

MUSIC-type imaging of small perfectly conducting cracks with an unknown frequency

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Abstract. Multiple Signal Classification (MUSIC) is a famous non-iterative detection algorithm in inverse scattering problems. However, when the applied frequency is unknown, inaccurate locations are identified via MUSIC. This fact has been confirmed through numerical simulations. However, the reason behind this phenomenon has not been investigated theoretically. Motivated by this fact, we identify the structure of MUSIC-type imaging functionals with unknown frequency, by establishing a relationship with Bessel functions of order zero of the first kind. Through this, we can explain why inaccurate results appear.

1. Introduction

The inverse scattering problem for the reconstruction of a single, perfectly conducting crack in $\mathbb{R}^2$, satisfying a Dirichlet boundary condition, has been studied in [1]. In this remarkable research, a Newton-type iterative reconstruction algorithm is proposed. In general, for a successful application, a good initial guess close to the unknown crack is required in order to guarantee convergence. Furthermore, the algorithm generally requires a large amount of computational time, and it is difficult to extend to the reconstruction of multiple cracks.

As an alternative, non-iterative algorithms have also been developed. Among them, Multiple Signal Classification (MUSIC)-type algorithms are applied to various problems for the imaging of small electromagnetic scatterers, cracks, and extended targets. For more details, one can refer to [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and references therein. However, to obtain a good result, the value of the applied frequency must be known. If not, inaccurate results are extracted via MUSIC. Until now, this fact has only been investigated through the results of numerical simulations. For this reason, a rigorous mathematical theory about the structure of MUSIC-type imaging functionals must be established, in order to diagnose this phenomenon.

In this contribution, we identify the structure of MUSIC-type imaging functionals of small, perfectly conducting cracks with an unknown frequency, by finding a relationship with Bessel functions of order zero of the first kind. This is based on the fact that the far-field pattern can be presented as an asymptotic expansion formula in the presence of a small crack. The derived structure illuminates the theoretical reason for the appearance of inaccurate locations.

The paper is organized as follows. In section 2, two-dimensional direct scattering problems and MUSIC-type imaging algorithms are briefly introduced. In section 3, the structure of MUSIC-type imaging functionals without information about the applied frequency is identified, and the corresponding results of a numerical simulation are exhibited. A short conclusion is given in Section 4.



2. Direct scattering problems and MUSIC-type imaging algorithm

Let Γ_m , $m = 1, 2, \dots, M$, be a linear crack with a small length $2h$ and center $\mathbf{z}_m$. We denote Γ be the collection of all of the Γ_m and assume that the Γ_m are sufficiently separated from each other. In this paper, we consider the case of so-called Transverse Magnetic (TM) polarization. Let $u(\mathbf{x}, \boldsymbol{\theta})$ be the time-harmonic total field, satisfying the Helmholtz equation

$$\nabla^2 u(\mathbf{x}, \boldsymbol{\theta}) + k^2 u(\mathbf{x}, \boldsymbol{\theta}) = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma, \quad (1)$$

with a Dirichlet boundary condition $u(\mathbf{x}, \boldsymbol{\theta}) = 0$ on Γ . Here, $\boldsymbol{\theta}$ is an incident direction on the unit circle $\mathbb{S}^1$, and $k = 2\pi/\lambda$ denotes a strictly positive wavenumber, with wavelength λ . We assume that k^2 is not an eigenvalue of (1) and $h \ll \lambda$.

Note that $u(\mathbf{x}, \boldsymbol{\theta})$ be decomposed as $u(\mathbf{x}, \boldsymbol{\theta}) = u_{\text{inc}}(\mathbf{x}, \boldsymbol{\theta}) + u_{\text{scat}}(\mathbf{x}, \boldsymbol{\theta})$, where $u_{\text{inc}}(\mathbf{x}, \boldsymbol{\theta}) = e^{ik\boldsymbol{\theta} \cdot \mathbf{x}}$ is the given incident field, and $u_{\text{scat}}(\mathbf{x}, \boldsymbol{\theta})$ is the unknown scattered field satisfying the Sommerfeld radiation condition uniformly in all directions $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$. Let $u_\infty(\hat{\mathbf{x}}, \boldsymbol{\theta})$ denote the far-field pattern of $u_{\text{scat}}(\mathbf{x}, \boldsymbol{\theta})$. Based on [5], the $u_\infty(\hat{\mathbf{x}}, \boldsymbol{\theta})$ can be represented as

$$u_\infty(\hat{\mathbf{x}}, \boldsymbol{\theta}) = -\frac{2\pi}{\ln(h/2)} \sum_{m=1}^M u_{\text{inc}}(\mathbf{z}_m, \boldsymbol{\theta}) \overline{u_{\text{inc}}(\mathbf{z}_m, \hat{\mathbf{x}})} + O\left(\frac{1}{|\ln h|^2}\right) \approx -\frac{2\pi}{\ln(h/2)} \sum_{m=1}^M e^{ik(\boldsymbol{\theta} - \hat{\mathbf{x}}) \cdot \mathbf{z}_m}. \quad (2)$$

We apply (2) in order to establish a MUSIC-type imaging functional. For this, one uses the eigenvalue structure of the MSR matrix $\mathbb{K} = [u_\infty(\hat{\mathbf{x}}_j, \boldsymbol{\theta}_l)]_{j,l=1}^N$. Suppose that $\boldsymbol{\theta}_j = -\boldsymbol{\theta}_j$ for all j . Then, $\mathbb{K}$ is a complex symmetric matrix, but is not Hermitian. Therefore, instead of Eigenvalue Decomposition, we perform a Singular Value Decomposition (SVD) of $\mathbb{K}$ (see [7]), with

$$\mathbb{K} = \sum_{m=1}^M \sigma_m \mathbf{U}_m \mathbf{V}_m^*, \quad (3)$$

where the superscript $*$ denotes the Hermitian conjugate. Then, $\{\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_M\}$ is the basis for the signal space of $\mathbb{K}$. Therefore, one can define the projection operator onto the null (or noise) subspace, $\mathbf{P}_{\text{noise}} : \mathbb{C}^{N \times 1} \rightarrow \mathbb{C}^{N \times 1}$. This projection is given explicitly by

$$\mathbf{P}_{\text{noise}} := \mathbb{I}_N - \sum_{m=1}^M \mathbf{U}_m \mathbf{U}_m^*, \quad (4)$$

where $\mathbb{I}_N$ denotes the $N \times N$ identity matrix. For any point $\mathbf{x} \in \mathbb{R}^2$, we define a test vector $\mathbf{f}(\mathbf{x}; k) \in \mathbb{C}^{N \times 1}$, as $\mathbf{f}(\mathbf{x}; k) = \frac{1}{N} [e^{ik\boldsymbol{\theta}_1 \cdot \mathbf{x}}, e^{ik\boldsymbol{\theta}_2 \cdot \mathbf{x}}, \dots, e^{ik\boldsymbol{\theta}_N \cdot \mathbf{x}}]^T$. With this, a MUSIC-type imaging functional $\mathcal{I}(\mathbf{x}; k)$ can be constructed as follows:

$$\mathcal{I}(\mathbf{x}; k) = |\mathbf{P}_{\text{noise}}(\mathbf{f}(\mathbf{x}; k))|^{-1}. \quad (5)$$

Then, the map of $\mathcal{I}(\mathbf{x}; k)$ will have peaks of large magnitudes at $\mathbf{z}_m \in \Gamma$.

3. Structure of MUSIC-type imaging functional and numerical results

Based on the above, one must know the exact value of k . Without this, one cannot detect the locations of the Γ_m exactly. This fact has been identified in various research, but has so far relied on the results of numerical simulations. Motivated by this, we explore the structure of (5) with an unknown frequency. Since the exact value of k is unknown, we choose a fixed value η , and apply the corresponding test vector $\mathbf{f}(\mathbf{x}; \eta) = \frac{1}{N} [e^{i\eta\boldsymbol{\theta}_1 \cdot \mathbf{x}}, e^{i\eta\boldsymbol{\theta}_2 \cdot \mathbf{x}}, \dots, e^{i\eta\boldsymbol{\theta}_N \cdot \mathbf{x}}]^T$ to (5), such that

$$\mathcal{I}(\mathbf{x}; \eta) = |\mathbf{P}_{\text{noise}}(\mathbf{f}(\mathbf{x}; \eta))|^{-1}. \quad (6)$$

Then, we can obtain the following result. A detailed derivation is to appear in an extended version of this work.

Theorem 3.1. For sufficiently large N , ω , and η , $\mathcal{I}(\mathbf{x}; \eta)$ is of the form:

$$\mathcal{I}(\mathbf{x}; \eta) \approx \left(1 - \sum_{m=1}^M J_0(|\eta \mathbf{x} - k \mathbf{z}_m|) \right)^{-1/2},$$

where J_0 denotes the Bessel function of order zero and of the first kind.

Proof. Based on [12], $\mathbf{f}(\mathbf{z}_m k) \approx \mathbf{U}_m$ for all $m = 1, 2, \dots, M$. It follows that

$$\mathbf{P}_{\text{noise}}(\mathbf{f}(\mathbf{x}; \eta)) = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{i\eta \boldsymbol{\theta}_1 \cdot \mathbf{x}} \\ \vdots \\ e^{i\eta \boldsymbol{\theta}_N \cdot \mathbf{x}} \end{bmatrix} - \frac{1}{N\sqrt{N}} \sum_{m=1}^M \begin{bmatrix} e^{ik\boldsymbol{\theta}_1 \cdot \mathbf{z}_m} \sum_{n=1}^N e^{i\boldsymbol{\theta}_n \cdot (\eta \mathbf{x} - k \mathbf{z}_m)} \\ \vdots \\ e^{ik\boldsymbol{\theta}_N \cdot \mathbf{z}_m} \sum_{n=1}^N e^{i\boldsymbol{\theta}_n \cdot (\eta \mathbf{x} - k \mathbf{z}_m)} \end{bmatrix}.$$

Since N is sufficiently large, the following relationship holds for $\boldsymbol{\theta}_n, \boldsymbol{\theta} \in \mathbb{S}^1$, and $\mathbf{x} \in \mathbb{R}^2$:

$$\frac{1}{N} \sum_{n=1}^N e^{ik\boldsymbol{\theta}_n \cdot \mathbf{x}} = \frac{1}{2\pi} \int_{\mathbb{S}^1} e^{ik\boldsymbol{\theta} \cdot \mathbf{x}} d\boldsymbol{\theta} = J_0(k|\mathbf{x}|),$$

where J_0 denotes the Bessel function of order zero of the first kind (see [15]). Therefore,

$$\mathbf{P}_{\text{noise}}(\mathbf{f}(\mathbf{x}; \eta)) = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{i\eta \boldsymbol{\theta}_1 \cdot \mathbf{x}} - \sum_{m=1}^M e^{ik\boldsymbol{\theta}_1 \cdot \mathbf{z}_m} J_0(|\eta \mathbf{x} - k \mathbf{z}_m|) \\ \vdots \\ e^{i\eta \boldsymbol{\theta}_N \cdot \mathbf{x}} - \sum_{m=1}^M e^{ik\boldsymbol{\theta}_N \cdot \mathbf{z}_m} J_0(|\eta \mathbf{x} - k \mathbf{z}_m|) \end{bmatrix}.$$

With this, we arrive at

$$|\mathbf{P}_{\text{noise}}(\mathbf{f}(\mathbf{x}; \eta))|^2 = \frac{1}{N} \sum_{n=1}^N \left\{ 1 - \sum_{m=1}^M (\Phi_1 + \bar{\Phi}_1) + \left(\sum_{m=1}^M \Phi_2 \right) \left(\sum_{m=1}^M \bar{\Phi}_2 \right) \right\},$$

where $\Phi_1 := e^{i\boldsymbol{\theta}_n \cdot (\eta \mathbf{x} - k \mathbf{z}_m)} J_0(|\eta \mathbf{x} - k \mathbf{z}_m|)$ and $\Phi_2 := e^{ik\boldsymbol{\theta}_n \cdot \mathbf{z}_m} J_0(|\eta \mathbf{x} - k \mathbf{z}_m|)$. Since ω and η are sufficiently large we can obtain the following through a tedious calculation:

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N \Phi_1 &= \frac{1}{N} \sum_{n=1}^N \sum_{m=1}^M e^{i\boldsymbol{\theta}_n \cdot (\eta \mathbf{x} - k \mathbf{z}_m)} J_0(|\eta \mathbf{x} - k \mathbf{z}_m|) = J_0(|\eta \mathbf{x} - k \mathbf{z}_m|)^2 \\ \frac{1}{N} \sum_{n=1}^N \Phi_2 \bar{\Phi}_2 &= \frac{1}{N} \sum_{n=1}^N \sum_{m=1}^M \sum_{m'=1}^M e^{ik\boldsymbol{\theta}_n \cdot (\mathbf{z}_m - \mathbf{z}_{m'})} J_0(|\eta \mathbf{x} - k \mathbf{z}_m|) J_0(|\eta \mathbf{x} - k \mathbf{z}_{m'}|) = J_0(|\eta \mathbf{x} - k \mathbf{z}_m|)^2. \end{aligned}$$

Therefore, $|\mathbf{P}_{\text{noise}}(\mathbf{f}(\mathbf{x}; \eta))|$ can be represented as

$$|\mathbf{P}_{\text{noise}}(\mathbf{f}(\mathbf{x}; \eta))| = \left(1 - \sum_{m=1}^M J_0(|\eta \mathbf{x} - k \mathbf{z}_m|) \right)^{1/2}.$$

This completes the proof. □

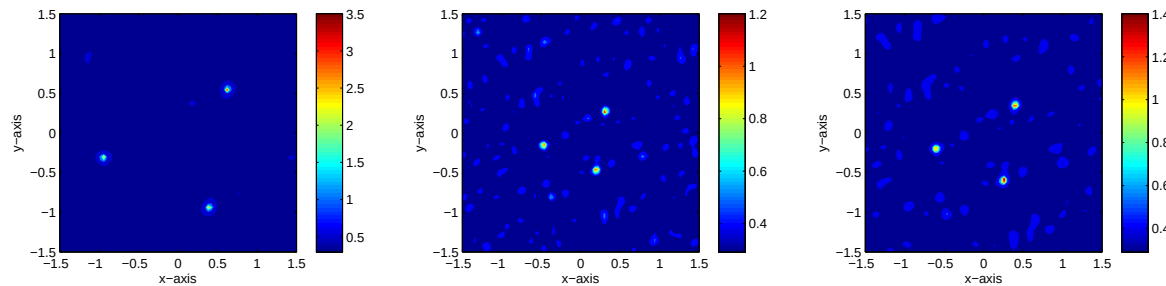


Figure 1. Maps of $\mathcal{I}(\mathbf{z}; 10)$ (left), $\mathcal{I}(\mathbf{z}; 20)$ (center), and $\mathcal{I}(\mathbf{z}; k)$ (right).

Note that $J_0(x)$ has a maximum value of 1 at $x = 0$. Therefore, the map of $\mathcal{I}(\mathbf{x}; \eta)$ will have plots of the magnitude (theoretically $+\infty$) at $\mathbf{x} = (k/\eta)\mathbf{z}_m$, instead of at the true location $\mathbf{z}_m$. This provides the theoretical reason for why inexact locations of cracks are extracted.

Now, let us consider the results of a numerical simulation, with the true value of $k = 2\pi/0.4$. Figure 1 displays maps of $\mathcal{I}(\mathbf{x}; \eta)$ for three small cracks. Following the result from Theorem 3.1, locations of $(k/\eta)\mathbf{z}_m$ are identified, instead of $\mathbf{z}_m$. This means that we can identify the existence of cracks, but it is impossible to find the true locations without *a priori* information about k .

4. Conclusion

Based on an asymptotic expansion formula, the structure of MUSIC-type imaging functionals with unknown frequencies is derived, by establishing a relationship with Bessel functions of order zero. On the basis of the identified structure, we were able to confirm the reason for the appearance of inaccurate locations of cracks.

Based on the result of the current contribution, we expect it is possible to develop an algorithm to detect the exact locations of cracks. The current contribution deals with cracks with a Dirichlet boundary condition. Along the same lines, an extension to cracks with a Neumann boundary condition would make an interesting topic for future research.

Acknowledgments

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education(No. NRF-2014R1A1A2055225) and the research program of Kookmin University in Korea.

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