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From infinitesimal symmetries to deformed symmetries of Lax-type equations

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Abstract. Using the procedure initiated in [14], we deform Lax-type equations though a scaling of the time parameter. This gives an equivalent (deformed) equation which is integrable in terms of power series of the scaling parameter. We then describe a regular Frölicher Lie group of symmetries of this deformed equation

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Introduction

In [14], one of the main advances of this work is to get a (non formal) principal bundle where the concept of holonomy makes sense rigorously. The geometric objects under consideration are diffeological or Frölicher groups, which are regular in the sense that the exponential map exists and is smooth. Diffeological spaces, developped by Souriau and his coworkers [2, 7, 10, 16] are generalizations of manifolds. Independently, Frölicher spaces give a more rigid framework, that also generalize the notion of manifolds [1, 3, 8]. The comparison of the two frameworks has been made independently in [12] and in [18], see e.g. [14]. The aim of this paper is to show how this framework can apply to the theory of Lax equations. A Lax equation [9] is a formally integrable equation of the type : $\partial_t L = [P, L]$ where P, L are in most cases differential, pseudo-differential, or difference operators. This equation integrates heuristically as a classical equation on a group of matrices: there should have an unique solution, up to the initial value $L(0)$, given by $L(t) = Ad_{ExpP(t)}L(0)$. Unfortunately, very often, the operator $ExpP(t)$ exists only at a formal level. We propose to apply a scaling $t \mapsto qt$ to the time variable. The operator $P(t)$ is changed into an operator $P_q(t)$ which is a monomial of order 1 in the $q-$variable, adapting the ideas of [14]. This allows the machinery of $q-$deformed operators: the algebras considered are now Lie algebras of (smooth) regular Lie groups. As a simple consequence, we get smoothness of the unique solution $L_q(t)$ with respect to $P(t)$ and $L(0)$; another consequence is that the full space of symmetries is a diffeological or Frölicher group, and that a class of symmetries of the $q-$deformed Lax equation obey also a Lax-type equation $\partial_t S_q = [ad_{P_q}, S_q]$. This equation is here interpreted as a holonomy equation, which integrates by virtue of the results of [14]. These symmetries S_q are rigorously constructed when L_q is a q-deformed formal pseudo-differential operator.

1. Preliminaries: Frölicher Lie groups of formal series

We now turn to key results from [14]:

Theorem 1.1 *Let $(A_n)_{n\in\mathbb{N}^*}$ be a sequence of complete locally convex (Frölicher) vector spaces which are regular, equipped with a graded smooth multiplication operation on $\bigoplus_{n\in\mathbb{N}^*} A_n$, i.e. a multiplication such that $A_n.A_m \subset A_{n+m}$, smooth with respect to the corresponding Frölicher structures. Then, the set $1+\mathcal{A} = \left\{1+\sum_{n\in\mathbb{N}^*} a_n | \forall n \in \mathbb{N}^*, a_n \in A_n\right\}$ is a Frölicher Lie group, with regular Frölicher Lie algebra $\mathcal{A} = \left\{\sum_{n\in\mathbb{N}^*} a_n | \forall n \in \mathbb{N}^*, a_n \in A_n\right\}$. Moreover, the exponential map defines a bijection $\mathcal{A} \to 1+\mathcal{A}$.*

We mimick and extend the procedure used in [14].

Theorem 1.2 *Let $\mathcal{A} = \bigoplus_{i\in I} \mathcal{A}_i$ be a Frölicher $I-$graded regular algebra. Let G be a regular Frölicher Lie group, acting on $\mathcal{A}$ componentwise. Then, $G \oplus A$ is a regular Frölicher Lie group.*

Proof. Considering the exact sequence $0 \to 1+\mathcal{A} \to G \oplus \mathcal{A} \to G \to 0$; there is a (global) slice $G \to G \oplus \{0_{\mathcal{A}}\}$.

In our work of Lax-type equations, we use the following group from [14]:Let M be a compact manifold without boundary. We denote by $\mathcal{FC}l$ be the space of formal classical pseudo-differential operators acting on $C^\infty(M,\mathbb{R})$. We denote by $\mathcal{FC}l^*$ the groups of the units of the algebras $\mathcal{FC}l$. Let q be a formal parameter. We define the algebra of formal series $\mathcal{FC}l_q = \left\{\sum_{t\in\mathbb{N}^*} q^k a_k | \forall k \in \mathbb{N}^*, a_k \in \mathcal{FC}l\right\}$. This is obviously an algebra, graded by the order (the valuation) into the variable q. Thus, setting $\mathcal{A}_n = \{q^n a_n | a_n \in \mathcal{FC}l\}$, we can set $\mathcal{A} = Cl_q(M,E)$ and state the following consequence of Theorem 1.1: Let $\mathcal{FC}l^{0,*}$ be the Lie group of invertible pseudo-differential operators of order 0. This group is known to be a regular Lie group since Omori, but the most efficient proof is actually in [5], to our knowledge. We remark a short exact sequence of Frölicher Lie groups: $0 \to 1+\mathcal{FC}l_q \to \mathcal{FC}l^{0,*}+\mathcal{FC}l_q \to \mathcal{FC}l^{0,*} \to 0$, which satisfies the conditions of Theorem 1.2. Thus, we have the following:

Theorem 1.3 *The group $1+\mathcal{FC}l_q$ is a regular Frölicher Lie group with regular Frölicher Lie algebra $\mathcal{FC}l_q$, and $\mathcal{FC}l^{0,*}+\mathcal{FC}l_q$ is a regular Frölicher Lie group with Lie algebra $\mathcal{FC}l^0+\mathcal{FC}l_q$.*

2. On Lax equations and their symmetries

A PDE is of Lax type if there is a representation of the solutions $u(t,x) \in C^\infty(\mathbb{R}\times M,\mathbb{C})$ in terms of Lax operators, i.e. a smooth map $u \mapsto L(u) \in \mathcal{FPDO}$ (formal pseudo-differential operators) (very often, L is a differential operator), and another smooth map $u \mapsto P(u) \in \mathcal{FPDO}$ which satisfy a **Lax equation** such that u is a solution of the initial PDE if and only if the following equation is fulfilled:

$$\begin{cases} \partial_t L(t) &= [P(t), L(t)] \\ L(0) &= \text{fixed operator (initial value)} \end{cases} \tag{2.1}$$

(here and in the sequel, we write L and P instead of $L(u)$ and $P(u)$ when it carries no ambiguity) The couple (L,P) is called a **Lax pair.** If the path P is a smooth path of the Lie algebra $\mathfrak{g}$ of a regular Lie group G, if G acts on a Fréchet algebra of operators $\mathcal{B}$ that contains $L(0)$, the path

$$L(t) = Ad_{Exp_G P(t)} L(0) \tag{2.2}$$

is a solution of equation 2.1, yet very often a formal solution.

Example: the KdV equation. The KdV equation reads as $\partial_t u = 6u\partial_x u - \partial_x^3 u$ where $u(t,x) \in C^\infty(\mathbb{R}^2,\mathbb{R})$ ad has a Lax pair $L = -\partial_x^2 + u$ and $P = -4\partial_x^3 + 3(\partial_x u + u\partial_x)$. the operator P is of order 4, so that there is no Lie group G such that $Exp_G P(t)$ exists. Moreover, P depends on u (essentially because the KdV equation is non linear).

Let us note $\mathcal{S}$ the set of solutions of the initial PDE, which is assumed non empty and equipped with the diffeology spanned by (see [11] for the link between diffeological spaces and Fréchet manifolds):

- the trace diffeology as a subset of $C^\infty(\mathbb{R}\times M,\mathbb{C})$,
- and the pull-back of the diffeology of $\mathcal{F}PDO$ from the maps $u\mapsto L(u)$ and $u\mapsto P(u)$.

The **total set of symmetries** of the initial PDE is the group $Diff(\mathcal{S})$, which is a diffeological group. This space of symmetries is actually, to our knowledge, not studied. Instead of working with $\mathcal{S}$, we work with $L(\mathcal{S})$. If $\mathcal{S}$ and $L(\mathcal{S})$ are (diffeologically) isomorphic, then $Diff(\mathcal{S})$ and $Diff(L(s))$ are also isomorphic. We now restrict ourselves to smooth linear maps acting on the vector space spanned by $L(\mathcal{S})$ and $P(\mathcal{S})$ in $\mathcal{F}PDO$. Let us write formally the action of such a symmetry S: if $L(t)$ is a solution of 2.1, $S(t).L(t)$ is also a solution, from which we get

$$(\partial_t S).L(t) = [ad_P, S]\,.L(t) \tag{2.3}$$

Here, the map S is a smooth map $S : L(\mathcal{S}) \to \mathcal{L}(Span(L(\mathcal{S})))$. The map $\phi : S \mapsto (\partial_t S).L(t) - [ad_P, S]\,.L(t)$ is linear and the (restricted groups of) symmetries of (2.3) are the zeros of ϕ. This relation is now linear in S which allows to pass to infinitesimal symmetries, if the algebra of symmetries under consideration is equipped with the functional diffeology. We get here symmetries that are not in general exactly the ones described in [15], where *projectable* symmetries are the symmetries coming from changing of coordinates, i.e. infinitesimal symmetries in $Vect(\mathbb{R}\times M)$.

3. Integration, symmetries and time scaling

We only assume that both P and L are in a fixed Fréchet algebra $\mathcal{A}$ with unit element, or in a $c^\infty-$algebra if one prefers to work in the convenient setting [8]. Let us now build a corresponding Lax equation in $\mathcal{A}[[q]]$. We consider the paths $P(qt)$ and $L(qt)$ obtained by time scaling $t\mapsto qt$. Then, $\partial_t L(qt) = q(\partial_t L)(qt) = [qP(qt), L(qt)]$ for a fixed parameter q. We note by $L_q(t) = L(qt)$ and by $P_q(t) = qP(qt)$. We get the following equation:

$$\left\{\begin{array}{lcl} \partial_t L_q(t) & = & [P_q(t), L_q(t)] \\ L(0) & = & \text{fixed operator (initial value) in } \mathcal{A} \end{array}\right. \tag{3.1}$$

Let val_q be the valuation of formal series in $\mathcal{A}[[q]]$ with respect to the q variable. We remark that $val_q L_q = 0$ and $val_q P_q = 1$. We note by $\mathcal{A}[[q]]_{>0}$ the ideal made of formal series S such that $val_q S > 0$.

Theorem 3.1 *The solutions of equation 3.1 in $\mathcal{A}[[q]]$ are such that: $L_q(t) = exp(P_q)(t).L(0).(exp(P_q)(t))^{-1}$ where the the map exp is the group exponential $\mathcal{A}[[q]]_{>0} \to Id + \mathcal{A}[[q]]_{>0}$.*

The proof is a straightforward consequence of basic results on Lie groups. The serie $exp(P_q)(t)$, read as $exp(P_q)(t) = \sum_{i=0}^\infty a_i(q)$ where $a_i(q) = \int_{t\geq s_1\geq\ldots\geq s_i\geq 0}\left[\prod_{j=1}^i P_q(s_j)\right](ds)^i$

Let us now look for symmetries of a Lax equation. A symmetry is a path S of linear invertible operators on $\mathcal{A}$ such that, Assuming smoothness, we shall quickly go into more restricted classes of symmetries along the lines of the last section. Applying the time scaling, we get, with the obvious notations:

$$(\partial_t S_q).L_q(t) = \left[ad_{P_q}, S_q\right].L_q(t) \tag{3.2}$$

The map $S\to S_q$ is an homomorphism from the group of symmetries of (2.1) to the group of symmetries of (3.1), and it appears to us that there should exist symmetries of (3.1) that are not induced from symmetries of (2.1). The map $\phi_q : S_q \mapsto (\partial_t S_q).L_q(t) - \left[ad_{P_q}, S_q\right].L_q(t)$ is linear and the symmetries of (3.2) are the zeros of ϕ_q. Such a problem appears non relevant to the methods of resolution of this paper, and we leave the question of solving these two equations open. Let us now turn to a special class of solutions.

4. Symmetries as holonomy elements

Let us now simplify this equation, avoiding the L_q- term. Then, we get another Lax-type equation

$$\partial_t S_q = \left[ad_{P_q}, S_q\right] \tag{4.1}$$

and we can remark that the operator ad_{P_q} is an inner derivation of A, which is of order 1 in q since P_q is of order 1. Let $In(A)$ be the Lie algebra of inner derivations of A. Let $In_q(A)$ be the $q-$graded algebra of operators spanned by qIn_A, endowed with the push-forward Frölicher structure from A.

We have to check:

(i) In_q is a smooth regular algebra

(ii) $Id_A + In_q(A)$ is a regular Frölicher Lie group with Lie algebra $In_q(A)$.

Let us remark that (ii) is a straightforward consequence of (i) and Theorem 1.1. Now, we recall that smoothness in $In(A)$ is induced by smoothness in A. Moreover, the inclusion $In(A) \to C^\infty(A, A)$ is smooth in the Frölicher sense [8]. So that $In_q(A)$ is a smooth algebra, where the composition is smooth and bilinear. Finally the only checkpoint is that the paths $ad_{a(t)}ad_{b(t)}...$ are integrable . This is in particular true in algebras of formal pseudo-differential operators, using the rules of composition of formal symbols. We can now apply the procedure that we used for equations (3.1): the exponential $exp_{Id_A+In_q(A)}$ exists and

$$S_q(t) = Exp_{Id_A+In_q(A)}(ad_{P_q}).S_q(0).\left(Exp_{Id_A+In_q(A)}(ad_{P_q})\right)^{-1}$$

is the unique solution to equation (3.2) with initial value $S_q(0)$. We now analyze equation 4.1: it extends to the path space of $C^\infty(M, \mathbb{R})[[q]] \times A_q$ which can be viewed as a trivial vector bundle. Setting $\nabla = d + ad_{P_q}$, we get a smooth connection on this fiber bundle. By the last discussion, ad_{P_q} takes values in a regular Frölicher group, and hence any path on $C^\infty(M)[[q]]$ lifts to a smooth path on A_q by holonomy theorem [14], and for any linear map S(0) that transforms an initial value $L(0)$ into another initial solution $S(0).L(0)$, we get a smooth path of operators $t \mapsto S_q(t)$ such that, if $t \mapsto L_q(t)$ is a smooth path which is solution of 3.1, the path $t \mapsto S_q(t).L_q(t)$ is also a solution of 3.1.

[1] Cherenack, P.; Ntumba, P. 2001 *Demonstratio Math.* **34** no 1 161-180
[2] Donato, P. 1984 *Revtements de groupes diffrentiels* Thse de doctorat d'état, Université de Provence, Marseille
[3] Frölicher, A; Kriegl, A 1988 *Linear spaces and differentiation theory* Wiley series in Pure and Applied Mathematics, Wiley Interscience
[4] Gilkey, P 1984 *Invariance theory, the heat equation and the Atiyah-Singer index theorem* Publish or Perish
[5] Glöckner, H 2002 *Studia Math.* **153**, no2 147-177
[6] Hector, G., and E. Macias-Virgos, Diffeological groups, Research Exp. Math. 25 (2002), 247260
[7] Iglesias-Zemmour, P. 2013 *Diffeology* Mathematical Surveys and Monographs, **185**, AMS
[8] Kriegl, A.; Michor, P.W. 2000 *The convenient setting for global analysis* Math. surveys and monographs **53**, AMS
[9] Lax, P. 1968 *Comm. Pure Appl. Math.* **21** no5 467490
[10] Leslie, J. 2003 *J. Lie Theory* **13**, 427-442
[11] Losik, M. V. 1992 *Russ. Math.* **36**, 3137.
[12] Magnot, J-P. 2006 *C. R. Math. Acad. Sci., Soc. R. Can.* **28**, no. 4, 121127
[13] Magnot, J-P. 2008 *Bull. Sci. Math.* **132** no. 2, 112-127
[14] Magnot, J-P. 2013 *Int. J. Geom. Meth. Mod. Phys.* **10**, No. 9, Article ID 1350043
[15] Olver, P. 1993 *Applications of Lie groups to differential equations* 2nd edition, GTM **107**, Springer (1993)
[16] Souriau, J.M. 1985 *Astérisque*, Hors Série, 341-399
[17] Robart, T. 1997 *Can. J. Math.* **49** (4) 820-839
[18] Watts, J.; *Diffeologies, differentiable spaces and symplectic geometry* PhD thesis arXiv:1208.3634v1