

# A necessary condition for applying MUSIC algorithm in limited-view inverse scattering problem

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**Abstract.** Throughout various results of numerical simulations, it is well-known that Multiple Signal Classification (MUSIC) algorithm can be applied in the limited-view inverse scattering problems. However, the application is somehow heuristic. In this contribution, we identify a necessary condition of MUSIC for imaging of collection of small, perfectly conducting cracks. This is based on the fact that MUSIC imaging functional can be represented as an infinite series of Bessel function of integer order of the first kind. Numerical experiments from noisy synthetic data supports our investigation.

## 1. Introduction

From the pioneering work of Multiple Signal Classification (MUSIC) algorithm for estimating the locations of a number of point-like scatterers, it has been remarkably developed and successfully applied in various inverse scattering problems for finding locations of small scatterers or shapes of crack-like inhomogeneities/extended targets, refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

Almost every works are focused on the application of MUSIC in full-view inverse scattering problems. In these results, it is confirmed that MUSIC produces very good results in full-view problems but, theoretical reason of its effectiveness for imaging of curve-like perfectly conducting crack is recently developed in [14, 15]. In these works, a relationship between MUSIC-type imaging functional and Bessel function of integer order of the first kind is investigated and the effectiveness of MUSIC and some undiscovered phenomenons are clearly identified. However, in the limited-view inverse scattering problem, MUSIC produces poor results. This is recognized as a general property of MUSIC, and this fact has been identified via various numerical examples, refer to [2].

Motivated this, theoretical reason of this limitation of MUSIC has been identified and examined that MUSIC can be applicable for detecting locations of point-like scatterers or cracks of small length and extended thin electromagnetic inhomogeneities if the range of incident and observation directions is not too narrow, refer to [16, 17]. However, when one tries to apply MUSIC for imaging extended perfectly conducting crack, the selection of range of incident and observation directions significantly affects the imaging performance. However, this phenomenon is heuristically discovered so that theoretical treatment of MUSIC in limited-view problem should be considered carefully.

In this contribution, we explore a necessary condition of MUSIC in limited-view inverse scattering problem for imaging of arc-like perfectly conducting crack in two-dimensional space.



This is based on the relationship with MUSIC-type imaging function and infinite series of Bessel functions of integer order.

This paper is organized as follows. In Section 2, we briefly introduce the two-dimensional direct scattering problem and the structure of MUSIC-type imaging function. In Section 3, we discover a necessary condition for applying MUSIC in limited-view problem. In Section 4 results of numerical simulation are exhibited in order to support our discover.

## 2. Direct scattering problem and structure of MUSIC

Let  $\Gamma$  be a smooth curve describes crack and we denote  $u(\mathbf{x})$  be the time-harmonic total field which satisfies

$$\Delta u(\mathbf{x}) + k^2 u(\mathbf{x}) = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma$$

(with positive wave number  $k$ ) and a Dirichlet boundary condition  $u(\mathbf{x}) = 0$  on  $\Gamma$ . Note that  $u(\mathbf{x})$  can be decomposed as  $u(\mathbf{x}) = u_{\text{inc}}(\mathbf{x}) + u_{\text{scat}}(\mathbf{x})$ , where  $u_{\text{inc}}(\mathbf{x}) = e^{ik\boldsymbol{\theta} \cdot \mathbf{x}}$  is given incident field with incident direction  $\boldsymbol{\theta} \in \mathbb{S}_*^1$  and  $u_{\text{scat}}(\mathbf{x})$  is scattered field which satisfies the Sommerfeld radiation condition. Here,  $\mathbb{S}_*^1$  is a connected, proper subset of two-dimensional unit circle  $\mathbb{S}^1$ . Let  $u_\infty(\boldsymbol{\vartheta}, \boldsymbol{\theta})$  be the far-field pattern of  $u_{\text{scat}}(\mathbf{x})$ . Then, it satisfies

$$u_{\text{scat}}(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} \left\{ u_\infty(\boldsymbol{\vartheta}, \boldsymbol{\theta}) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right) \right\}$$

uniformly in all directions  $\boldsymbol{\vartheta} = \mathbf{x}/|\mathbf{x}|$  and  $|\mathbf{x}| \rightarrow \infty$ .

Now, let us introduce MUSIC-type imaging algorithm. Before starting, we assume that  $\Gamma$  is divided into  $M$  different segments of size of order half the wavelength  $\lambda/2$ . Then, based on the Rayleigh resolution limit from far-field data, any detail less than one-half of the wavelength cannot be seen, and only one point, say  $\mathbf{x}_m$ , at each segment is expected to contribute at the image space of the response matrix  $\mathbb{K}$  (see [11, 12] for instance). Having in mind this, we consider the collected Multi-Static Response (MSR) matrix such that

$$\mathbb{K} := [K_{jl}(\boldsymbol{\vartheta}_j, \boldsymbol{\theta}_l)]_{j,l=1}^N = [u_\infty(-\boldsymbol{\theta}_j, \boldsymbol{\theta}_l)]_{j,l=1}^N,$$

where  $\{\boldsymbol{\vartheta}_j \in \mathbb{S}_*^1 : j = 1, 2, \dots, N\}$  and  $\{\boldsymbol{\theta}_l \in \mathbb{S}_*^1 : l = 1, 2, \dots, N\}$  are set of observation and incident directions, respectively. Then, the singular value decomposition of  $\mathbb{K}$  can be written as

$$\mathbb{K} = \mathbf{U} \mathbf{D} \mathbf{V}^* \approx \sum_{m=1}^M \sigma_m \mathbf{U}_m \mathbf{V}_m^*.$$

Note that the first  $M$  left-singular vectors,  $\{\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_M\}$ , provide an orthonormal basis for  $\mathbb{K}$ . Hence, one can define a projection operator onto the null (or noise) subspace:

$$\mathbf{P}_{\text{noise}} := \mathbb{I}_N - \sum_{m=1}^M \mathbf{U}_m \mathbf{U}_m^*.$$

For any point  $\mathbf{z} \in \mathbb{R}^2$ , define a vector  $\mathbf{f}(\mathbf{z}) \in \mathbb{C}^{N \times 1}$  such that  $\mathbf{f}(\mathbf{z}) = [e^{ik\boldsymbol{\theta}_1 \cdot \mathbf{z}}, e^{ik\boldsymbol{\theta}_2 \cdot \mathbf{z}}, \dots, e^{ik\boldsymbol{\theta}_N \cdot \mathbf{z}}]^T$ . Then, an image of  $\mathbf{z} \in \Gamma$ , follows from computing

$$\mathcal{I}(\mathbf{z}) = |\mathbf{P}_{\text{noise}}(\mathbf{f}(\mathbf{z}))|^{-1}.$$

Based on [1, 4, 11, 12], map of  $\mathcal{I}(\mathbf{z})$  will exhibit large peaks at  $\mathbf{z} = \mathbf{x}_m \in \Gamma$ .

Unfortunately, throughout above description, it is very hard to answer that why MUSIC yields unexpected results in limited-view inverse scattering problem. For this, the structure of MUSIC-type imaging functional  $\mathcal{I}(\mathbf{z})$  has been derived in [17] as follows.

**Lemma 2.1** *Assume that  $N$  and  $k$  are sufficiently large. Then,  $\mathcal{I}(\mathbf{z})$  can be written*

$$\mathcal{I}(\mathbf{z}) = \left\{ 1 - \sum_{m=1}^M \left| J_0(k|\mathbf{z} - \mathbf{x}_m|) + \frac{\mathcal{D}(k|\mathbf{z} - \mathbf{x}_m|)}{\theta_N - \theta_1} \right|^2 \right\}^{-1},$$

where  $\mathbf{z} - \mathbf{x}_m = r_m[\cos(\phi_m), \sin(\phi_m)]$  and

$$\mathcal{D}(k|\mathbf{z} - \mathbf{x}_m|) := 4 \sum_{n=1}^{\infty} \frac{i^n}{n} J_n(k|\mathbf{z} - \mathbf{x}_m|) \sin\left(\frac{n(\theta_N - \theta_1)}{2}\right) \cos\left(\frac{n(\theta_N + \theta_1 - 2\phi_m)}{2}\right). \quad (1)$$

Based on the property of Bessel functions, the terms  $J_0(k|\mathbf{z} - \mathbf{x}_m|)$  and  $\mathcal{D}(k|\mathbf{z} - \mathbf{x}_m|)$  will contribute to and disturb the imaging performance, respectively. Therefore, we can conclude that  $\mathcal{I}(\mathbf{z})$  is highly influenced by the range of incident directions.

### 3. A necessary condition for applying MUSIC in limited-view problems

Based on the result in Lemma 2.1, it is expected that if the term  $\mathcal{D}(k|\mathbf{z} - \mathbf{x}_m|)$  is disappear, very good results can be obtained via MUSIC. Note that  $\mathbf{z}$  is arbitrary and  $\mathbf{x}_m$  is unknown. This means that we cannot make  $J_n(k|\mathbf{z} - \mathbf{x}_m|) = 0$  in (1) so, we must find a condition of the range of incident directions such that

$$\sin\left(\frac{n(\theta_N - \theta_1)}{2}\right) \cos\left(\frac{n(\theta_N + \theta_1 - 2\phi_m)}{2}\right) \equiv 0.$$

Note that if  $\theta_N - \theta_1 = \pi$  and  $\theta_N + \theta_1 - 2\phi_m = \pi$  then,  $\mathcal{D}(k|\mathbf{z} - \mathbf{x}_m|)$  is identically zero. Hence, if one can set the range  $\theta_1 = \phi_m$  and  $\theta_N = \phi_m + \pi$ , good results can be obtained via MUSIC.

Based on above observation, a necessary condition for applying MUSIC in limited-view inverse scattering problem: let  $\psi(\mathbf{x}_m)$  denotes the slope of tangential line at  $\mathbf{x}_m \in \Gamma$ . Then, for obtaining good results, the range of incident directions  $\theta_1$  and  $\theta_N$  must satisfy

$$\theta_1 = \min\{\psi(\mathbf{x}_m)\} \quad \text{and} \quad \theta_N = \pi + \max\{\psi(\mathbf{x}_m)\} \quad \text{for} \quad \mathbf{x}_m \in \Gamma, \quad m = 1, 2, \dots, M. \quad (2)$$

A detailed description is to appear in an extended version of this contribution.

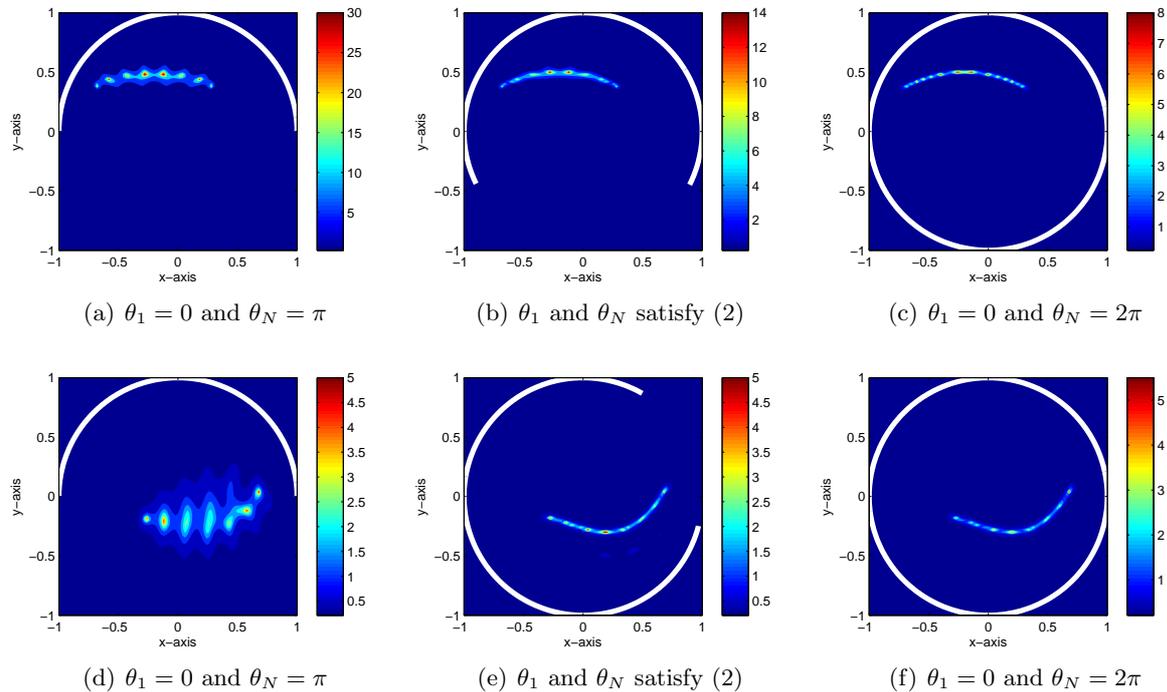
### 4. Results of numerical simulation

In this section, we exhibit some numerical results. For this,  $\lambda = 0.4$  is applied and two  $\Gamma_s$  are chosen

$$\begin{aligned} \Gamma_1 &= \{[s - 0.2, -0.5s^2 + 0.5] : -0.5 \leq s \leq 0.5\} \\ \Gamma_2 &= \{[s + 0.2, s^3 + s^2 - 0.3] : -0.5 \leq s \leq 0.5\}. \end{aligned}$$

Figure 1 shows maps of  $\mathcal{I}(\mathbf{z})$  with various range of directions when the cracks are  $\Gamma_1$  and  $\Gamma_2$ . This result shows that, an approximate shape of  $\Gamma_1$  can be identified when  $\theta_N - \theta_1 = \pi$  (see Figure 1(a)), and almost complete shape of crack can be identified when  $\theta_1$  and  $\theta_N$  satisfy (2), refer to Figure 1(b). On the basis of the result in [15], complete shape of  $\Gamma_1$  can be imaged via the map of  $\mathcal{I}(\mathbf{z})$  (see Figure 1(c)) when  $\theta_N - \theta_1 = 2\pi$ .

Now, let us consider the imaging of  $\Gamma_2$ . Opposite to the previous result, we cannot recognize the shape of  $\Gamma_2$  via the map of  $\mathcal{I}(\mathbf{z})$  with  $\theta_1 = 0$  and  $\theta_N = \pi$  (see Figure 1(d)). On the basis of the result in Figure 1(e), we can observe that if we select  $\theta_1$  and  $\theta_N$  so as to satisfy (2), the shape of  $\Gamma_2$  can be recognized via the map of  $\mathcal{I}(\mathbf{z})$ . Same as the result in Figure 1(c), complete shape of  $\Gamma_2$  can be imaged via the map of  $\mathcal{I}(\mathbf{z})$  when  $\theta_0 = 0$  and  $\theta_N = 2\pi$ , refer to Figure 1(f).



**Figure 1.** Maps of  $\mathcal{I}(z)$  with various range of directions for  $\Gamma_1$  (top row) and  $\Gamma_2$  (bottom row).

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