Hadronic Observables from Dyson–Schwinger and Bethe–Salpeter equations

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Abstract. In these proceedings we present a mini-review on the topic of the Dyson–Schwinger/Bethe–Salpeter approach to the study of relativistic bound-states in physics. In particular, we present a self-contained discussion of their derivation, as well as their truncation such that important symmetries are maintained.

1. Introduction

In Quantum Chromodynamics (QCD) the only observable objects are hadrons, which appear as bound-states of the elementary (but not observed) quark and gluon degrees of freedom. Consequently, most phenomenological aspects of QCD are essentially non-perturbative problems and require an appropriate framework for their study. One such approach is the combination of Dyson–Schwinger (DSE) and Bethe-Salpeter (BSE) equations that provide a means to study the non-perturbative properties of hadrons – at the microscopic level – without abandoning *a priori* the principles of QCD as a scale dependent continuum quantum field theory. We devote these proceedings to a concise exposition of known results about how this framework can be systematically constructed as well as a description of the main technical issues faced upon solving the DSEs and BSEs in combination.

We refrain, for reasons of space, from including here results for observable data that can already be found in the literature [1–3]. Meson spectra have been thoroughly studied in [4–28] and references therein. Baryons have been studied in the quark-diquark approximation [29–37], and also as three-body objects [38–43]. Other observables of enormous experimental interest in deep inelastic scattering such as PDAs, PDFs, GPDs are also starting to be investigated [44–48]. Most of these calculations were performed upon truncating the (anti)quark-quark interaction to a single gluon exchange, the so-called Rainbow-Ladder truncation. This can be seen as the leading order interaction mechanism in the systematic expansion to be described below. While it provides accurate results for ground-state pseudoscalar and vector mesons, and also for ground state baryons, it shows clear deficiencies in all other channels. For this reason, current interest lies is the inclusion of interaction mechanisms beyond the leading one [17, 49–61], and in the extension to glueball [62–64] and tetraquark bound-states [65].

The main purpose, therefore, of this contribution is to make it apparent that one is not pursuing a blind hunt for missing interaction mechanisms, but that there is a step-by-step programme for their inclusion in a systematic manner.

2. Dyson–Schwinger and Bethe–Salpeter equations

A complete description of a continuum quantum field theory, and in particular of QCD, is given when all the (infinitely many) Green's functions of the theory are known. Functional methods deal with these without abandoning the realm of continuum physics, thus providing an approach complementary to that of lattice calculations. In particular, Dyson-Schwinger equations constitute an infinite set of coupled, non-linear integral equations for the full Green's functions of the theory. We define in this section the basic concepts to be used later on.

2.1. Functional Methods

The generating functional for full Green's functions corresponding to a classical Euclidean action S$[\phi]$ is given by

$$Z[J] = \int \mathcal{D}\phi e^{-i(S[\phi]+J_i\phi_i)} = \langle \exp\left(-iJ_i\phi_i\right)\rangle \ , \tag{1}$$

where $J_i\phi_i$ denotes $\int d^4x J_a(x)\phi_a(x)$ with i a superindex that subsumes possible discrete (collectively denoted a) and continuous indices x. This functional is normalised such that $Z[0] = 1$. The $\langle\cdot\rangle$ indicate the weighted functional average with sources subsequently set to zero. In the presence of external sources one writes $\langle\cdot\rangle_J$.

The generating functional for connected Green's functions is given by

$$W[J] = -i\ln Z[J] \ , \tag{2}$$

which can be shown by a Taylor expansion. Furthermore, the generating functional for one-particle irreducible vertex functions (the effective action) is obtained by a Legendre transform

$$\Gamma[\phi^c] = W[J] - \phi_i^c J_i \ , \qquad \text{with } \phi_i^c = \langle\phi_i\rangle_J = \frac{\delta W[J]}{\delta J_i} \ . \tag{3}$$

2.2. Dyson–Schwinger Equation

The observation that the integral of a total derivative vanishes, true so long as the functional measure is invariant under spacetime translations of the field variables

$$\left\langle \frac{\delta}{\delta\phi_i}S[\phi] - J_i \right\rangle_J = 0 \ , \tag{4}$$

can be used to obtain the Dyson–Schwinger equations (DSEs). Equivalently, one can view the DSEs as a consequence of the Ward-Identity associated with translational invariance. These DSEs are the quantum field theoretic equivalent of the classical equations of motion. Upon a vertex expansion we generate the infinite tower of non-linear integral equations that relate the fundamental Green's functions of the quantum field theory to one-another. In a very dense notation [66] the DSEs for proper n-point Green's functions may be obtained from

$$\frac{\delta\Gamma[\phi]}{\delta\phi_i} - \frac{\delta S}{\delta\phi_i}\left[\phi + \frac{\delta^2 W}{\delta J\delta J}\frac{\delta}{\delta\phi}\right] = 0 \ , \tag{5}$$

by taking vacuum expectation values $\langle\cdot\rangle$ of the nth derivative of the functional.

2.3. Bound state equations

Consider an n-particle Green's function $G^{(n)}(p_1,\ldots,p_n)$ describing the evolution of an n-particle system (each carrying momentum p_i), and its amputated counterpart the scattering scattering matrix $T^{(n)}(p_1,\ldots,p_n)$

$$G^{(n)} = G_0^{(n)} + G_0^{(n)}T^{(n)}G_0^{(n)} \ , \tag{6}$$

with $G_0^{(n)}$ the disconnected product of n-propagators. Then we may obtain $T^{(n)}$ from the Dyson equation

$$T^{(n)} = K + KG_0^{(n)}T^{(n)},\qquad(7)$$

with K the 2, 3,-, ..., n-particle irreducible interaction kernel. When the system forms a bound state, the momentum-space function $T^{(n)}$ develops a pole at $P^2 = -M^2$ (in Euclidean spacetime) with $P^\mu = \sum_i^n p_i^\mu$ the total momentum. At the pole we define

$$T^{(n)} \sim \mathcal{N}\frac{\Psi\bar{\Psi}}{P^2 + M^2},\qquad(8)$$

where $\mathcal{N}$ is a state-dependent normalisation factor, Ψ is the bound-state Bethe-Salpeter amplitude and $\bar{\Psi}$ its charge conjugate. Inserting this ansatz into the Dyson equation for the T-matrix and equating residues yields the homogeneous Bethe-Salpeter equation

$$\Psi = KG_0^{(n)}\Psi,\qquad(9)$$

or for its conjugate

$$\bar{\Psi}K^{-1} = \bar{\Psi}G_0^{(n)}.\qquad(10)$$

It can also be shown that the correct normalisation condition for Bethe-Salpeter amplitudes is [67, 68]

$$\bar{\Psi}\left(\frac{dG_0^{(n)}}{dP^2} - i\frac{dK^{-1}}{dP^2}\right)\Psi = 1.\qquad(11)$$

Finally, Bethe-Salpeter amplitudes are the amputated versions of the Bethe-Salpeter wave functions $\varphi = G_0^{(n)}\Psi$. Using this notation, we can rewrite (9) as

$$\Psi = K\varphi,\qquad(12)$$

The interaction kernels K contain a sum of infinitely many terms. If they are known, one can then solve the BSEs and obtain all the information about the bound states of the theory.

3. Deriving the Bethe-Salpeter kernel
The effective action provides a systematic means to derive (generalised) Bethe-Salpeter kernels. This not only enables new interaction mechanisms to be included in a controlled way, but ensures that relevant symmetries are maintained even when approximations are made. Here, we outline the main ideas behind this procedure for the case of two- and three-body bound-state equations following Refs. [69–71].

3.1. Local case
We start with the case of local fields, introducing the main ideas applicable to the bi- and tri-local fields necessary for the study of bound-state equations. First, let us consider the generating functional for a purely fermionic action and introduce source terms $J_k(x)$ for a local operator $\mathcal{O}_k(x) = \{\psi(x), \bar{\psi}(x), \psi(x)^2, \ldots\}$, with k an index describing all discrete indices of the operator and, also, distinguishing the different possible local operators

$$Z[J] = \mathcal{N}\int \mathcal{D}\psi e^{-i\left(\mathcal{S}[\psi]+\int J_k(x)\mathcal{O}_k(x)\right)},\qquad(13)$$

3

where $\mathcal{N} = 1/Z[J]\big|_{J=0}$. Defining, as above, the generating functional for connected Green's functions as $W[J] = -i \log Z[J]$, the one-particle irreducible (1PI) effective action is given by the Legendre transformation

$$\Gamma[\Psi] = W[J] - \int d^4x J_k(x)\Psi_k(x)\,, \tag{14}$$

with the *classical* (super)field defined as $\Psi_k(x) = \delta W[J]/\delta J_k(x)$. It can be determined in the absence of sources through satisfaction of the stationary condition

$$\frac{\delta\Gamma[\Psi]}{\delta\Psi_k} = 0\,. \tag{15}$$

Consider a solution of this stationary condition, $\Psi_k^{(0)}(x)$. We call it *stable* if there exists solution $\Psi_k^{(1)}(x) = \Psi_k^{(0)}(x) + \Delta\Psi_k(x)$ with $\Delta\Psi_k(x)$ *suitably infinitesimal*. Expanding around $\Psi_k^{(0)}(x)$ yields

$$\frac{\delta\Gamma[\Psi]}{\delta\Psi_k}\bigg|_{\Psi_k=\Psi_k^{(1)}} = \frac{\delta\Gamma[\Psi]}{\delta\Psi_k}\bigg|_{\Psi_k=\Psi_k^{(0)}} - \int d^4x' \frac{\delta^2\Gamma[\Psi]}{\delta\Psi_k(x)\delta\Psi_\ell(x')}\bigg|_{\Psi_k=\Psi_k^{(0)}} \Delta\Psi_\ell(x') + \cdots = 0\,, \tag{16}$$

with a minus sign from exchanging the order of derivatives. To first order this equation implies

$$\int d^4x' \frac{\delta^2\Gamma[\Psi]}{\delta\Psi_k(x)\delta\Psi_\ell(x')}\bigg|_{\Psi_k=\Psi_k^{(0)}} \Delta\Psi_\ell(x') = 0\,. \tag{17}$$

That is, the solution $\Psi_k^{(0)}(x)$ is stable if $\Delta\Psi_k(x)$ is an eigenvector of $\delta^2\Gamma/\delta\Psi\delta\Psi$ with zero eigenvalue. To interpret this, let us recall the following identity

$$\int d^4x'' \frac{\delta^2\Gamma[\Psi]}{\delta\Psi_\ell(x)\delta\Psi_k(x'')} \frac{\delta^2W[J]}{\delta J_k(x'')\delta J_m(x')} = -\delta_{\ell m}\delta^{(4)}(x-x')\,. \tag{18}$$

Now, consider an eigenvector $\xi_k(x)$ of $\delta^2W/\delta J\delta J$ with eigenvalue $1/\lambda$

$$\int d^4x' \frac{\delta^2W[J]}{\delta J_k(x)\delta J_\ell(x')} \xi_\ell(x') = \frac{1}{\lambda}\xi_k(x)\,. \tag{19}$$

Using the identity (18) we arrive at

$$\int d^4x' \frac{\delta^2\Gamma[\Psi]}{\delta\Psi_k(x)\delta\Psi_\ell(x')} \xi_\ell(x') = \lambda\xi_k(x)\,. \tag{20}$$

Therefore, $\xi_k(x)$ is also an eigenvector of $\delta^2\Gamma/\delta\Psi\delta\Psi$ with eigenvalue λ. In particular, the perturbation $\Delta\Psi_k(x)$ corresponds to an eigenvector with $\lambda = 0$, wich in turn is related to a pole in $\delta^2W/\delta J\delta J$. In summary, stable solutions for $\Psi_k(x)$ are associated to poles of the connected Green's functions given by $\delta^2W/\delta J\delta J$ (e.g. propagators if $\mathcal{O}(x) = \psi(x)$).

3.2. Bilocal case: Two-body bound-states
For the study of two body problems we add a bilocal term $K_{rs}^k(x,y)\mathcal{O}_{rs}^k(x,y)$ to the partition function (13), with k again distinguishing the different bilocal operators and the other discrete indices described by r, s. Specialising to the case of fermion-antifermion bound states (mesons) we limit the discussion to $\mathcal{O}_{rs}(x,y) = \psi_r(x)\bar{\psi}_s(y)$ and antisymmetric sources $K_{rs}(x,y) =$

$-K_{sr}(y, x)$. Then, in addition to the classical field $\Psi_k(x)$ we have the connected two-point Green's function

$$\frac{\delta W[J, K]}{\delta K_{ab}(x, y)} = \frac{1}{2} \left(\Psi_a(x)\bar{\Psi}_b(y) + G_{ab}(x, y) \right) . \tag{21}$$

The 2PI effective action is then defined as

$$\Gamma[\Psi, G] = W[J, K] - \int d^4x J_k(x)\Psi_k(x) - \frac{1}{2} \int d^4x d^4y K_{ab}(x, y)\Psi_a(x)\bar{\Psi}_b(y) \tag{22}$$

$$- \frac{1}{2} \int d^4x d^4y K_{ab}(x, y)G_{ab}(x, y) , \tag{23}$$

which admits, in addition to (15) and in the absence of sources, the stationary condition

$$\frac{\delta\Gamma[\Psi, G]}{\delta G_{ab}(x, y)} = 0 . \tag{24}$$

This is equivalent to the Dyson–Schwinger equation for the fermion propagator, as we show below.

Following the ideas of previous section, a solution $G_{ab}^{(0)}(x, y)$ of (24) is stable if there exists a perturbed solution $G_{ab}^{(1)}(x, y) = G_{ab}^{(0)}(x, y) + \Delta G_{ab}(x, y)$ determined by a non-trivial solution of

$$\int d^4x' d^4y' \left. \frac{\delta^2\Gamma[\Psi, G]}{\delta G_{ab}(x, y)\delta G_{a'b'}(x', y')} \right|_{G=G^{(0)}} \Delta G_{a'b'}(x', y') = 0 . \tag{25}$$

We show below that this is indeed equivalent to the usual Bethe-Salpeter equation for a fermion-antifermion bound state. Nevertheless this can be seen here, analogous to the local case, through the fact that a solution of (25) is related to a pole in $\delta^2 W/\delta K\delta K$, *viz.* to a pole in a four-point Green's function.

3.2.1. Relation to the Bethe-Salpeter equation   It is known [72] that the 2PI effective action can be written as

$$\Gamma[\Psi, G] = \mathrm{S}[\Psi] + iTr\log G - iTrG_0^{-1}G + \Gamma_2[\Psi, G] , \tag{26}$$

where $\Gamma_2[\Psi, G]$ contains two-particle irreducible diagrams only and G_0 is the classical propagator. The stationary condition for G gives, as promised above, the Dyson-Schwinger equation for the propagator. Indeed, taking a functional derivative with respect to the propagator G

$$\frac{\delta\Gamma[\Psi, G]}{\delta G_{ab}(x, y)} = iG_{ab}^{-1}(x, y) - iG_{0,ab}^{-1}(x, y) + \frac{\delta\Gamma_2[\Psi, G]}{\delta G_{ab}(x, y)} = 0 , \tag{27}$$

where we used $\delta Tr\log G = G^{-1}\delta G$. Defining the self-energy as $\Sigma = -i\delta\Gamma_2/\delta G$ we can rewrite the stationary condition for G as

$$G_{ab}^{-1}(x, y) = G_{0,ab}^{-1}(x, y) - \Sigma_{ab}(x, y) , \tag{28}$$

which is the gap equation for the fermion propagator. If $G^{(0)}$ is one of the solutions of the gap equation, we can take one further functional derivative with respect to the propagator G and rewrite (25) as

$$0 = \int d^4x' d^4y' \left. \frac{\delta^2\Gamma[\Psi, G]}{\delta G_{ab}(x, y)\delta G_{a'b'}(x', y')} \right|_{G=G^0} \Delta G_{a'b'}(x', y')$$

$$= \int d^4x' d^4y' \left(-G_{ac}^{(0)}(x, x')G_{db}^{(0)}(y, y') + K_{ab;cd}(x, y; x', y') \right) \Delta G_{cd}(x', y') , \tag{29}$$

where we used $\delta M_{ij}^{-1}/\delta M_{kl} = -M_{ik}^{-1}M_{lj}^{-1}$. This is precisely the Bethe-Salpeter equation (12) for a Bethe-Salpeter wave function ΔG, with the quark-antiquark interaction kernel given by

$$K_{ab;cd}(x,y;x',y') = -\left.\frac{\delta\Sigma_{ab}(x,y)}{\delta G_{cd}(x',y')}\right|_{G=G^{(0)}} . \tag{30}$$

One then sees immediately that the interaction kernel is obtained by functionally *cutting* propagator lines from the self-energy. Note that the solution $G = G^{(0)}$ is inserted into the self-energy only after the cutting has been performed.

3.2.2. Goldstone bosons. Chiral symmetry. We show here that when the meson BSE is derived using the methods outlined above, the spontaneous breaking of chiral symmetry is accompanied by the appearance of a pseudoscalar massless bound state (the Goldstone boson). More details can be found in [73]. Consider that under a global SU(2) chiral transformation

$$\Psi' = e^{i\gamma_5\tau^a\theta}\Psi \qquad , \qquad G' = e^{i\gamma_5\tau^a\theta}Ge^{i\gamma_5\tau^{\dagger a}\theta} , \tag{31}$$

the 2PI effective action (we ignore the dependence on the fields, as it plays no role in this discussion) is invariant $\Gamma[G'] = \Gamma[G]$. One then has, for infinitesimal transformations

$$\delta\Gamma[B] = \frac{\theta\delta\Gamma}{\delta G_{a'b'}(x',y')}\{i\gamma_5\tau^a, G(x',y')\}_{a'b'} = 0 . \tag{32}$$

Taking one further derivative with respect to G yields

$$\frac{\delta^2\Gamma}{\delta G_{ab}(x,y)G_{a'b'}(x',y')}\{i\gamma_5\tau^a, G(x',y')\}_{a'b'} + \frac{\delta\Gamma}{\delta G_{ac}(x,y)}\tau^a\gamma_{5,cb} + \tau^a\gamma_{5,ac}\frac{\delta\Gamma}{\delta G_{cb}(x,y)} = 0 . \tag{33}$$

When we set G to the solution of the stationary condition, $G = G^{(0)}$, the equation simplifies

$$\left.\frac{\delta^2\Gamma}{\delta G_{ab}(x,y)G_{a'b'}(x',y')}\right|_{G=G^{(0)}}\{\gamma_5\tau^a, G^{(0)}(x',y')\}_{a'b'} = 0 . \tag{34}$$

If chiral symmetry is spontaneously broken then $\{\gamma_5\tau^a, G^{(0)}(x,y)\}$ is non-vanishing, which means that there exists a solution of the BSE with pseudoscalar quantum numbers. Moreover, since $G^{(0)}(x,y) = G^{(0)}(x-y)$, the solution corresponds to one that has vanishing total momentum after transforming to momentum space.

The key observation here is that, even after the effective action is truncated to some loop order, as long as the truncated action is invariant and both the quark self-energy and the meson BSE kernel are derived by taking functional derivatives of it, Goldstone's theorem will hold without fine tuning. We finally note that the arguments given here translate identically to any global symmetry of the effective action.

3.3. Tri-local: Three-body bound-states
The extension of the above formulae to the three-body bound state case is rather straightforward [70] if one adds to the partition function a source term for the trilocal operators of interest, $R_{rst}^k(x,y,z)V_{rst}^k(x,y,z)$. After a Legendre transformation, the effective action acquires an explicit dependence on the three-body vertex V. Note that depending on whether source terms for the proper tri-vertices of the theory are added or not, one is dealing with the 2PI or the 3PI effective action, supplemented with an extra vertex for the three-body bound state.

Focusing here on the case of a baryonic bound state, we introduce sources R and $\bar{R}$ for the operators $V = \psi_r(x)\psi_s(y)\psi_t(z)$ and $\overline{V} = \bar{\psi}_r(x)\bar{\psi}_s(y)\bar{\psi}_t(z)$, respectively. Using the stability arguments laid above for a would-be solution $V^{(0)}$ of the stationary condition $\delta\Gamma/\delta V = 0$ then lead to the following three-body bound state equation (see [70] for a detailed derivation)

$$\int d^4x' d^4y' d^4z' \frac{\delta^2\Gamma[\Psi, G, V]}{\delta\overline{V}_{rst}(x,y,z)\delta V_{r's't'}(x',y',z')}\bigg|_{G=G^{(0)}, V=V^{(0)}} \Delta V_{r's't'}(x',y',z') = 0 \,. \qquad (35)$$

Note that only mixed derivatives with respect to V and $\overline{V}$ do not vanish identically when one sets $V = V^{(0)}$ and $\overline{V} = \overline{V}^{(0)}$.

It is important to comment here that for the important cases of mixed quark-flavour states, and indeed mixed states in general, the procedure just outlined proceeds identically, only with the introduction of mixed propagators and vertices. Generally speaking, one could say that for each bound state of interest, one inserts the appropriate vertex in the effective action, takes functional derivatives with respect to it, and sets it to its vacuum value at the end. The fact that for flavour-diagonal mesons this vertex coincides with the quark propagator is therefore merely accidental.

4. Internal structure: form factors

The prototypical experiment for probing the internal structure of hadrons consists of (in)elastic scattering of a particle on a hadron. Such interactions are mediated by gauge fields, thus the theoretical study of hadrons must couple gauge fields so that relevant symmetries are preserved, as well as maintaining internal consistency of the theoretical framework. At the level of Green's functions, a method coined *gauging of equations* exists that ensures current conservation [74–76].

The method of *gauging* can be best understood through the introduction of a new source term in the partition function, $-\mathcal{J}^\mu(x)A_\mu(x)$, where A^μ is the external, non-dynamical gauge field and $\mathcal{J}^\mu$ is the current that couples to it. For example, consider the $2n$-point fermionic Green's function

$$G^{(2n)}(x_1 \ldots x_n; x_1' \ldots x_n') = \langle 0|T\left[\psi^1(x_1)\ldots\psi^n(x_n)\bar{\psi}^1(x_1')\ldots\bar{\psi}^n(x_n')\right]|0\rangle \,. \qquad (36)$$

Then, its coupling to an external gauge field A_μ would be

$$G^{(2n),\mu}(x_1 \ldots x_n; x_1' \ldots x_n'; y) = \langle 0|T\left[\psi^1(x_1)\ldots\psi^n(x_n)\bar{\psi}^1(x_1')\ldots\bar{\psi}^n(x_n')\mathcal{J}^\mu(y)\right]|0\rangle \,, \qquad (37)$$

which is called the *gauged Green's function*. It can be obtained by applying a functional derivative with respect to A^μ to the original Green's function (we drop from now on the superindices indicating the number of particles)

$$G^\mu = -\left.\frac{\delta}{\delta A_\mu}G\right|_{A=0} \,, \qquad (38)$$

and setting the external field to zero. It hence follows the ordinary rules for derivatives when acting, for instance, on products of functions. The proper n-body vertex, J^μ, for the coupling of the n-body system to the field A_μ is defined as

$$G^\mu = G_0 J^\mu G_0 \,. \qquad (39)$$

with G_0 again the product of full propagators. In particular, for $n = 1$ this defines the proper (fermion-photon) vertex Γ^μ
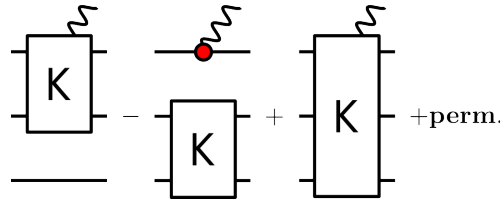
$$S^\mu = S\Gamma^\mu S \,, \qquad (40)$$

Figure 1. Gauging of the three-body kernel, K^μ, as illustrated in (49). Note the minus sign which prevents the over counting of diagrams.

as a result of gauging the full propagator $G^{(2)} = S$. A useful relation follows from gauging $(SS^{-1})^\mu$ and using the identity $\mathbb{1}^\mu = 0$

$$(S^{-1})^\mu = -S^{-1}S^\mu S^{-1} = -\Gamma^\mu \ . \tag{41}$$

For the following it is in general more convenient to work with the amputated Green's function, *i.e.* the scattering matrix T defined in (6). Consider then a hadron described by the following Dyson equation

$$T = K + KG_0 T \ , \tag{42}$$

with K the interaction kernels derived using the prescriptions given above. We can gauge this equation, following the rules of differentiation, to obtain

$$T^\mu = K^\mu + K^\mu G_0 T + KG_0^\mu T + KG_0 T^\mu \ , \tag{43}$$

which can be rewritten using (7) as

$$\begin{aligned} T^\mu &= (\mathbb{1} - KG_0)^{-1}\left(K^\mu + K^\mu G_0 T + KG_0^\mu T\right) \\ &= T\left(K^{-1}K^\mu K^{-1} + G_0^\mu\right)T \ . \end{aligned} \tag{44}$$

At the bound state poles, one can introduce a bound-state electromagnetic current J^μ in a similar fashion as in (8)

$$T^\mu \sim \frac{\Psi_f}{P_f^2 + M_f^2} J^\mu \frac{\bar\Psi_i}{P_i^2 + M_i^2} \ , \tag{45}$$

where $M_{i,f}$ and $\Psi_{i,f}$ are the initial and final bound-state masses and amplitudes, respectively. From (44), (9) and (10), we arrive at

$$J^\mu = \bar\Psi_f\left(G_0^\mu + G_0 K^\mu G_0\right)\Psi_i \ . \tag{46}$$

Many of these details can be illustrated with a three-body system. Then, G_0 is the product of three full propagators S and thus its gauged analogue generates three impulse-like diagrams

$$G_0^\mu = (S_1 S_2 S_3)^\mu = S_1^\mu S_2 S_3 + S_1 S_2^\mu S_3 + S_1 S_2 S_3^\mu = \chi_1^\mu S_2 S_3 + S_1 \chi_2^\mu S_3 + S_1 S_2 \chi_3^\mu \ . \tag{47}$$

Here $\chi^\mu = S^f \Gamma^\mu S^i$ and the superscripts i, f denote that the propagators are evaluated before and after the *momentum injection* from the external field. The interaction kernel K must be decomposed into the sum of its two- and three-particle irreducible terms

$$K = \sum_{\text{perm.}} K_{\text{(2PI)}} S^{-1} + K_{\text{(3PI)}} \ . \tag{48}$$

8

Then, using (41) its gauged version is

$$K^\mu = \sum_{\text{perm.}} K^\mu_{\text{(2PI)}} S^{-1} - \sum_{\text{perm.}} K_{\text{(2PI)}} \chi^\mu + K^\mu_{\text{(3PI)}} \ . \tag{49}$$

It is interesting to note how the gauging procedure has automatically introduced a relative sign between those terms in which the external field interacts with the spectator line and those in which it interacts with the irreducible kernel; this ensures the absence of the over counting of diagrams [75, 76]. The irreducible kernels themselves must be gauged once they are expressed in terms of the elementary degrees of freedom. A diagrammatic representation of this equation is shown in Fig. 1.

The application to two-body states is entirely analogous, with the simplification that K^μ in (46) is directly the gauged two-particle irreducible kernel, $K^\mu_{\text{(2PI)}}$. Also, the generalisation to the coupling of two external fields by *gauging twice* has been presented in [77, 78].

5. Application to QCD
In this section we apply the previously introduced formalism to QCD and derive the quark self-energy, quark-gluon vertex and meson Bethe-Salpeter kernel from the truncated 2PI and 3PI effective actions.
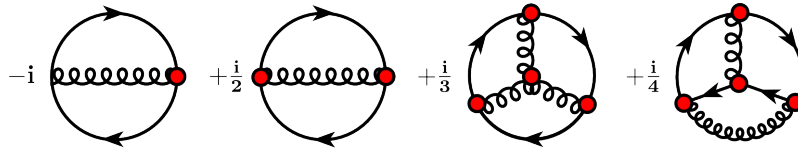

Figure 2. The terms of the 3PI effective action at 3-loop relevant to the quark, quark-gluon vertex and meson Bethe-Salpeter kernel.

5.1. 3PI Effective Action
As an example, let us derive the quark self-energy, quark-gluon vertex, and meson Bethe-Salpeter kernel from the 3PI effective action at three-loop order. We start with the effective action, given in Fig. 2, wherein we have kept only those terms relevant to the discussion at hand. Note also that at least a three-loop expansion is required in order to obtain, upon the inclusion of a *baryonic vertex* in the effective action, a non-trivial baryon BSE kernel.

The quark self-energy is given by (28), while solving the stationary condition $\delta\Gamma[\Psi, G, V]/\delta V^\mu_{ab} = 0$ gives the quark-gluon vertex DSE

$$\Sigma = -i \frac{\delta\Gamma_2[\Psi, G, V]}{\delta G_{ab}} = \qquad \tag{50}$$

$$0 = \frac{\delta\Gamma_2[\Psi, G, V]}{\delta V^\mu_{ab}}\bigg|_{G=G^{(0)}, V=V^{(0)}} \implies \qquad \tag{51}$$

In the second line of Eq.(50) we have used the vertex DSE (51) to simplify the form of the self-energy contribution. Note that in doing so, the self-energy is no longer a function of V, but rather $V^{(0)}$ which now implicitly depends upon the quark propagators G. Thus, if the BSE kernel is obtained by a further functional derivative of the simplified self-energy, the functional dependence of $V^{(0)}$ on G must be resolved.

Thus, we use the first line in Eq. (50) to derive the Bethe-Salpeter kernel. Taking one further functional derivative of the quark self-energy with respect to the quark (which diagrammatically amounts to cutting one quark line) yields

$$-K = \frac{\delta \Sigma}{\delta S} =$$

$$\tag{52}$$

where we can once again use the quark-gluon vertex DSE (51) to simplify the kernel in the last step.

It is interesting to note here the appearance of a ladder exchange that features two dressed vertices. Additionally, at this order in the truncation the doubly-dressed gluon exchange must necessarily be accompanied by a crossed-ladder exchange in the BSE kernel in order to preserve chiral symmetry and any other global symmetries of the system.

5.2. 2PI Effective Action

It is enlightening to compare the 3PI effective action at 3-loop to the 2PI effective action at the same order. We can read this off from Fig. 2 by replacing the dressed vertices with bare ones. Then, the quark self-energy is

$$\Sigma = -i\frac{\delta \Gamma_2[\Psi, G]}{\delta G_{ab}} =$$

$$\tag{53}$$

from which the quark-gluon vertex can be inferred

$$\implies$$

$$\tag{54}$$

In a similar fashion to the above, we can write down the corresponding Bethe-Salpeter kernel by taking one further functional derivative of the quark self-energy with respect to the quark

(resolving the quark dependence of the vertex if needs be) to find

$$-K = \frac{\delta\Sigma}{\delta S} = \quad - \quad + \quad + \quad + \quad + \quad + \tag{55}$$

$$= \quad + \quad - \quad - \quad + \quad .$$

This truncation is essentially the one employed in Refs. [79, 80]; the application to baryons is reported in Ref. [61].

Notice here that (55) is structurally quite different from (52) in that the ladder exchanges always contain one perturbative vertex; this could be remedied by including, for example, 4-loop terms in the 2PI effective action.

All of this can be compared to the rainbow-ladder truncation which follows from the 2PI effective action at two-loop order. Then only the two-loop terms (with vertices bare) of Fig. 2 are required, yielding

$$\Sigma = - \quad , \qquad -K = \quad - \quad . \tag{56}$$

Note that only this simplistic truncation lacks the flavour dependence that the two-body kernel necessarily features (due to implicit and explicit dependence on the quark propagator). To make such a truncation viable, the bare vertices are renormalisation-group improved (i.e. dressed such that perturbative anomalous dimensions are recovered). This accounts for the lack of interaction strength provided by a single gluon-exchange, and is important for both phenomenology and chiral dynamics.

6. Some technical remarks
We comment in this last section upon some technical aspects that must be taken into account when attempting to solve, in practice, the DSE/BSE system.

6.1. Covariant decomposition of amplitudes
The quantum numbers, such a spin, parity, etc. of the bound state to be studied can be enforced by restricting the tensor structure of the Bethe-Salpeter amplitudes to have the correct symmetries for these quantum numbers.

For the description of a two-fermion bound-state, we need to provide a covariant decomposition $\psi^{\mathcal{I}}_{\alpha\beta}$. The spinor indices of the two fermions are α, β, while for total spin J, $\mathcal{I}$ is a product of J Lorentz indices. For spin $J = 0$ it is well-known that a suitable representation is [81, 82]

$$\mathbb{1}, \ \gamma^\mu, \ \gamma^{[\mu}\gamma^{\nu]}, \ \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}, \ \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} \ , \tag{57}$$

that is often simplified through the introduction of γ_5. Saturating these with the quark relative momentum k^μ and imposing positive parity yields $D_i = \{\mathbb{1}, \slashed{k}\}$. Then, the general decomposition for a state of zero spin is

$$\Gamma(k, P) = \begin{pmatrix} \mathbb{1} \\ \gamma_5 \end{pmatrix} D_i \, \Lambda_\pm \ . \tag{58}$$

Here, the first term selects the overall parity of the state and $\Lambda_\pm = \left(\mathbb{1} \pm \hat{P}\right)/2$ is a positive/negative energy projector that introduces the (normalised) total bound-state momentum P. To introduce total angular momentum J, we couple the spin-zero state Γ to the two possible angular momentum tensors $Q^{\mu_1\cdots\mu_J}$ and $T^{\mu_1\cdots\mu_J}$

$$\Gamma^{\mu_1\cdots\mu_J}(k,P) = \begin{pmatrix} Q^{\mu_1\cdots\mu_J} \\ T^{\mu_1\cdots\mu_J} \end{pmatrix} \Gamma(k,P) . \tag{59}$$

These tensors are given by the traceless part of the symmetrised J-fold tensor products of a transversal projector transforming like a vector [83].

It is convenient to introduce the transverse projector $T_P^{\mu\nu} = \delta^{\mu\nu} - P^\mu P^\nu/P^2$, and to denote its application using subscripts as follows: $k_T^\mu = T_P^{\mu\nu} k^\nu$; $\gamma_t^\mu = T_k^{\mu\alpha} T_P^{\alpha\nu} \gamma^\nu$. If we define the symmetrised J-fold tensor products

$$\tilde{Q}^{\mu_1\cdots\mu_J} = k_T^{\{\mu_1}\cdots k_T^{\mu_J\}} , \quad \tilde{T}^{\mu_1\cdots\mu_J} = \gamma_t^{\{\mu_1} k_T^{\mu_2}\cdots k_T^{\mu_J\}} , \tag{60}$$

then the angular momentum tensors $Q^{\mu_1\cdots\mu_J}$ and $T^{\mu_1\cdots\mu_J}$ are the traceless part thereof [19,24,84].

To describe a three-fermion bound-state we need to provide a covariant decomposition for $\psi^{\mathcal{I}}_{\alpha\beta\gamma}$. The spinor indices α, β, γ correspond to the three fermionic legs, while for total spin-$k+1/2$, $\mathcal{I}$ is composed of one spinor and k Lorentz indices.

Let us begin with a spin-$1/2$ baryon where $\mathcal{I} = \delta$ carries the incoming baryon spin index. Saturating the matrices (57) with the two relative quark momenta k^μ and q^μ and selecting positive parity gives $D_i = \left\{\mathbb{1}, \not{k}_T, \not{q}_t, \not{k}_T\not{q}_t\right\}$. Then

$$\psi_{\alpha\beta;\gamma}{}^\delta(k,q,P) = \begin{pmatrix} \mathbb{1} & \otimes & \mathbb{1} \\ \gamma_5 & \otimes & \gamma_5 \\ \gamma_T^\rho & \otimes & \gamma_T^\rho \\ \gamma_T^\rho\gamma_5 & \otimes & \gamma_T^\rho\gamma_5 \end{pmatrix} \begin{pmatrix} D_i & \otimes & D_j \end{pmatrix} \begin{pmatrix} \Lambda_\pm\gamma_5 C & \otimes & \Lambda_+ \end{pmatrix} . \tag{61}$$

The left tensor product denotes the outgoing quark legs with indices α, β and hence warrants the inclusion of $\gamma^5 C$, with $C = \gamma^4\gamma^2$ the charge conjugation matrix. The right tensor product describes the outgoing quark leg, index γ, and the incoming baryon spin-index δ; the Λ_+ here selects the positive energy baryon. Not all elements are linearly independent; it can be checked that a linearly independent subspace of 64 elements can be constructed [38,85].

The generalisation to a state of spin-$k + 1/2$ (with k integer) is obtained by extension of this basis

$$\psi_{\alpha\beta;\gamma}{}^{\delta\nu_1\cdots\nu_k}(k,q,P) = \begin{pmatrix} M^{\mu_1\cdots\mu_k} & \otimes & \mathbb{P}^{\mu_1\cdots\mu_k\nu_1\cdots\nu_k} \end{pmatrix} \psi_{\alpha\beta;\gamma}{}^\delta(k,q,P) , \tag{62}$$

$$M^{\mu_1\cdots\mu_k} = \begin{cases} \gamma_T^{\mu_1}\cdots\gamma_T^{\mu_k}\gamma_5 \\ \gamma_T^{\mu_1}\cdots\hat{k}_T^{\mu_k}\gamma_5 \\ \hat{k}_T^{\mu_1}\cdots\hat{q}_t^{\mu_k}\gamma_5 \\ \cdots \end{cases} , \tag{63}$$

with $M^{\mu_1\cdots\mu_k}$ representing all combinations of k products of $\gamma_T^{\mu_i}$, $k_T^{\mu_i}$ and $q_t^{\mu_i}$. Here, $\mathbb{P}^{\mu_1\cdots\mu_k\nu_1\cdots\nu_k}$ is the generalised Rarita-Schwinger projector. A linearly independent subspace spans $64(k+1)$ elements. As an example, we give the Rarita-Schwinger projector for spin-$3/2$

$$\mathbb{P}^{\mu_1\nu_1} = T_P^{\mu_1\nu_1} - \frac{1}{3}\gamma_T^{\mu_1}\gamma_T^{\nu_1} , \tag{64}$$

where note that we omitted the here redundant positive energy projector Λ_+. In this case 128 linearly independent basis-elements can be constructed [42].
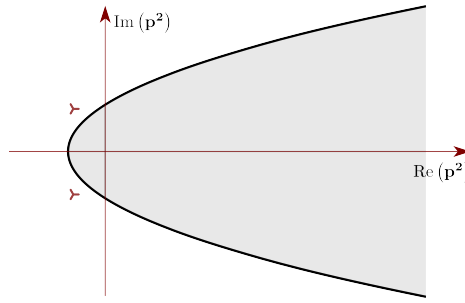
Figure 3. A sketch of the bounded parabolic region of the complex plane probed by the quark propagators in the Bethe-Salpeter equation. Crosses symbolise the appearance of complex conjugate poles in the timelike complex region.

6.2. Euclidean spacetime and quarks in the complex plane

Calculations using DSEs and BSEs are mainly performed in Euclidean momentum space (for studies using Minkowski spacetime see [86, 87] and references therein). In particular, this means that for a bound state of mass M, the total momentum P in the rest frame is

$$P = (0, 0, 0, i\,M) \ . \tag{65}$$

Let us focus, for simplicity, on the case of a mesonic bound state. If the relative momentum between the constituent quarks is k, the momentum of the constituents can be written as

$$p_\pm = k \pm \frac{P}{2} \ . \tag{66}$$

The propagators for these constituents are functions of $p_\pm^2$. Using (66) and assuming that the relative momentum k is real, one can see that

$$p_\pm^2 = (t \pm i\,M)^2 \tag{67}$$

for some real and positive parameter t. This is the parametric equation for a parabola. Therefore, the quark dressing functions need to be known in a parabolic region of the complex plane (see Fig. 3). It is a general feature of the analytic structure of the quark propagators in the complex plane to have complex conjugate poles in the timelike complex region. These poles pose a limitation on the maximum mass of the bound state that can be calculated as well as the spacelike region for which form factors can be studied (see, e.g. [88]). The possibility of parametrising the quark propagators by analytic functions that allow better control over its singularities has been explored, *e.g.* in [48, 89, 90].

7. Conclusions

In these proceedings we have collected the key aspects for a systematic study of hadronic properties such as masses and form-factors using the combination of Dyson-Schwinger and Bethe-Salpeter equations. The use of nPI effective action techniques proves to be a powerful resource in this respect.

It should be clear from the presentation that the now ubiquitous rainbow-ladder truncation is the first term in a systematic expansion of the Bethe-Salpeter kernel using the effective action; in particular it appears when a renormalisation-group improved (RGI) 2PI effective action at two-loops is used. Present investigations [61] are aimed at extending the calculation of meson and baryon spectra using kernels derived from a three-loop truncation of the effective action.

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