

Canonical quantisation via conditional symmetries of the closed FLRW model coupled to a scalar field

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Abstract. We study the classical, quantum and semiclassical solutions of a Robertson-Walker spacetime coupled to a massless scalar field. The Lagrangian of these minisuperspace models is singular and the application of the theory of Noether symmetries is modified to include the conditional symmetries of the corresponding (weakly vanishing) Hamiltonian. These are found to be the simultaneous symmetries of the supermetric and the superpotential. The quantisation is performed adopting the Dirac proposal for constrained systems. The innovation in the approach we use is that the integrals of motion related to the conditional symmetries are promoted to operators together with the Hamiltonian and momentum constraints. These additional conditions imposed on the wave function render the system integrable and it is possible to obtain solutions of the Wheeler-DeWitt equation. Finally, we use the wave function to perform a semiclassical analysis following Bohm and make contact with the classical solution. The analysis starts with a modified Hamilton-Jacobi equation from which the semiclassical momenta are defined. The solutions of the semiclassical equations are then studied and compared to the classical ones in order to understand the nature and behaviour of the classical singularities.

1. Introduction

In this paper, we examine the classical, quantum and semiclassical behaviour of the FLRW spacetime metric coupled to a massless scalar field with the method of conditional symmetries which was first introduced in [1, 2]. The theory of variational symmetries has been extended to include cases in which the lapse function is not gauge fixed. This supplies the superspace with an additional degree of freedom and the theory with a gauge freedom, which leads to the appearance of additional symmetries. At the classical level, the importance of the method consists in solving first-order differential equations instead of the second-order Einstein equations. This method is presented in section 2. At the quantum level, it can be considered as an approach to the problem of time of quantum gravity in which one first quantises and then selects time variable [3]. However, we adopt a more ambitious analysis and exploit the quantum solution to perform a semiclassical analysis in order to gain insight on the behaviour of the classical singularities. This analysis is performed in section 3. Finally, we discuss the results in section 4.



2. Classical analysis for the massless field $\lambda = 0$ in FLRW with non flat $k \neq 0$ spatial metric

The model we study is the FLRW closed universe coupled to a massless scalar field with the following spacetime metric

$$ds^2 = -N^2(t)dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 \sin^2 \theta + r^2 \sin^2 \theta d\varphi^2 \right] \quad (1)$$

where $N(t)$ is the lapse function which is not gauge fixed, $a(t)$ is the scale factor and k is the spatial curvature, which in our case takes the value $k = 1$. The total action of the gravitational plus matter system is $S_{tot} = S_{grav} + S_{mat} = \int d^4x \sqrt{-g} [R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi]$ from which we obtain the total Lagrangian of the system after discarding a term of total derivative is equal to

$$L = 6Nka - \frac{6a\dot{a}^2}{N} + \frac{a^3 \dot{\phi}^2}{2N} \quad (2)$$

This Lagrangian has the singular form

$$L = \frac{1}{2N} G_{\alpha\beta}(q) \dot{q}^\alpha \dot{q}^\beta - N\mathcal{V}(q) \quad (3)$$

where $G_{\alpha\beta}(q)$ is the supermetric defined on the configuration space of the dependent variables $q^\alpha(t)$ and $\mathcal{V}(q)$ the superpotential. The presence of the lapse function makes this Lagrangian explicitly invariant under the transformation $t = f(\tilde{t})$ which in turn leads to the existence of additional symmetries on the superspace called conditional symmetries [1]. In the constant superpotential parametrisation of the Lagrangian which can be obtained by the change $N \rightarrow n = \frac{N}{V(a,\phi)}$, the conditional symmetries become the usual symmetries of the supermetric plus the scaling symmetry. This leads to the existence of conserved quantities corresponding to the Killing fields of the supermetric and a rheonomic integral, $Q_i = \kappa_i, Q_h = \kappa_h + \int n(t)dt$.

The supermetric of our model in this parametrisation becomes

$$\tilde{G}_{\alpha\beta} = \begin{pmatrix} -72ka & 0 \\ 0 & 6ka^4 \end{pmatrix} \quad (4)$$

and it has three Killing vector fields generating the symmetries of the configuration space (a, ϕ)

$$\xi_1 = \left(\frac{e^{\phi/\sqrt{3}}}{a}, -\frac{2\sqrt{3}e^{\phi/\sqrt{3}}}{a^2} \right), \quad \xi_2 = \left(\frac{e^{-\phi/\sqrt{3}}}{a}, \frac{2\sqrt{3}e^{-\phi/\sqrt{3}}}{a^2} \right), \quad \xi_3 = (0, 1) \quad (5)$$

plus one homothetic field

$$\xi_h = \left(\frac{a}{4}, 0 \right) \quad (6)$$

The system of equations of the conserved quantities becomes

$$-\frac{12e^{\frac{\phi}{\sqrt{3}}} ka \left(6\dot{a} + \sqrt{3}a\dot{\phi} \right)}{n} = \kappa_1, \quad \frac{12e^{-\frac{\phi}{\sqrt{3}}} ka \left(-6\dot{a} + \sqrt{3}a\dot{\phi} \right)}{n} = \kappa_2 \quad (7)$$

$$\frac{6ka^4 \dot{\phi}}{n} = \kappa_3, \quad -\frac{18ka^3 \dot{a}}{n} = \kappa_h + \int n(t)dt \quad (8)$$

The solution of the system cannot be fully specified because it contains a gauge freedom resulting from the freedom of the lapse function. In order to find the full solution, we choose to gauge fix

the matter field $\phi = \ln t$. After performing suitable coordinate transformations to absorb the redundant constants, the final form of the spacetime metric is

$$ds^2 = -\frac{\lambda}{4\sqrt{t}(1+t)^3}dt^2 + \frac{\lambda\sqrt{t}}{(1-r^2)(1+t)}dr^2 + \frac{\lambda\sqrt{t}r^2}{1+t}d\theta^2 + \frac{\lambda\sqrt{t}r^2\sin^2\theta}{1+t}d\varphi^2 \quad (9)$$

where we have set $\lambda = -\frac{\kappa_3}{\sqrt{3}k^{3/2}}$. This solution satisfies the Euler-Lagrange equations of the Lagrangian (2) and therefore it is valid. This metric has a singularity for early and late times, that is when $t \rightarrow 0$ and $t \rightarrow \infty$. We will canonically quantise the system and then perform the semiclassical analysis in order to study whether the singularity persists for the semiclassical metric as well.

3. Canonical quantisation and semiclassical analysis

For the quantisation of our system we follow the scheme of canonical quantisation of constrained systems [4] extended in [1] to include the quantisation of the conserved quantities Q_i according to the relation $\hat{Q}_i \equiv -\frac{i}{2\mu}(\mu\xi_i^\alpha\partial_\alpha + \partial_\alpha\mu\xi_i^\alpha)$, with eigenvalues equal to their classical values κ_i and measure on the Hilbert space of states equal to $\mu(a, \phi) = 6\sqrt{3}a^3k$. The quantum operators \hat{Q}_i cannot be imposed simultaneously because of the condition $c_{ij}^n\kappa_n = 0$. Therefore the compatible algebras are the two-dimensional (\hat{Q}_1, \hat{Q}_2) and the one-dimensional \hat{Q}_3 . The case of the one-dimensional algebra \hat{Q}_2 is not considered since it already belongs to a higher dimensional subalgebra and gives a trivial result. Following the above mentioned steps and solving the quantum equations

$$\hat{Q}_i\Psi(\alpha, \phi) = \kappa_i\Psi(\alpha, \phi), \quad \hat{H}\Psi(\alpha, \phi) = 0 \quad (10)$$

where $i = 1, 2, 3$ we find that the wave functions are respectively

$$\Psi_{12}(a, \phi) = A_{12}e^{i\frac{1}{4}a^2e^{-\frac{\phi}{\sqrt{3}}(\kappa_1+\kappa_2e^{\frac{2\phi}{\sqrt{3}}})}} \quad (11)$$

$$\Psi_3(a, \phi) = e^{i\phi\kappa_3 - \frac{\sqrt{3}}{2}\pi\kappa_3}(c_1e^{\sqrt{3}\pi\kappa_3}I_{-i\sqrt{3}\kappa_3}(6a^2\sqrt{k})\Gamma(1 - i\sqrt{3}\kappa_3) + c_2I_{i\sqrt{3}\kappa_3}(6a^2\sqrt{k})\Gamma(1 + i\sqrt{3}\kappa_3)) \quad (12)$$

In order to make contact with the classical solutions, we perform a semiclassical analysis based on the Bohmian approach [5] first presented in the context of minisuperspace models in [2, 6]. To this end, the wave function is written in the polar form $\Psi(q_i) = \Omega(q_i)e^{iS(q_i)}$ where $\Omega(q_i)$ is the amplitude and $S(q_i)$ is the phase of the wave function, which satisfies a modified Hamilton-Jacobi equation with an additional quantum potential term [5] of the form $U(x) \equiv -\frac{1}{2}\square\Omega$. When this quantum term vanishes we obtain the classical solution. The Hamilton-Jacobi equation also defines the quantum canonical momenta defined as $p_i = \frac{\partial S}{\partial q_i}$ from which the semiclassical equations are formed by equating them with the classical canonical momenta defined by the Lagrangian, thus having $\frac{\partial S}{\partial q_i} = \frac{\partial L}{\partial q_i}$. For the case of the two-dimensional algebra (Q_1, Q_2) , the wavefunction is $S_{12} = \frac{1}{4}a^2e^{-\frac{\phi}{\sqrt{3}}(\kappa_1 + \kappa_2e^{\frac{2\phi}{\sqrt{3}}})}$ and the coefficient A_{12} constant, thus leading to the vanishing of the quantum potential. Therefore, it is expected that the solution will not differ from the classical one. This indeed turns out to be the case when we solve the system of semiclassical equations $\frac{\partial L}{\partial q_i} = \frac{\partial S_{12}}{\partial q_i}$ and the classical singularity is not resolved.

In the case of the one-dimensional algebra however things turn out to be different. First, the wave function does not have the assumed polar form. In order to write it in this form, two approximation limits are considered, one for small values of the scale factor and one for

large. This represents early and late times of the universe respectively. For early times, the wave function is approximated by (up to constants)

$$\Psi_{3_{sm}} \approx e^{i\kappa_3\phi} \cos\left(\sqrt{3}\kappa_3 \ln\left(3a^2\sqrt{k}\right)\right) \quad (13)$$

while for late ones

$$\Psi_{3_{la}} \approx e^{i\kappa_3\phi} \frac{\cosh\left(6a^2\sqrt{k}\right)}{\sqrt{a^2\sqrt{k}}\sqrt{3\pi}} \quad (14)$$

The quantum potential in both cases is non-zero and equal to $V_{sm} = \frac{\kappa_3^2}{12ka^4}$ for early times and $V_{la} = -1 - \frac{1}{144ka^4}$ for late times. Even though the quantum potential is not the same in the two limits, the phase function is common, $S = \kappa_3\phi$ and the solution will be the same at the two limits. The semiclassical equations $-\frac{72ka^2\dot{a}}{n} = 0$, $\frac{6ka^4\dot{\phi}}{n} = \kappa_3$ have a gauge freedom for the scalar field which we select to be such that the lapse function $N(t)$ of the semiclassical element is the same as the classical one, that is

$$\phi(t) = \alpha - \frac{8 \times 3^{3/4} t^{\sqrt{3}/2} \sqrt{-\frac{48kt^2/\sqrt{3}}{\kappa_1} - \frac{\kappa_1}{3} \kappa_1 \kappa_3^{3/2}} \left(-3 + \sqrt{1 + \frac{144kt^2/\sqrt{3}}{\kappa_1^2}} {}_2F_1\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, -\frac{144kt^2/\sqrt{3}}{\kappa_1^2}\right)\right)}{c^3(144kt^2/\sqrt{3}\kappa_1 + \kappa_1^3)} \quad (15)$$

The semiclassical line element will then be

$$ds^2 = -\frac{\lambda}{4\sqrt{t}(1+t)^3} dt^2 + \frac{1}{1 - (-\frac{4}{\lambda})^{2/3} r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad (16)$$

This spacetime metric does not contain any singularities since invariants do not diverge for $t \rightarrow 0$ and $t \rightarrow \infty$. Therefore, for this case the singularities of the classical metric vanish.

4. Conclusions

The FLRW cosmological model filled with a massless scalar field was studied at a classical and quantum level, exploiting the presence of additional symmetries. The innovation in this method is that the lapse function is not gauge fixed. Under the presence of the conditional symmetries the classical solution can be obtained by a first-order system, which is much easier to be integrated. This method also has advantages on the quantisation, since the integrals of motion are imposed as conditions on the wave function and the system of constraints is integrable. The final new element in our viewpoint first introduced in [6] is the semiclassical analysis performed in the context of a minisuperspace model following Bohm. The result was that, for the one-dimensional subalgebra the spacetime is not the same as the classical one and the singularities disappear for early and late times, exactly at the limits where the classical singularities existed. This is encouraging for obtaining interesting results for other more complicated cosmological models such as the Bianchi cosmologies.

References

- [1] Christodoulakis T, Dimakis N, Terzis P A, Doulis G, Grammenos T *et al.* 2013 *J.Geom.Phys.* **71** 127–138 (*Preprint* 1208.0462)
- [2] Christodoulakis T, Dimakis N, Terzis P A and Doulis G 2014 (*Preprint* 1405.0363)
- [3] Isham C 1992 (*Preprint* gr-qc/9210011)
- [4] Dirac P A M 1964 *Lectures on Quantum Mechanics* Belfer Graduate School Sci. Mono. (New York, NY: Belfer Graduate School of Science)
- [5] Bohm D 1952 *Phys.Rev.* **85** 166–179
- [6] Christodoulakis T, Dimakis N, Terzis P A, Vakili B, Melas E *et al.* 2014 *Phys.Rev.* **D89** 044031 (*Preprint* 1309.6106)