

# Fermions embedded in a scalar-vector kink-like smooth potential

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**Abstract.** The behaviour of massive fermions is analyzed with scalar and vector potentials. A continuous chiral-conjugation transformation decouples the equation for the upper component of the Dirac spinor provided the vector coupling does not exceed the scalar coupling. It is shown that a Sturm-Liouville perspective is convenient for studying scattering as well as bound states. One possible isolated solution (excluded from the Sturm-Liouville problem) corresponding to a bound state might also come into sight. For potentials with kink-like profiles, beyond the intrinsically relativistic isolated bound-state solution corresponding to the zero-mode solution of the massive Jackiw-Rebbi model in the case of no vector coupling, a finite set of bound-state solutions might appear as poles of the transmission amplitude in a strong coupling regime. It is also shown that the possible isolated bound solution disappears asymptotically as the magnitude of the scalar and vector coupling becomes the same. Furthermore, we show that due to the sizeable mass gain from the scalar background the high localization of the fermion in an extreme relativistic regime is conformable to comply with the Heisenberg uncertainty principle.

## 1. Introduction

The Dirac Hamiltonian with a mixing of scalar potential and time component of vector potential in a four-dimensional space-time is invariant under an SU(2) algebra when the difference between the potentials, or their sum, is a constant [1]. The near realization of these symmetries may explain degeneracies in some heavy meson spectra (spin symmetry) [2, 3] or in single-particle energy levels in nuclei (pseudospin symmetry) [3–38]. When these symmetries are realized, the energy spectrum does not depend on the spinorial structure, being identical to the spectrum of a spinless particle [39]. Despite the absence of spin effects in 1+1 dimensions, many attributes of the spin and pseudospin symmetries in four dimensions are preserved.

In a pioneering work, Jackiw and Rebbi [40] have shown that massless fermions coupled to scalar fields with kink-like profiles in 1+1 dimensions develops quantum states with fractional fermion number due to the zero-mode solution. This phenomenon has been seen in certain polymers such as polyacetylene [41–44]. Recently the complete set of solutions for the kink-like scalar potential behaving like  $\tanh x/\lambda$  has been considered for massless fermions in Ref. [45], and for massive fermions in Ref. [46]. The complete set of solutions for massive fermions under the influence of a kink-like scalar potential added by the time component of a vector potential with the same functional form was considered in Refs. [47] and [48], in Ref. [47] for the background field behaving like  $\operatorname{sgn} x$ , and in Ref. [48] for the background behaving like  $\tanh x/\lambda$ .



In Refs. [47] and [48], it has been shown that the Dirac equation with a scalar potential plus a time component of vector potential of the same functional form is manageable if the vector coupling does not exceed the scalar coupling, and that the bound states for mixed scalar-vector potentials with kink-like profiles are intrinsically relativistic solutions. Furthermore, it has been shown that the fermion can be confined in a highly localized region of space under a very strong field without any chance for spontaneous pair production related to Klein's paradox.

Here we shall outline the scalar-vector mixing framework developed in Refs. [47]-[48] with the Sturm-Liouville perspective plus its isolated solution. We show that the isolated solution disappears asymptotically as one approaches the conditions for the realization of the so-called spin and pseudospin symmetries in four dimensions. After a general consideration of the Sturm-Liouville problem for an arbitrary kink-like potential, we concentrate our attention on the isolated solution which corresponds to the zero-mode solution of the Jackiw-Rebbi model in the case of a pure scalar coupling. It is shown that such an isolated solution is an intrinsically relativistic solution even if the fermion is massive. Then we use the smooth step potential  $\tanh x/\lambda$  to show in detail how the additional mass acquired by the fermion from the scalar background can acquiesce highly localized states without violating the Heisenberg uncertainty principle.

## 2. Mixed scalar-vector interactions

Consider the Lagrangian density for a massive fermion

$$L = \bar{\Psi} (i\hbar c \gamma^\mu \partial_\mu - Imc^2 - V) \Psi \quad (1)$$

where  $\hbar$  is the constant of Planck,  $c$  is the velocity of light,  $I$  is the unit matrix,  $m$  is the mass of the free fermion and the square matrices  $\gamma^\mu$  satisfy the algebra  $\{\gamma^\mu, \gamma^\nu\} = 2I g^{\mu\nu}$ . The spinor adjoint to  $\Psi$  is defined by  $\bar{\Psi} = \Psi^\dagger \gamma^0$ . In 1+1 dimensions  $\Psi$  is a  $2 \times 1$  matrix and the metric tensor is  $g^{\mu\nu} = \text{diag}(1, -1)$ . For vector and scalar interactions the matrix potential is written as

$$V = \gamma^\mu A_\mu + IV_s \quad (2)$$

We say that  $A_\mu$  and  $V_s$  are the vector and scalar potentials, respectively, because the bilinear forms  $\bar{\Psi} \gamma^\mu \Psi$  and  $\bar{\Psi} I \Psi$  behave like vector and scalar quantities under a Lorentz transformation, respectively. Eq. (1) leads to the Hamiltonian form for the Dirac equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H \Psi \quad (3)$$

with the Hamiltonian given as

$$H = \gamma^5 c \left( p_1 + \frac{A_1}{c} \right) + IA_0 + \gamma^0 (mc^2 + V_s) \quad (4)$$

where  $\gamma^5 = \gamma^0 \gamma^1$ . Requiring  $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$ , one finds the continuity equation  $\partial_\mu J^\mu = 0$ , where the conserved current is  $J^\mu = c \bar{\Psi} \gamma^\mu \Psi$ . The positive-definite function  $J^0/c = |\Psi|^2$  is interpreted as a position probability density and its norm is a constant of motion. This interpretation is completely satisfactory for single-particle states [49]. The space component of the vector potential can be gauged away by defining a new spinor just differing from the old by a phase factor so that we can consider  $A_1 = 0$  without loss of generality.

Assuming that the potentials are time independent, one can write  $\Psi(x, t) = \psi(x) \exp(-iEt/\hbar)$  in such a way that the time-independent Dirac equation becomes  $H\psi = E\psi$ . Meanwhile  $J^\mu = c \bar{\psi} \gamma^\mu \psi$  is time independent and  $J^1$  is uniform.

From now on, we use an explicit representation for the  $2 \times 2$  matrices  $\gamma$  as

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2 \quad (5)$$

in such a way that  $\gamma^5 = \sigma_1$ . Here,  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  stand for the Pauli matrices.

### 2.1. Nonrelativistic limit

The Lorentz nature of the potentials does no matter in a weak-coupling regime. Indeed, fermions (antifermions) are subject to the effective potential  $V_s + A_0$  ( $V_s - A_0$ ) with energy  $E \approx +mc^2$  ( $-mc^2$ ) so that a mixed potential with  $A_0 = -V_s$  ( $A_0 = +V_s$ ) is associated with free fermions (antifermions) in a nonrelativistic regime [47]-[48]. The changes of signs of  $A_0$  and  $E$  as well as the invariance of the sign of  $V_s$  when one exchanges the roles of fermions and antifermions are justified by the charge-conjugation transformation.

### 2.2. Extreme relativistic limit

For potentials localized in the range  $\lambda$ , quantum effects appear when  $\lambda$  is comparable to the Compton wavelength  $\lambda_C = \hbar/mc$ , and relativistic quantum effects are expected when  $\lambda$  is of the same order or smaller than the Compton wavelength. The fermion is under an extreme relativistic regime if  $\lambda \ll \lambda_C$ . When the fermion localized in the region  $\Delta x$  reduces its extension (increasing the intensity of the potential, for example) then the uncertainty in the momentum must expand, in consonance with the Heisenberg uncertainty principle. Nevertheless, the maximum uncertainty in the momentum is comparable with  $mc$  requiring that is impossible to localize a fermion in a region of space less than or comparable with half of its Compton wavelength (see, for example, [50, 51]). Nevertheless, due to the mass gain granted by its interaction with the scalar-field background, those bound states could be highly localized without any chance of spontaneous pair production.

### 2.3. Charge conjugation

The charge-conjugation operation is accomplished by the transformation  $\psi \rightarrow \sigma_1 \psi^*$  followed by  $A_0 \rightarrow -A_0$ ,  $V_s \rightarrow V_s$  and  $E \rightarrow -E$  [52]. As a matter of fact,  $A_0$  distinguishes fermions from antifermions but  $V_s$  does not, and so the spectrum is symmetrical about  $E = 0$  in the case of a pure scalar potential.

### 2.4. Chiral conjugation

The chiral-conjugation operation  $\psi \rightarrow \gamma^5 \psi$  (according to Ref. [53]) is followed by the changes of the signs of  $V_s$  and  $m$ , but not of  $A_0$  and  $E$  [52]. One sees that the charge-conjugation and the chiral-conjugation operations interchange the roles of the upper and lower components of the Dirac spinor.

### 2.5. Continuous chiral conjugation

The unitary operator

$$U(\theta) = \exp \left( -\frac{\theta}{2} i \gamma^5 \right) \quad (6)$$

where  $\theta$  is a real quantity such that  $0 \leq \theta \leq \pi$ , allows one to write

$$h\phi = E\phi \quad (7)$$

where

$$\phi = U\psi, \quad h = UHU^{-1} \quad (8)$$

with

$$h = \sigma_1 cp_1 + IA_0 + \sigma_3 (mc^2 + V_s) \cos \theta - \sigma_2 (mc^2 + V_s) \sin \theta \quad (9)$$

It is instructive to note that the transformation preserves the form of the current in such a way that  $J^\mu = c\bar{\phi}\gamma^\mu\phi$ . An additional important feature of the continuous chiral transformation (see, e.g., [54]) induced by (6) is that it is a symmetry transformation when  $m = V_s = 0$ .

In terms of the upper and the lower components of the spinor  $\phi$ , the Dirac equation decomposes into:

$$\hbar c \frac{d\phi_{\pm}}{dx} \pm (mc^2 + V_s) \sin \theta \phi_{\pm} = i [E \pm (mc^2 + V_s) \cos \theta - A_0] \phi_{\mp} \quad (10)$$

Furthermore,

$$\frac{J^0}{c} = |\phi_+|^2 + |\phi_-|^2, \quad \frac{J^1}{c} = 2\text{Re}(\phi_+^* \phi_-) \quad (11)$$

## 2.6. Special mixing

Choosing

$$A_0 = V_s \cos \theta \quad (12)$$

one has

$$\hbar c \frac{d\phi_+}{dx} + (mc^2 + V_s) \sin \theta \phi_+ = i (E + mc^2 \cos \theta) \phi_- \quad (13a)$$

$$\hbar c \frac{d\phi_-}{dx} - (mc^2 + V_s) \sin \theta \phi_- = i [E - (mc^2 + 2V_s) \cos \theta] \phi_+ \quad (13b)$$

Note that due to the constraint represented by (12), the vector and scalar potentials have the very same functional form and the parameter  $\theta$  in (6) measures the dosage of vector coupling in the vector-scalar admixture in such a way that  $|V_s| \geq |A_0|$ . Note also that when the mixing angle  $\theta$  goes from  $\pi/2 - \varepsilon$  to  $\pi/2 + \varepsilon$  the sign of the spectrum undergoes an inversion under the charge-conjugation operation whereas the spectrum of a massless fermion is invariant under the chiral-conjugation operation. Combining charge-conjugation and chiral-conjugation operations makes the spectrum of a massless fermion to be symmetrical about  $E = 0$  in spite of the presence of vector potential.

We now split two classes of solutions depending on whether  $E$  is equal to or different from  $-mc^2 \cos \theta$ .

## 2.7. The class $E \neq -mc^2 \cos \theta$

For  $E \neq -mc^2 \cos \theta$ , using the expression for  $\phi_-$  obtained from (13a), viz.

$$\phi_- = \frac{-i}{E + mc^2 \cos \theta} \left[ \hbar c \frac{d\phi_+}{dx} + (mc^2 + V_s) \sin \theta \phi_+ \right] \quad (14)$$

one finds

$$J^1 = \frac{2\hbar c^2}{E + mc^2 \cos \theta} \text{Im} \left( \phi_+^* \frac{d\phi_+}{dx} \right) \quad (15)$$

Inserting (14) into (13b) one arrives at the following second-order differential equation for  $\phi_+$ :

$$-\frac{\hbar^2}{2} \frac{d^2 \phi_+}{dx^2} + V_{\text{eff}} \phi_+ = E_{\text{eff}} \phi_+ \quad (16)$$

where

$$V_{\text{eff}} = \frac{\sin^2 \theta}{2c^2} V_s^2 + \frac{mc^2 + E \cos \theta}{c^2} V_s - \frac{\hbar \sin \theta}{2c} \frac{dV_s}{dx} \quad (17)$$

and

$$E_{\text{eff}} = \frac{E^2 - m^2 c^4}{2c^2} \quad (18)$$

Therefore, the solution of the relativistic problem for this class is mapped into a Sturm-Liouville problem for the upper component of the Dirac spinor. In this way one can solve the Dirac problem for determining the possible discrete or continuous eigenvalues of the system by recurring to the solution of a Schrödinger-like problem because  $\phi_+$  is a square-integrable function.

Notice that the effective potential in (17) behaves like  $V_s$  in the case of scalar and vector potentials of the same magnitude. For the case of a pure scalar coupling ( $E \neq 0$ ), it is also possible to write a second-order differential equation for  $\phi_-$  just differing from the equation for  $\phi_+$  in the sign of the term involving  $dV_s/dx$ , namely,

$$-\frac{\hbar^2}{2} \frac{d^2 \phi_{\pm}}{dx^2} + \left( \frac{V_s^2}{2c^2} + mV_s \mp \frac{\hbar}{2c} \frac{dV_s}{dx} \right) \phi_{\pm} = E_{\text{eff}} \phi_{\pm} \quad (19)$$

which is of the form of supersymmetric quantum mechanics, as has already been appreciated in the literature [55, 56].

### 2.8. The class $E = -mc^2 \cos \theta$

Defining

$$v(x) = \int^x dy V_s(y) \quad (20)$$

the solutions for (13a) and (13b) with  $E = -mc^2 \cos \theta$  are

$$\phi_+ = N_+ \quad (21a)$$

$$\phi_- = N_- - 2 \frac{i}{\hbar c} N_+ [mc^2 x + v(x)] \cos \theta \quad (21b)$$

for  $\sin \theta = 0$ , and

$$\phi_+ = N_+ \exp \left\{ -\frac{\sin \theta}{\hbar c} [mc^2 x + v(x)] \right\} \quad (22a)$$

$$\phi_- = N_- \exp \left\{ +\frac{\sin \theta}{\hbar c} [mc^2 x + v(x)] \right\} + i\phi_+ \cot \theta \quad (22b)$$

for  $\sin \theta \neq 0$ .  $N_+$  and  $N_-$  are normalization constants. It is instructive to note that there is no solution for scattering states. Both set of solutions present a space component for the current equal to  $J^1 = 2c \text{Re}(N_+^* N_-)$  and a bound-state solution demands  $N_+ = 0$  or  $N_- = 0$ , because  $\phi_+$  and  $\phi_-$  are square-integrable functions vanishing as  $|x| \rightarrow \infty$ . It is remarkable that the eigenenergy does not depend on the magnitude of the potential but the eigenspinor does. Note also that

$$\phi_{\pm} = N_{\pm} \exp \left\{ \mp \frac{1}{\hbar c} [mc^2 x + v(x)] \right\} \quad (23)$$

in the case of a pure scalar coupling ( $E = 0$ ) so that either  $\phi_+ = 0$  or  $\phi_- = 0$ . There is no bound-state solution for  $\sin \theta = 0$ , and for  $\sin \theta \neq 0$  the existence of a bound state solution depends on the asymptotic behaviour of  $v(x)$  [57, 58].

## 3. Kink potentials

Now we consider a kink-like potential with the asymptotic behaviour  $V_s(x) \rightarrow \pm v_0$  as  $x \rightarrow \pm x_0$ , with  $v_0 = \text{constant}$  and  $x_0 \gg \lambda$  ( $\lambda$  is the range of the interaction centered on the origin).

We turn our attention to scattering states for fermions with  $E \neq -mc^2 \cos \theta$  coming from the left. Then,  $\phi$  for  $x \rightarrow -\infty$  describes an incident wave moving to the right and a reflected wave moving to the left, and  $\phi$  for  $x \rightarrow +\infty$  describes a transmitted wave moving to the right or an evanescent wave. The upper components for scattering states are written as

$$\phi_+ = \begin{cases} A_+ e^{+ik_-x} + A_- e^{-ik_-x}, & \text{for } x \rightarrow -\infty \\ B_{\pm} e^{\pm ik_+x}, & \text{for } x \rightarrow +\infty \end{cases} \quad (24)$$

where

$$\hbar ck_{\pm} = \sqrt{(E \mp v_0 \cos \theta)^2 - (mc^2 \pm v_0)^2} \quad (25)$$

Note that  $k_+$  is a real number for a progressive wave and an imaginary number for an evanescent wave ( $k_-$  is a real number for scattering states). Therefore,

$$J_{\geq}^1(-\infty) = \frac{2\hbar c^2 k_-}{E + mc^2 \cos \theta} (|A_{\pm}|^2 - |A_{\mp}|^2), \quad \text{for } E \gtrless -mc^2 \cos \theta \quad (26)$$

and

$$J_{\geq}^1(+\infty) = \pm \frac{2\hbar c^2 \text{Re } k_+}{E + mc^2 \cos \theta} |B_{\pm}|^2, \quad \text{for } E \gtrless -mc^2 \cos \theta \quad (27)$$

Note also that

$$J_{\geq}^1(-\infty) = J_{\text{inc}} - J_{\text{ref}} \quad (28)$$

and

$$J_{\geq}^1(+\infty) = J_{\text{tran}} \quad (29)$$

where  $J_{\text{inc}}$ ,  $J_{\text{ref}}$  and  $J_{\text{tran}}$  are nonnegative quantities characterizing the incident, reflected and transmitted waves, respectively. Note also that the roles of  $A_+$  and  $A_-$  are exchanged as the sign of  $E + mc^2 \cos \theta$  changes. In fact, if  $E > -mc^2 \cos \theta$ , then  $A_+ e^{+ik_-x}$  ( $A_- e^{-ik_-x}$ ) will describe the incident (reflected) wave, and  $B_- = 0$ . On the other hand, if  $E < -mc^2 \cos \theta$ , then  $A_- e^{-ik_-x}$  ( $A_+ e^{+ik_-x}$ ) will describe the incident (reflected) wave, and  $B_+ = 0$ . Therefore, the reflection and transmission amplitudes are given by

$$r_{\geq} = \frac{A_{\mp}}{A_{\pm}}, \quad t_{\geq} = \frac{B_{\pm}}{A_{\pm}}, \quad \text{for } E \gtrless -mc^2 \cos \theta \quad (30)$$

To determine the transmission coefficient we use the current densities  $J_{\geq}^1(-\infty)$  and  $J_{\geq}^1(+\infty)$ . The  $x$ -independent space component of the current allows us to define the reflection and transmission coefficients as

$$R_{\geq} = \frac{|A_{\mp}|^2}{|A_{\pm}|^2}, \quad T_{\geq} = \frac{\text{Re } k_+}{k_-} \frac{|B_{\pm}|^2}{|A_{\pm}|^2}, \quad \text{for } E \gtrless -mc^2 \cos \theta \quad (31)$$

Notice that  $R_{\geq} + T_{\geq} = 1$  by construction. The possibility of bound states requires a solution with an asymptotic behaviour given by (24) with  $k_{\pm} = i|k_{\pm}|$  and  $A_+ = B_- = 0$ , or  $k_{\pm} = -i|k_{\pm}|$  and  $A_- = B_+ = 0$ , to obtain a square-integrable  $\phi_+$ . On the other hand, if one considers the transmission amplitude  $t$  in (30) as a function of the complex variables  $k_{\pm}$  one sees that for  $k_{\pm} > 0$  one obtains the scattering states whereas the bound states would be obtained by the poles lying along the imaginary axis of the complex  $k$ -plane.

As for  $E = -mc^2 \cos \theta$ , the existence of a bound-state solution requires  $|v_0| > mc^2$  so that the eigenspinor behaves asymptotically as

$$\phi \sim \begin{pmatrix} 1 \\ i \cot \theta \end{pmatrix} f \quad (32)$$

for  $v_0 > mc^2$ , and

$$\phi \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} f \quad (33)$$

for  $v_0 < -mc^2$ . Here,

$$f = \exp \left\{ -\frac{\sin \theta}{\hbar c} [|v_0| + mc^2 \operatorname{sgn}(v_0 x)] |x| \right\} \quad (34)$$

Armed with the knowledge about asymptotic solutions and with the definition of the transmission coefficient we should proceed for searching solutions on the entire region of space. Nevertheless, we can not tell much more about the problem until the potential function is specified.

#### 4. The smooth step potential

Now the scalar potential takes the form

$$V_s = v_0 \tanh x/\lambda \quad (35)$$

where the skew positive parameter  $\lambda$  is related to the range of the interaction which makes  $V_s$  to change noticeably in the interval  $-\lambda < x < \lambda$ , and  $v_0$  is the height of the potential at  $x = +\infty$ . Notice that as  $\lambda \rightarrow 0$ , the case of an extreme relativistic regime, the smooth step approximates the sign potential already considered in Ref. [47].

As commented before, there is no isolated solution from the Sturm-Liouville problem when  $\sin \theta = 0$ , and the existence of a well-behaved isolated solution when  $\sin \theta \neq 0$  makes

$$v(x) = \lambda v_0 \ln(\cosh x/\lambda) \quad (36)$$

and requires  $|v_0| > mc^2$ :

$$\phi = \begin{pmatrix} 1 \\ i \cot \theta \end{pmatrix} N_> f \quad (37)$$

for  $v_0 > mc^2$ , and

$$\phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} N_< f \quad (38)$$

for  $v_0 < -mc^2$ . Here,

$$f = \frac{\exp(-\alpha_1 x)}{\cosh^{\alpha_2} x/\lambda} \quad (39)$$

where

$$\alpha_1 = \frac{\operatorname{sgn}(v_0) mc \sin \theta}{\hbar}, \quad \alpha_2 = \frac{\lambda |v_0| \sin \theta}{\hbar c} \quad (40)$$

The normalization condition

$$\int_{-\infty}^{+\infty} dx (|\phi_+|^2 + |\phi_-|^2) = 1 \quad (41)$$

and the integral tabulated (see the formula 3.512.1, or 8.380.10, in Ref. [59])

$$\int_0^\infty dx \frac{\cosh 2\beta_1 x}{\cosh^{2\beta_2} \gamma x} = \frac{2^{2\beta_2}}{4\gamma} B\left(\beta_2 + \frac{\beta_1}{\gamma}, \beta_2 - \frac{\beta_1}{\gamma}\right) \quad (42)$$

where  $B(z_1, z_2)$  is the beta function [60], allow one to determine  $N_{\geq}$ . In the way indicated one can find the position probability density [48]

$$|\phi|^2 = \frac{2f^2}{2^{2\alpha_2} \lambda B(\alpha_+, \alpha_-)} \quad (43)$$

where

$$\alpha_{\pm} = \alpha_2 \pm \lambda \alpha_1 \quad (44)$$

Therefore, a massive fermion tends to concentrate at the left (right) region when  $v_0 > 0$  ( $v_0 < 0$ ), and tends to avoid the origin more and more as  $\sin \theta$  decreases. A massless fermion has a position probability density symmetric around the origin. One can see that the best localization occurs for a pure scalar coupling. In fact, the fermion becomes delocalized as  $\sin \theta$  decreases.

The expectation value of  $x$  and  $x^2$  is given by

$$\langle x \rangle = -\frac{4}{2^{2\alpha_2} \lambda B(\alpha_+, \alpha_-)} \int_0^\infty dx \frac{x \sinh 2\alpha_1 x}{\cosh^{2\alpha_2} x/\lambda} \quad (45)$$

and

$$\langle x^2 \rangle = \frac{4}{2^{2\alpha_2} \lambda B(\alpha_+, \alpha_-)} \int_0^\infty dx \frac{x^2 \cosh 2\alpha_1 x}{\cosh^{2\alpha_2} x/\lambda} \quad (46)$$

Defining

$$\Delta(\alpha) = \psi(\alpha_+) - \psi(\alpha_-) \quad (47a)$$

$$\Sigma^{(1)}(\alpha) = \psi^{(1)}(\alpha_+) + \psi^{(1)}(\alpha_-) \quad (47b)$$

where

$$\psi(z) = \frac{d \ln \Gamma(z)}{dz} \quad (48)$$

is the digamma (psi) function and

$$\psi^{(1)}(z) = \frac{d\psi(z)}{dz} \quad (49)$$

is the trigamma function [60], printed in a boldface type to differ from the Dirac eigenspinor, and using a pair of integrals tabulated in [48],  $\langle x \rangle$  and  $\langle x^2 \rangle$  can be simplified to

$$\langle x \rangle = -\frac{\lambda}{2} \Delta(\alpha) \quad (50)$$

and

$$\langle x^2 \rangle = \frac{\lambda^2}{4} \Sigma^{(1)}(\alpha) + \langle x \rangle^2 \quad (51)$$

and hence the fermion is confined within an interval

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \quad (52)$$

given by

$$\Delta x = \frac{\lambda}{2} \sqrt{\Sigma^{(1)}(\alpha)} \quad (53)$$

With the help of a few approximate formulas for the special functions in Ref. [48], one obtains the values for  $\langle x \rangle$  and  $\Delta x$  either in the case of  $\sin \theta \simeq 0$  or  $\lambda \ll \lambda_C$ :

$$\langle x \rangle \simeq -\operatorname{sgn}(v_0) \frac{\hbar c}{\sin \theta} \frac{mc^2}{v_0^2 - m^2 c^4} \quad (54a)$$

$$\Delta x \simeq \frac{\hbar c}{\sqrt{2} \sin \theta} \frac{\sqrt{v_0^2 + m^2 c^4}}{v_0^2 - m^2 c^4} \quad (54b)$$

One can also see that when  $\lambda \gg \lambda_C$  or  $|v_0| \gg mc^2$

$$\langle x \rangle \simeq \frac{\lambda}{2} \ln \frac{|v_0| - \text{sgn}(v_0) mc^2}{|v_0| + \text{sgn}(v_0) mc^2} \quad (55a)$$

$$\Delta x \simeq \sqrt{\frac{\lambda \hbar c}{2 \sin \theta} \frac{|v_0|}{v_0^2 - m^2 c^4}} \quad (55b)$$

Again one can see that the fermion becomes delocalized as  $\sin \theta$  decreases and that the best localization occurs for a pure scalar coupling. More than this,  $\langle x \rangle \rightarrow -\infty$  and  $\Delta x \rightarrow \infty$  as  $|v_0| \rightarrow mc^2$ , and besides  $\langle x \rangle \rightarrow 0$  and  $\Delta x \rightarrow 0$  as  $|v_0| \rightarrow \infty$ .

Those last results show that  $\Delta x$  reduces its extension with rising  $|v_0|$  or  $\sin \theta$ , or decreasing  $\lambda$ , in such a way that  $\Delta x$  can be arbitrarily small for a potential strong enough or short-range enough. The impasse related to the Heisenberg uncertainty principle can be broken by resorting to the concepts of effective mass and effective Compton wavelength. Indeed, if one defines an effective mass as

$$m_{\text{eff}} = m \sqrt{1 + \left( \frac{v_0}{mc^2} \right)^2} \quad (56)$$

and an effective Compton wavelength  $\lambda_{\text{eff}} = \hbar / (m_{\text{eff}} c)$ , one will find

$$\Delta x = \frac{\sqrt{2} \lambda_{\text{eff}}}{4 \sin \theta} \sqrt{(\alpha_+^2 + \alpha_-^2) \Sigma^{(1)}(\alpha)} \quad (57)$$

It follows that the high localization of fermions, related to high values of  $|v_0|$  or small values of  $\lambda$ , never menaces the single-particle interpretation of the Dirac theory even if the fermion is massless ( $m_{\text{eff}} = |v_0|/c^2$ ). As a matter of fact, (54b) furnishes

$$(\Delta x)_{\min} \simeq \frac{\lambda_{\text{eff}}}{\sqrt{2} \sin \theta} \quad (58)$$

for  $|v_0| \gg mc^2$  or  $\lambda \ll \lambda_C$ .

## 5. Final remarks

After reviewing the use of a continuous chiral transformation for solving the Dirac equation in the background of scalar and vector potential developed in Refs. [47]-[48], we have done an extension for arbitrary kink-like potentials which generalizes the Jackiw-Rebbi [40] model not only for considering massive fermions but also for taking into account an additional vector coupling. Concentrating our attention on the “zero-mode” solution, we have shown that, due to the sizeable additional mass acquired by the fermion resulting from its interaction with the scalar-field background, those bound states can be highly localized by very strong potentials without any chance of spontaneous pair production. This fact is convincing because the scalar is stronger than the vector coupling, and so the conditions for Klein’s paradox are never reached.

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## References

- [1] Bell J and Ruegg H 1975 *Nucl. Phys. B* **98** 151
- [2] Page P R, Goldman T and Ginocchio J N 2001 *Phys. Rev. Lett.* **86** 204
- [3] Ginocchio J N 2005 *Phys. Rep.* **414** 165
- [4] Ginocchio J N 1997 *Phys. Rev. Lett.* **78** 436

- [5] Ginocchio J N and Leviatan A 1998 *Phys. Lett. B* **425** 1
- [6] Lalazissis G A, Gambhir Y K, Maharana J P, Warke C S and Ring P 1998 *Phys. Rev. C* **58** R45
- [7] Meng J, Sugawara-Tanabe K, Yamaji S, Ring P and Arima A 1998 *Phys. Rev. C* **58** R628
- [8] Sugawara-Tanabe K and Arima A 1998 *Phys. Rev. C* **58** R3065
- [9] Ginocchio J N 1999 *Phys. Rep.* **315** 231
- [10] Marcos S, Savushkin L N, López-Quelle M and Ring P 2000 *Phys. Rev. C* **62** 054309
- [11] Alberto P, Fiolhais M, Malheiro M, Delfino A and Chiapparini M 2001 *Phys. Rev. Lett.* **86** 5015
- [12] Marcos S, Lpez-Quelle M, Niembro R, Savushkin L and Bernardos P 2001 *Phys. Lett. B* **513** 30
- [13] Ginocchio J N and Leviatan A 2001 *Phys. Rev. Lett.* **87** 072502
- [14] Alberto P, Fiolhais M, Malheiro M, Delfino A and Chiapparini M 2002 *Phys. Rev. C* **65** 034307
- [15] Ginocchio J N 2002 *Phys. Rev. C* **66** 064312
- [16] Ti-Sheng C, Hong-Feng L, Jie M, Shuang-Quan Z and Shan-Gui Z 2003 *Chinese Phys. Lett.* **20** 358
- [17] Zhou S G, Meng J and Ring P 2003 *Phys. Rev. Lett.* **91** 262501
- [18] Mao G 2003 *Phys. Rev. C* **67** 044318
- [19] Lisboa R, Malheiro M, de Castro A S, Alberto P and Fiolhais M 2004 *Phys. Rev. C* **69** 024319
- [20] Leviatan A 2004 *Phys. Rev. Lett.* **92** 202501
- [21] Guo J Y and Sheng Z Q 2005 *Phys. Lett. A* **338** 90
- [22] Alberto P, Lisboa R, Malheiro M and de Castro A S 2005 *Phys. Rev. C* **71** 034313
- [23] Guo J Y, Wang R D and Fang X Z 2005 *Phys. Rev. C* **72** 054319
- [24] Guo J Y, Fang X Z and Xu F X 2005 *Nuc. Phys. A* **757** 411
- [25] Berkdemir C 2006 *Nucl. Phys. A* **770** 32
- [26] Xu Q and Zhu S J 2006 *Nucl. Phys. A* **768** 161
- [27] He X, Zhou S, Meng J, Zhao E and Scheid W 2006 *Eur. Phys. J. A* **28** 265
- [28] Jolos R and Voronov V 2007 *Phys. Atomic Nuclei* **70** 812
- [29] Chun-Yan S, Jiang-Ming Y and Jie M 2009 *Chinese Phys. Lett.* **26** 122102
- [30] Liang H, Hui Long W, Meng J and Van Giai N 2010 *Eur. Phys. J. A* **44** 119
- [31] Lisboa R, Malheiro M, Alberto P, Fiolhais M and de Castro A S 2010 *Phys. Rev. C* **81** 064324
- [32] Chun-Yan S and Jiang-Ming Y 2010 *Chinese Phys. C* **34** 1425
- [33] Liang H, Zhao P, Zhang Y, Meng J and Giai N V 2011 *Phys. Rev. C* **83** 041301
- [34] Chun-Yan S, Jiang-Ming Y and Jie M 2011 *Chinese Phys. Lett.* **28** 092101
- [35] Lu B N, Zhao E G and Zhou S G 2012 *Phys. Rev. Lett.* **109** 072501
- [36] de Castro A S and Alberto P 2012 *Phys. Rev. A* **86** 032122
- [37] Alberto P, de Castro A S and Malheiro M 2013 *Phys. Rev. C* **87** 031301
- [38] Alberto P, Castro A, Fiolhais M, Lisboa R and Malheiro M 2014 *J. Phys.: Conf. Series* **490** 012069
- [39] Alberto P, de Castro A S and Malheiro M 2007 *Phys. Rev. C* **75** 047303
- [40] Jackiw R and Rebbi C 1976 *Phys. Rev. D* **13** 3398
- [41] Su W P, Schrieffer J R and Heeger A J 1979 *Phys. Rev. Lett.* **42** 1698
- [42] Goldstone J and Wilczek F 1981 *Phys. Rev. Lett.* **47** 986
- [43] Jackiw R and Semenoff G 1983 *Phys. Rev. Lett.* **50** 439
- [44] Niemi A J and Semenoff G W 1986 *Phys. Rep.* **135** 99
- [45] Charmchi F and Gousheh S 2014 *Phys. Rev. D* **89** 025002
- [46] Charmchi F and Gousheh S 2014 *Nucl. Phys. B* **883** 256
- [47] Castilho W M and de Castro A S 2014 *Ann. Phys. (NY)* **340** 1
- [48] Castilho W M and de Castro A S 2014 *Ann. Phys. (NY)* **346** 164
- [49] Thaller B 1992 *The Dirac Equation (Texts and Monographs in Physics)* (Berlin: Springer)
- [50] Greiner W 1990 *Relativistic quantum mechanics: wave equations* (Berlin: Springer)
- [51] Strange P 1998 *Relativistic Quantum Mechanics: with applications in condensed matter and atomic physics* (Cambridge: Cambridge University Press)
- [52] de Castro A S, Alberto P, Lisboa R and Malheiro M 2006 *Phys. Rev. C* **73** 054309
- [53] Watanabe S 1957 *Phys. Rev.* **106** 1306
- [54] Touschek B 1957 *Il Nuovo Cimento* **5** 754
- [55] Cooper F, Khare A, Musto R and Wipf A 1988 *Ann. Phys. (NY)* **187** 1
- [56] Nogami Y and Toyama F M 1993 *Phys. Rev. A* **47** 1708
- [57] de Castro A S 2005 *Ann. Phys. (NY)* **316** 414
- [58] de Castro A S and Hott M 2005 *Phys. Lett. A* **342** 53
- [59] Gradshteyn I and Ryzhik I 1980 *Table of Integrals, Series and Products* 2nd ed (San Diego: Academic)
- [60] Abramowitz M and Stegun I A (eds) 1965 *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables (Dover Books on Mathematics)* 9th ed (Dover: Dover Publications)