

Gravitational mass and Newton's universal gravitational law under relativistic conditions

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Abstract. We discuss the predictions of Newton's universal gravitational law when using the gravitational, m_g , rather than the rest masses, m_o , of the attracting particles. According to the equivalence principle, the gravitational mass equals the inertial mass, m_i , and the latter which can be directly computed from special relativity, is an increasing function of the Lorentz factor, γ , and thus of the particle velocity. We consider gravitationally bound rotating composite states, and we show that the ratio of the gravitational force for gravitationally bound rotational states to the force corresponding to low ($\gamma \approx 1$) particle velocities is of the order of $(m_{Pl}/m_o)^2$ where m_{Pl} is the Planck mass $(\hbar c/G)^{1/2}$. We also obtain a similar result, within a factor of two, by employing the derivative of the effective potential of the Schwarzschild geodesics of GR. Finally, we show that for certain macroscopic systems, such as the perihelion precession of planets, the predictions of this relativistic Newtonian gravitational law differ again by only a factor of two from the predictions of GR.

1. Introduction

Since the formulation [1] and frequent experimental confirmation [2] of the theory of General Relativity it is generally accepted that Newton's universal gravitational law is not accurate enough under relativistic conditions [2]. For example it cannot describe the mercury perihelion precession, whereas the theory of General Relativity (GR) is "not inconsistent" with it [3, 4]. Also the deflection of light passing near massive objects predicted by Newton's gravitational law is half the value predicted by GR which is in good agreement with experiment [3].

In testing Newton's gravitational law little attention has been paid in the differences between the rest, relativistic, inertial and gravitational masses of the particles involved.

Inertial and gravitational masses are equal to each other, according to the equivalence principle [5], but the ratios of inertial or relativistic mass to the rest mass are unbounded as the particle velocity approaches the speed of light [6].

Consequently the following quite important question arises when using Newton's universal gravitational law under conditions where the Lorentz factor, γ , is significantly larger than unity:

Should one use the particle rest mass, m_o , the relativistic mass, γm_o , or the gravitational mass, m_g , which is equal to the inertial mass, m_i ? This is the question discussed in the present



paper and although it appears to be obvious that one should use the gravitational mass, m_g , rather than the rest mass, m_o , or the relativistic mass, γm_o , one may wonder why this point has been discussed only recently [7]. There appear to be two reasons for this, first that it has been discussed for years that Newton's gravitational law fails under relativistic conditions, which is certainly true when using rest or relativistic masses [2, 7], as also shown here, and second that the velocities for the planets of our planetary system are of the order $10^{-4}c$, so that the corresponding Lorentz factors are of the order 10^{-8} and thus the differences between relativistic and gravitational masses are very small. This, however, is not the case for many other astronomical systems [8, 9].

2. Rest, relativistic, inertial and gravitational mass

The inertial, m_i , and thus the gravitational mass, m_g , is related to the rest mass, m_o , via

$$m_g = m_i = \gamma^3 m_o, \quad (1)$$

where $\gamma(= (1 - v^2/c^2)^{-1/2})$ is the Lorentz factor. This result was first obtained in A. Einstein's pioneering special relativity paper for linear particle motion [6]. It has been shown recently to remain valid for arbitrary particle motion, including circular motion [7]. Consequently, Newton's universal gravitational law for the gravitational force between two particles of rest mass m_o , each, and velocities v_1 , and v_2 relative to a laboratory observer, is

$$F = \frac{Gm_{g,1}m_{g,2}}{r^2} = \frac{Gm_o^2\gamma_1^3\gamma_2^3}{r^2}, \quad (2)$$

and for equal particle velocities v_1 and v_2 reduces to

$$F = \frac{Gm_o^2\gamma^6}{r^2}. \quad (3)$$

This equation has been found to predict the existence of gravitationally confined states with many of the properties of hadrons [7, 10] and to show that the rest masses of quarks are very small, in the mass range of neutrinos [7, 13]. The ratio F/F_o of the force keeping two or three-particle systems in orbit, divided by the same force computed for $\gamma \approx 1$, is $(1/2)(m_{Pl}/m_o)^2$, exactly half of the value predicted by GR [7].

3. SR treatment of the perihelion precession

According to Newton's 2nd law the equation of motion of a body or particle of rest mass m_o is given by

$$F = m_o a, \quad (4)$$

where a is the acceleration. For example, for linear motion we have [11]:

$$F = m_o a = m_o \frac{d^2 y}{dt^2} = m_o \frac{\gamma^{-1} d^2 y_o}{\gamma^2 dt_o^2} = m_o \gamma^{-3} \frac{d^2 y_o}{dt_o^2} = \gamma^{-3} m_o a_o. \quad (5)$$

If the force is a gravitational one, then using the equivalence principle of inertial and gravitational mass, m_i and m_g respectively, and accounting for $m_i = \gamma^3 m_o$ [6, 7, 11, 12] one obtains

$$F = \frac{GMm_o\gamma^3}{r^2}. \quad (6)$$

From (5) and (6) it follows

$$a_o = \gamma^6 GM/r^2. \quad (7)$$

The proper acceleration, i.e. the one perceived on the rotating planet, e.g. on mercury, is

$$a_o = GM/r^2. \quad (8)$$

It thus follows that one can perform the analysis as if moving on mercury ($\gamma = 1$) and simply replace GM by $GM\gamma^6$.

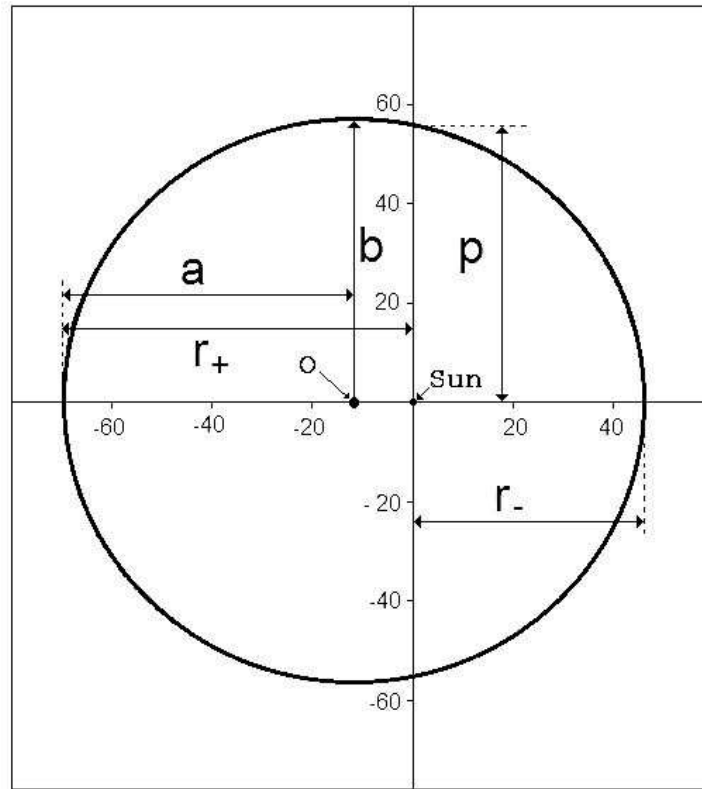


Figure 1. Schematic of the mercury elliptical orbit ($e=0.2054$), showing the definitions of the elliptical parameter (semilatus rectum) p , of the semimajor axis a , of the semiminor axis b , of the perihelion r_- and aphelion r_+ ; numbers indicate distances in 10^9 m.

The final expression obtained by GR for the perihelion precession of mercury or any other planet moving on an elliptical orbit is [2, 4, 14, 15, 16, 17]

$$\Delta\varphi = \frac{6\pi GM}{a(1-e^2)c^2} = \frac{6\pi GM}{pc^2}, \quad (9)$$

where a is the semimajor axis of the ellipse, p is the elliptic parameter, also known as semi-latus rectum, and e is the eccentricity (it is $e = 0$ for a cycle, $0 < e < 1$ for an ellipse, $e = 1$ for a parabola and $e > 1$ for a hyperbola) (Figure 1); p and a are related via

$$p = (1 - e^2)a. \quad (10)$$

One observes in the second equation (9) that when considering the set of all ellipses, including the cycle, of given ellipse parameter p , then $\Delta\varphi$ does not depend on the eccentricity e . Consequently in the SR treatment given here we simplify the computation by considering a

cycle of radius $r = p = a$. Actually, in order to keep the definition of perihelion uniquely defined, we consider an ellipse of eccentricity $e = \epsilon$, where ϵ is an arbitrarily small positive number, e.g. $\epsilon < 10^{-12}$, which is sufficient for the present analysis.

For $r = p = a$ the equation of motion of the rotating planet or, more generally object, is

$$\frac{m_o v^2}{p} = \frac{GM m_o \gamma^6}{p^2}, \quad (11)$$

obtained from the non-relativistic proper equation of motion observed on the planet, i.e.

$$\frac{m_o v^2}{p} = \frac{GM m_o}{p^2}, \quad (12)$$

by replacing GM with $GM\gamma^6$. Equation (11) yields

$$\frac{v^2}{\gamma^6} = \frac{GM}{p}, \quad (13)$$

which for $v \ll c$, as is the case here, thus $\gamma \approx 1$, gives

$$v^2 \approx GM/p, \quad (14)$$

and thus from the definition of γ it follows

$$\gamma \approx \frac{1}{(1 - GM/c^2 p)^{1/2}}, \quad (15)$$

thus

$$\gamma^3 \approx \frac{1}{(1 - GM/c^2 p)^{3/2}} \approx 1 + \frac{3GM}{2pc^2}, \quad (16)$$

and

$$\gamma - 1 \approx \frac{GM}{2pc^2}. \quad (17)$$

On the other hand, the rotational period, T , is given by

$$T = \frac{2\pi p}{v}, \quad (18)$$

and using eq. (14) one obtains

$$T = 2\pi \left(\frac{p^3}{GM\gamma^6} \right)^{1/2} = 2\pi \left(\frac{p^3}{GM} \right)^{1/2} \frac{1}{\gamma^3} = T_o [1 - 3(\gamma - 1)], \quad (19)$$

where T_o is the non-relativistic period corresponding to $\gamma = 1$. It thus follows from (19)

$$\Delta T = T - T_o = -3(\gamma - 1)T_o, \quad (20)$$

thus

$$\frac{\Delta T}{T_o} = -3(\gamma - 1), \quad (21)$$

and using (17)

$$\frac{\Delta T}{T_o} = -\frac{3GM}{2pc^2}. \quad (22)$$

Accounting for the constant angular velocity, it follows

$$\frac{\Delta T}{T_o} = \frac{\Delta \varphi}{2\pi}, \quad (23)$$

and thus it follows

$$\Delta \varphi = \frac{3\pi GM}{pc^2}, \quad (24)$$

which is exactly 1/2 of the value predicted by GR [14, 15, 16, 17].

Consequently, when using gravitational masses in the universal Newtonian gravitational law, the resulting eq. (3) provides very good agreement (within a factor of 2) with the results of general relativity (GR). It is interesting that the SR treatment leads to 1/2 of the GR treatment results both for fm size systems and for macroscopic astronomical systems.

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