

# Compactly supported kernels method of approximate particular solutions for solving elliptic problems

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**Abstract.** In this work the method of approximate particular solutions using compactly supported kernels is investigated. In the work of [1] the globally supported radial kernels for the construction of solution are used, and it is observed that for large scaled PDEs the differentiation matrix is ill-conditioned. We extended the work of [1] for compactly supported kernels, in order to solve large-scaled PDEs engineering sciences. The numerical scheme of the present method of approximate particular solutions is very accurate and simple in implementation. Three benchmark problems are solved by the present numerical scheme and the results are compared to other methods in the literature.

## 1. Introduction

For the approximate solution of various types of PDEs, the method of approximate particular solution (MAPS) is recently developed in [1]. In this approach the approximate particular solution is obtained in such a way to satisfy the differential equation as well as the boundary conditions [2]. In this numerical approach very simple and accurate kernel based numerical scheme is obtained. In this approach, due to the radial symmetry of the special differential operator, the integration is performed in the radial direction. In the present method due the



global structure the system matrix is dense and ill-conditioned, and is unable to solve large-scaled problems.

The finite element and finite difference methods are local in nature, and are very effective in solving large-scaled problems. Like these mesh-dependent methods there are also some kernel based methods which are local in their construction and are very effective for solving large system in irregular domains [3–7].

Apart from these methods there are also several methods to overcome this difficulty such as domain decomposition [8], the greedy algorithm [9], the extended precision arithmetic [10], the improved truncated singular valued decomposition [11], iterative methods [12], fast multipole expansion techniques [13].

Our purpose is to extend the MAPS for solving large-scaled problems. In our work we have adopted the idea of using locally supported kernel functions resulted a localized formulation in the construction of MAPS. In our procedure, we approximate the solution by a linear combination of locally supported kernels. Contrary to global method, the local method have sparse system matrix which can be solved easily and efficiently.

## 2. Description of the method

The idea to extend the method of approximate particular solution (MAPS) in [1] for locally supported radial kernel is similar to the construction of the DRBEM [14], in which the operator of Laplacian is retain as main operator on the left, and all the other terms are shifted to right side. We consider the following elliptic partial differential equation in 2D

$$\Delta u = f(\xi, \eta), (\xi, \eta) \in \Omega, \quad (1)$$

$$\mathcal{B}u = g(\xi, \eta), (\xi, \eta) \in \partial\Omega, \quad (2)$$

where  $\Delta$ , is a linear differential operator and  $\mathcal{B}$  is a boundary differential operator, and  $f(\xi, \eta)$  and  $g(\xi, \eta)$  are known functions. Using radial kernels, an approximate particular solution to (25)-(27) is given by

$$u = \sum_{i=1}^n \lambda_i \Phi(r_i), \quad (3)$$

where  $r_i = \|(\xi, \eta) - (\xi_i, \eta_i)\|$  and  $\{(\xi_i, \eta_i)\}_{i=1}^n$  are called the centers or trial points. In the method of approximate particular solution the equation defined by

$$\Delta \Phi = \phi, \quad (4)$$

can be solved analytically for kernel  $\Phi$  in (19) for a given radial kernel  $\phi$  [15]. Our aim is to solve this equation for locally supported radial kernels.

It was proved in ([16], Lemma 9.15, p. 130) that the transition  $\Phi \rightarrow -\Delta\Phi$  allows to generate new kernels under certain circumstances. Here, the construction explicit and transparent for the standard families of radial kernels. In this construction the radial form of the Laplacian are apply it to any kernel to generate new kernels.

If we write a radial kernel  $K$  in  $f$ -form with  $s = r^2/2 = \|\xi - \eta\|_2^2/2$ , its d-variate Laplacian follows via

$$\frac{\partial}{\partial \xi_j} K(\xi - \eta) = \frac{\partial s}{\partial \xi_j} \frac{d}{ds} f(s) = f'(s)(\xi_j - \eta_j) \quad (5)$$

$$\frac{\partial^2}{\partial \xi_j^2} K(\xi - \eta) = f''(s)(\xi_j - \eta_j)^2 + f'(s) \quad (6)$$

$$-\Delta K(\xi - \eta) = -2sf''(s) - 2df'(s) \tag{7}$$

the radial kernel defined by  $-\Delta\Phi$  is positive definite, if  $\Phi$  is positive definite. For conditionally positive definite  $\Phi$  of order  $m$ , the same argument applies, but the order of conditional positive definiteness of  $\Delta\Phi$  then is  $m - 1$  because of the new factor in the Fourier transform.

**Theorem 1.** [17] *The transition on radial kernels generates a radial kernel consisting of a weighted sum (31)-(7) of two radial kernels, if  $f$  is the  $f$ -form of, and if the action of  $-\Delta$  is valid on the kernel. If, furthermore, the class of kernels is invariant under taking pairs of forward and backward Fourier transforms in arbitrary dimensions. the resulting kernel is a weighted linear combination of two radial kernels of the same family.*

In the present study we choose the locally compactly kernel

$$\Phi(r, \varepsilon) = (1 - \varepsilon r)_+^8 (32(\varepsilon r)^3 + 25(\varepsilon r)^2 + 8\varepsilon r + 1) \tag{8}$$

then we recover generate the new kernel

$$\Delta\Phi(r, \varepsilon) = 44\varepsilon^2(1 - \varepsilon r)_+^6 (88(\varepsilon r)^3 + 3(\varepsilon r)^2 - 6\varepsilon r - 1) \tag{9}$$

From equation (30), we have

$$\Delta u = \sum_{i=1}^n \lambda_i \Delta\Phi(r_i) = \sum_{i=1}^n a_i \phi(r_i) \text{ in } \Omega \tag{10}$$

putting these values in (25)-(27), we get

$$\sum_{i=1}^n \lambda_i \phi(r_i) = f(\xi, \eta) \quad (\xi, \eta) \in \Omega \tag{11}$$

$$\sum_{i=1}^n \lambda_i \mathcal{B}\Phi(r_i) = g(\xi, \eta), \quad (\xi, \eta) \in \partial\Omega \tag{12}$$

For the numerical implementation, we need to choose two sets of interpolation points as shown in Figure 1. We let  $n_i$  be the number of interior points,  $\{(\xi_j, \eta_j)\}_1^n$  and  $n_b$  the number of boundary points,  $\{(\xi_j, \eta_j)\}_{n_i+1}^{n_i+n_b}$ . Furthermore,  $n$  denotes the total number of points; i.e.,  $n = n_i + n_b$ . By collocation method, we have

$$\sum_{i=1}^n \lambda_i \phi(r_{ij}) = f(\xi_j, \eta_j), \quad 1 \leq j \leq n_i \tag{13}$$

$$\sum_{i=1}^n \lambda_i \mathcal{B}\Phi(r_{ij}) = g(\xi_j, \eta_j), \quad n_i + 1 \leq j \leq n \tag{14}$$

where,  $r_{ij} = \|(\xi_j, \eta_j) - (\xi_i, \eta_i)\|$ . The above system of equations can be easily solved by standard matrix solver. Once  $\{\lambda_i\}_1^n$  is determined, the approximate particular solution becomes the approximate solution  $u$  of (25)-(27); i.e.,

$$u(\xi, \eta) = \sum_{i=1}^n \lambda_i \Phi(r_i) \tag{15}$$

**Problem 1.** We consider the Poisson problem with the Dirichlet boundary condition:

$$\Delta u(\xi, \eta) = -\pi^2 (\eta \sin(\pi\xi) + \xi \cos(\pi\eta)), \quad (\xi, \eta) \in \Omega. \quad (16)$$

$$u(\xi, \eta) = \eta \sin(\pi\xi) + \xi \cos(\pi\eta), \quad (\xi, \eta) \in \partial\Omega. \quad (17)$$

The parametric equation of the computational domain is given by

$$\Omega = \{(\xi, \eta) | \xi = \rho \cos \theta, \eta = \rho \sin \theta, 0 \leq \theta \leq 2\pi\}, \quad (18)$$

where

$$\rho = \left( \cos(3\theta) + \sqrt{2 - \sin^2(3\theta)} \right)^{1/3}. \quad (19)$$

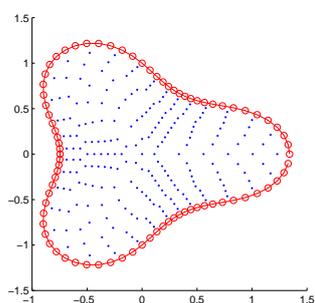
The analytic solution is given by s

$$u(\xi, \eta) = \eta \sin(\pi\xi) + \xi \cos(\pi\eta) \quad (20)$$

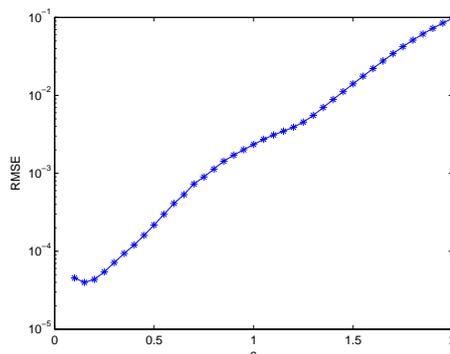
We solved the problem by MAPS using compactly supported radial kernels (8)-(9). The computational domain considered for this problem and the analytic solution in the extended domain are shown in Figures 1-2 respectively. We compute the  $L_\infty$  and the RMS error norms using CS-MAPS for different values of interior and boundary nodes. It can be seen from Figures 3-4, that with the increase in  $c$ , the sparsity of the differentiation matrix increases and at the same time the accuracy decreases. We incorporate more and more interpolation nodes with a small support, but at the expense of decrease in optimal accuracy. It is demonstrated from the table that these locally supported kernel-based MAPS is capable of solving large scaled Poisson's problems with reasonably acceptable accuracy.

**Table 1.** Comparison of the two method in terms of RMSE and  $L_\infty$  for compactly supported kernels.

$n_b$	20	40	80	120	1000
$n_i$	71	117	211	295	4832
CS-MAPS					
Support $c$	0.5	0.5	0.5	0.5	2
RMSE	6.349e-003	1.411e-003	2.171e-004	1.252e-004	1.026e-004
$L_\infty$	3.124e-002	4.341e-003	1.801e-003	1.077e-003	8.920e-004



**Figure 1.** Plot of computational domain for problem 1.



**Figure 2.** Plot of RMSE versus shape parameter for problem 1

**Problem 2.** In this problem we consider the inhomogeneous modified Helmholtz equation:

$$(\Delta - \lambda^2)u(\xi, \eta) = f(\xi, \eta), \quad (\xi, \eta) \in \Omega. \quad (21)$$

$$u(\xi, \eta) = g(\xi, \eta), \quad (\xi, \eta) \in \partial\Omega. \quad (22)$$

where  $f(\xi, \eta)$  and  $g(\xi, \eta)$  are chosen according to the following analytical solution. The parametric equation of the computational domain is given by

$$\Omega = \{(\xi, \eta) | \xi = \cos \theta, \eta = \sin \theta, 0 \leq \theta \leq 2\pi\}, \quad (23)$$

where

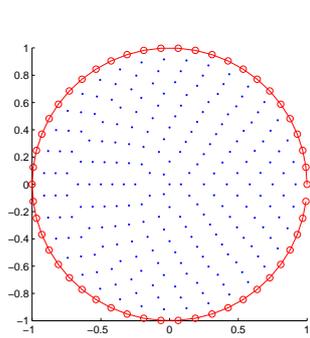
The analytic solution is given by

$$u(\xi, \eta) = \sin(\pi\xi) \cosh(\eta) + \cos(\pi\xi) \sinh(\eta) \quad (24)$$

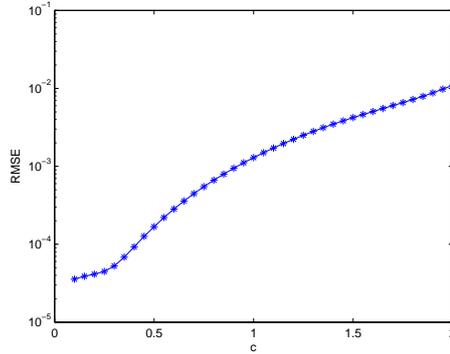
We solved this problem in a unit disk with CS-MAPS. For this problem the computational domain and the analytic solution in the extended domain are shown in Figures 5-6. We used the  $\lambda = 10$ , and different number of interior and boundary nodes. It should be noted that Helmholtz-type equations with large value of  $\lambda$  are mostly difficult. But with CS-MAPS the solution accuracy is excellent for large values of  $\lambda$ . Furthermore we note that a very large as well as a very small number of nodes can be used to achieve reasonably good accuracy. Here we solve  $(\Delta - \lambda^2)\Phi = \phi$ , for the radial kernel given in equation (8).

**Table 2.** Comparison of the two methods in terms of RMSE and  $L_\infty$  for compactly supported kernels.

$n_b$	50	100	150	200	500
$n_i$	170	295	436	933	5778
CS-MAPS					
Support $c$	1	1	1	1	4
RMSE	1.690e-003	7.285e-004	3.762e-004	9.741e-005	3.224e-003
$L_\infty$	8.033e-003	3.939e-003	2.172e-003	5.904e-004	4.839e-004



**Figure 3.** Plot of computational domain for problem 2.



**Figure 4.** Plot of RMSE versus shape parameter for problem 2,  $n_i = 211$ ,  $n_b = 50$

**Problem 3.** In this problem we consider the equation:

$$\Delta u(\xi, \eta) = f(\xi, \eta), \quad (\xi, \eta) \in \Omega. \quad (25)$$

$$u(\xi, \eta) = \sin(\eta^2 + \xi) - \cos(\eta - \xi^2), \quad (\xi, \eta) \in \partial\Omega_1. \quad (26)$$

$$\frac{\partial u(\xi, \eta)}{\partial \mathbf{n}} = (\nabla(\sin(\eta^2 + \xi) - \cos(\eta - \xi^2))) \cdot \mathbf{n}, \quad (\xi, \eta) \in \partial\Omega_2. \quad (27)$$

where  $f(\xi, \eta)$  is chosen according to the following analytical solution. The parametric equation of the computational domain is given by

$$\Omega = [0, 1]^2, \quad (28)$$

$$\partial\Omega_1 = \{(\xi, \eta) \mid \xi = 0, 0 \leq \eta \leq 1, \wedge \xi = 1, 0 \leq \eta \leq 1\} \quad (29)$$

$$\partial\Omega_2 = \{(\xi, \eta) \mid \eta = 0, 0 \leq \xi \leq 1, \wedge \eta = 1, 0 \leq \xi \leq 1\} \quad (30)$$

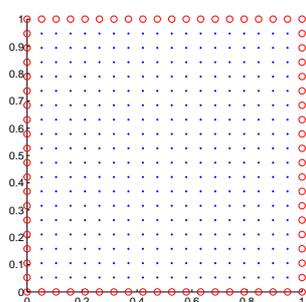
where The analytic solution is given by

$$u(\xi, \eta) = \sin(\eta^2 + \xi) - \cos(\eta - \xi^2) \quad (31)$$

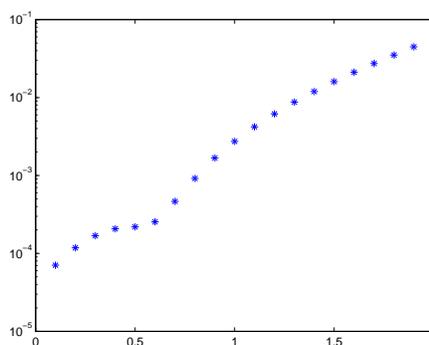
We solved this problem in a unit square  $[0, 1]^2$  with mixed boundary conditions. Along the boundary  $\Omega_1$  we applied Dirichlet boundary condition, while along the boundary  $\Omega_2$  we used Neumann boundary condition. We used here again the locally supported radial kernel (8). The results are shown in Table 3, and Figures 11-12, which demonstrate the capability of CS-MAPS with a very small as well as large number of nodes with mixed boundary conditions.

**Table 3.** Comparison of the two method in terms of RMSE and  $L_\infty$  for compactly supported kernels.

$n_b$	76	96	116	136	316
$n_i$	324	529	784	1089	6084
CS-MAPS					
Support $c$	0.6	0.6	0.6	0.6	2
RMSE	2.541e-004	1.106e-004	5.827e-005	2.646e-005	1.784e-004
$L_\infty$	8.682e-004	5.183e-004	3.636e-004	1.760e-004	2.317e-003



**Figure 5.** Plot of computational domain for problem 2.



**Figure 6.** Plot of RMSE versus shape parameter for problem 2,  $n_i = 324$ ,  $n_b = 76$

**Conclusions** In this work, we extended the work of [1] for compactly supported radial kernels. We used the work of [?] to solve  $\Delta\Phi = \phi$ . The solution for the  $\Phi$  was not very trivial for arbitrary radial kernel  $\phi$ . It was not known why  $\Phi$  be a radial kernel for a given arbitrary kernel  $\phi$ . Theorem 1 show that  $\Phi$  is again a radial kernel for a given arbitrary kernel  $\phi$ . The present study demonstrated that the recently developed MAPS [1] may be very successful for large data points with locally supported kernels.

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