$$

In $P_{t_{i+1}-t_i}^\epsilon$, we distinguish if $x_{t_{i-1}}$ and x_{t_i} are far or not. If they are we use the previous lemma. If they are close, we deduce a positive measures $|W_\epsilon|$ on polygonal paths ϕ_t which joins x_{t_i} to $x_{t_{i+1}}$. By the previous lemma, it remains to estimate $|W_\epsilon|[|1_{[x-\delta,x+\delta]^c}(\phi_1)]$. But $|W_\epsilon|$ is a positive measure, we have

$$|W_\epsilon[|[1_{[x-\delta,x+\delta]^c}](\phi_1)] \leq |W_\epsilon|[\exp[\frac{S(\phi)}{\epsilon}]1_{[x-\delta,x+\delta]^c}(\phi_1)] \exp[-\inf_{\phi_1 \in [x-\delta,x+\delta]^c} \frac{S(\phi)}{\epsilon}] \tag{35}$$

Therefore we have only to estimate $|W_\epsilon|[\exp[\frac{S(\phi)}{\epsilon}]1_{[x-\delta,x+\delta]^c}(\phi_1)]$. The sequel follows [19] p 152 ([16]. We can choose some p_i in finite numberrs such that if we put

$$L'(p) = \sup_i (L(p_i) + \frac{\partial}{\partial p} L(p_i)(p - p_i)) \tag{36}$$

we have for all polygonal paths considereded for a small χ

$$L(\frac{d\phi_t}{dt}) - L'(\frac{d\phi_t}{dt}) \leq \chi \tag{37}$$

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Let us put

$$S'(\phi) = \int_0^1 L'(\frac{d\phi_t}{dt})dt \tag{38}$$

Since $|W_\epsilon|$ is a positive measure, we have only to estimate the quantity

$$|W_\epsilon|[\exp[\frac{S'(\phi)}{\epsilon}]1_{[x-\delta,x+\delta]^c}(\phi_1)] \tag{39}$$

We remark that

$$\exp[\sup a_i] \leq \sum \exp[a_i] \tag{40}$$

Moreover

$$L'(p) = \sup(\xi_i p - H(\xi_i)) \tag{41}$$

where $\xi_i = \frac{\partial}{\partial p}L(p_i)$. Therefore it is enough to show that

$$\sup_{x,|\xi|<C} |P^\epsilon_{t_\delta}|[\exp[\frac{\xi}{\epsilon}(x'-x) - t_\delta H(\xi)]](x) \tag{42}$$

has a small blowing up when $\epsilon \to 0$. We do as in the previous lemma. We consider the generator of the semi group

$$f \to P^\epsilon_t[\exp[\frac{\xi}{\epsilon}(x'-x) - tH(\xi)]f](x) \tag{43}$$

Its symbol is

$$\frac{1}{\epsilon}H(\epsilon\sqrt{-1}\xi' + \xi) - \frac{1}{\epsilon}H(\xi) \tag{44}$$

Its asymptotic expansion in ϵ is

$$((H^{(1)}\sqrt{-1}\xi' + R^\epsilon_\xi(\xi') \tag{45}$$

The result follows as in the lemma.$\diamondsuit$

5. Conclusion

We have adapted the standard proof of large deviation estimates of jump processes of [19] (with the standard normalisation of semi-classical analysis [18]) to the case of a non-markov Lévy generator of big order. The main difference with [16] is that the classical gauge transform of Davies [21] induces a simple transformation on the symbol of the Lévy generator [20]

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