Strongly coupled quark gluon plasma in a magnetic field

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Abstract.
We derive the equation of state of a strongly coupled quark gluon plasma at finite temperature and high baryon density in a strong magnetic field. The formalism presented here is an extension of our previous works. We derive analytical expressions for the pression and energy density.

1. Introduction

After the analysis of RHIC and LHC data on heavy ion collisions, it became a consensus that the quark gluon plasma at high (but not asymptotic) temperatures is a strongly interacting system (which is now called strongly interacting quark gluon plasma or sQGP). This finding motivated a reanalysis of the previousy used equations of state (EOS), most of them based on the assumption of weak coupling between quarks and gluons. It is also reasonable to assume that the yet untested cold quark gluon plasma, which presumably exists in the core of dense stars, is also a strongly interacting system. In [1] we developed a mean field approach to the cold sQGP, which is inspired in the old relativistic mean field Walecka model. As in that model, because of the strong coupling and the large number of fermion sources, the vector boson is assumed to behave as a classical field. In contrast to the Walecka model, because of asymptotic freedom the low and high momentum modes of the gluon field have a different behavior. This justifies their separation into low ("soft") and high ("hard") momentum components already at the level of the Lagrangian. For the "hard" gluons we use the mean field approximation. The "soft" gluons are replaced by their vacuum expectation values, the in-medium gluon condensates. Using this approach we obtained the pressure and energy density of the sQGP. They both have a term similar to the bag constant usually found in the MIT bag model. Due to the strongly repulsive interaction of the vector field our EOS is much harder than the MIT one. The details of the derivation can be found in [1], where we discuss these approximations. In a subsequent paper, Ref.[2], we have applied our EOS [1] to the calculation of the structure of compact quark stars. We found that the inclusion of gluon interaction creates pressure and energy density large enough to generate stars wiht masses and radii consistent with the most recent astrophysical observations.

It is well accepted [3, 4, 5, 6, 7] that in astrophysical compact objects, such as magnetars, there is a strong magnetic field of intensity of $10^{12}\,G$ to $10^{18}\,G$. It is then natural to extend our formalism and include the effects of the magnetic field. This will be done in the next sections.

2. Equation of state

2.1. *Effective Lagrangian*

In what follows, we employ natural units: $\hbar = c = k_B = 1$ and the metric is given by $g_{\mu\nu} = \text{diag}(+,-,-,-)$. We first consider the homogeneous magnetic field in the Cartesian z direction:

$$\vec{B} = B\hat{z} \qquad \text{and hence} \qquad A_\mu = (0, yB, 0, 0) \tag{1}$$

Including this magnetic field in the QCD Lagrangian we obtain:

$$\mathcal{L}_{QCD} = +\sum_f \bar{\psi}_i^f \Big[i\gamma^\mu(\delta_{ij}\partial_\mu + i\delta_{ij}Q_f A_\mu - igT^a_{ij}G^a_\mu) - \delta_{ij}m_f \Big] \psi_j^f$$
$$-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2}$$

The summation in f runs over all quark flavors, m_f is the mass of the quark of flavor f, Q_f is the charge of the quark of flavor f, i and j are the color indices of the quarks, T^a are the SU(3) generators and f^{abc} are the SU(3) antisymmetric structure constants. The object $F^{a\mu\nu} = \partial^\mu G^{a\nu} - \partial^\nu G^{a\mu} + gf^{abc}G^{b\mu}G^{c\nu}$ is the gluon field tensor and the electromagnetic Lagrangian term is $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, with A^μ given by (1).

Performing the gluon field decomposition discussed above[1]:

$$G^{a\mu} = A^{a\mu} + \alpha^{a\mu}$$

and repeating the steps described in [1] we rewrite (2) as the effective Lagrangian:

$$\mathcal{L}_0 = \frac{{m_G}^2}{2}\alpha^a_0\alpha^a_0 - \frac{B^2}{2} - \mathcal{B}_{QCD}$$
$$+\sum_f \bar{\psi}_i^f \Big\{ i\gamma^\mu \Big[\delta_{ij}\partial_\mu + i\delta_{ij}Q_f A_\mu\Big] + g_h\gamma^0 T^a_{ij}\alpha^a_0 - \delta_{ij}m_f \Big\} \psi_j^f \tag{3}$$

where $\mathcal{B}_{QCD}$ is our equivalent of the bag constant given by $\mathcal{B}_{QCD} \equiv 9\,{\phi_0}^4/136$ and m_G is the dynamical gluon mass given by ${m_G}^2 \equiv 9\,{\mu_0}^2/32$, where μ_0 is an energy scale associated with $\langle A^2 \rangle$, which is the gluon condensate of dimension two [1]:

$$\langle A^2 \rangle \equiv \langle g_s^2 A^{a\mu} A^{b\nu} \rangle = \langle g_s^2 A^2 \rangle = -\frac{\delta^{ab} g^{\mu\nu}}{32}{\mu_0}^2 \tag{4}$$

and since $\langle g_s^2 A^2 \rangle < 0$ we always have ${m_G}^2 > 0$. The constant ϕ_0 is associated with $\langle F^2 \rangle$, which is the gluon condensate of dimension four [1]:

$$\mathcal{B}_{QCD} = b\phi_0^4 = \langle \frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} \rangle = \frac{\pi^2}{g_s^2}\langle F^2 \rangle \tag{5}$$

In the above expression we have two QCD coupling constants given by g_s and g_h. The coupling g_s is associated to the soft gluons, while g_h is associated to the hard gluons as in [1].

2.2. Equations of motion and Landau levels

The equations of motion calculated from (3) are:

$$\Big[i\gamma^{\mu}\Big(\partial_{\mu}+iQ_f A_{\mu}\Big)+g_h\gamma^0 T^a\alpha_0^a-m_f\Big]\psi^f=0 \tag{6}$$

$$m_G{}^2\alpha_0^a=-g_h\sum_f\rho_f^a=-g_h\,\rho^a \tag{7}$$

$$\partial_{\mu}F^{\mu\nu}=\sum_f Q_f(\bar{\psi}^f\gamma^{\nu}\psi^f) \tag{8}$$

The color vector current $j^{a\nu}$ is given by:

$$j^{a\nu}=\sum_f\bar{\psi}_i^f\gamma^{\nu}T_{ij}^a\psi_j^f \tag{9}$$

and its temporal component ρ^a which appears in (7) is:

$$j^{a0}=\rho^a=\sum_f\bar{\psi}_i^f\gamma^0 T_{ij}^a\psi_j^f=\sum_f{\psi_i^{\dagger}}^f T_{ij}^a\psi_j^f \tag{10}$$

In order to solve (6), we perform the steps described in [8] and write ψ_j^f as follows:

$$\psi_j^f=c_j e^{-iEt}\begin{pmatrix}\Phi_f\\ \chi_f\end{pmatrix} \tag{11}$$

where c_j is the quark color vector used in some textbooks [9]:

$$c_1=\begin{pmatrix}1\\0\\0\end{pmatrix}\ \text{ for red, }\ c_2=\begin{pmatrix}0\\1\\0\end{pmatrix}\ \text{ for blue, }\ c_3=\begin{pmatrix}0\\0\\1\end{pmatrix}\ \text{ for green} \tag{12}$$

From the above definitions it follows that:

$$c_i^{\dagger}\delta_{ij}c_j=c_1^{\dagger}c_1+c_2^{\dagger}c_2+c_3^{\dagger}c_3=3 \tag{13}$$

For future purposes we will replace the above sum by the following average:

$$c_i^{\dagger}\delta_{ij}c_j\to\frac{c_i^{\dagger}\delta_{ij}c_j}{\text{(number of quark colors)}}=\frac{c_1^{\dagger}c_1+c_2^{\dagger}c_2+c_3^{\dagger}c_3}{3}=1 \tag{14}$$

From the exact solution [8] of the Dirac equation (6) with magnetic field and hard gluon terms, we have the following expression for the eigenvalues:

$$\Big(E_{\nu}^f+g_h\mathcal{A}\Big)^2=m_f^2+k_z^2+(2\nu+1)|Q_f|B-Q_f Bs \tag{15}$$

where $\nu=0,1,2,3,4,5\ldots$ and $s=+1$ or $s=-1$, for the projection *up* or *down* of the spin states, respectively. The momentum component along the magnetic field direction is given by k_z. As in our previous work [1] the constant $\mathcal{A}$ in (15) is the "algebra valued" result, $\mathcal{A}=c_i^{\dagger}T_{ij}^a c_j\alpha_0^a$ with the implicit summation over $i,j=1,2,3$ and $a=1,\ldots,8$.

With the help of (12) we are able to calculate the relation between ρ^a previously identified in (10) and the net quark density ρ. We perform the product $\rho^a\rho^a$ considering the average over the number of $SU(3)$ generators, which is 8, as follows:

$$\rho^a\rho^a = \sum_f \rho^a_f \sum_{f'} \rho^a_{f'} \longrightarrow \langle \sum_f \rho^a_f \sum_{f'} \rho^a_{f'} \rangle = \frac{1}{8} \sum_f \rho^a_f \sum_{f'} \rho^a_{f'}$$

$$= \frac{1}{8} \sum_f (\psi^{\dagger\, f}_i T^a_{ij} \psi^f_j) \sum_{f'} (\psi^{\dagger\, f'}_k T^a_{kl} \psi^{f'}_l) = \frac{1}{8} \sum_f (c^\dagger_i T^a_{ij} c_j) \psi^{\dagger\, f} \psi^f \sum_{f'} (c^\dagger_k T^a_{kl} c_l) \psi^{\dagger\, f'} \psi^{f'}$$

The result $(c^\dagger_i T^a_{ij} c_j)(c^\dagger_k T^a_{kl} c_l) = 3$ is obtained from the Gell-Mann matrices and from the color vectors (12):

$$\rho^a\rho^a = \sum_f \rho^a_f \sum_{f'} \rho^a_{f'} = \frac{3}{8} \sum_f (\psi^{\dagger\, f} \psi^f) \sum_{f'} (\psi^{\dagger\, f'} \psi^{f'})$$

As $\sum_f (\psi^{\dagger\, f}\psi^f) = \sum_f \rho_f = \rho$, where $f = u, d, s$ and ρ is the total net quark density, we have:

$$\rho^a\rho^a = \frac{3}{8}\rho^2 \tag{16}$$

The baryon density ρ_B is related to net quark density through:

$$\rho_B = \frac{1}{3}\rho \tag{17}$$

Returning now to (15) and rescaling the single particle energy as:

$$\tilde{E}^f_\nu \equiv E^f_\nu + g_h \mathcal{A} \tag{18}$$

we are able to rewrite (15) as:

$$\left(\tilde{E}^f_\nu\right)^2 = m^2_f + k^2_z + \Big[2\nu + 1 - s \times sgn(Q_f)\Big] |Q_f| B \tag{19}$$

where $Q_f = sgn(Q_f) \times |Q_f|$. Defining $2\nu + 1 - s \times sgn(Q_f) \equiv 2n$, (19) becomes:

$$\tilde{E}^{f(\pm)}_n = \pm\sqrt{m^2_f + k^2_z + 2n|Q_f|B} \tag{20}$$

and n denotes the n^{th} Landau level. We note that, except for the rescaling (18), the equation above is the one usually found in the literature.

2.3. Thermodynamical quantities

We next follow the calculations presented in [6, 10, 11] and starting from (3) we arrive at the following thermodynamical potential:

$$\Omega = \Bigg[-\frac{{m_G}^2}{2} \alpha^a_0 \alpha^a_0 + \mathcal{B}_{QCD} + \frac{B^2}{2} \Bigg] V$$

$$+ T \sum_f \sum_{\vec{k}, s, n} \Big\{ ln\Big(1 - d_f\Big) + ln\Big(1 - \bar{d}_f\Big) \Big\} \tag{21}$$

where V is the volume, T is the temperature and the fermion distribution functions are:

$$d_f \equiv \frac{1}{1+e^{(\mathcal{E}_n^f-\nu_f)/T}} \qquad \text{and} \qquad \bar{d}_f \equiv \frac{1}{1+e^{(\mathcal{E}_n^f+\nu_f)/T}} \tag{22}$$

Using (20) in the evaluation of (21) the energy is now defined as:

$$\mathcal{E}_n^f = \sqrt{m_f^2 + k_z^2 + 2n|Q_f|B} \tag{23}$$

and the effective chemical potential for the quark f is defined as:

$$\nu_f \equiv \mu_f + g_h(c_i^\dagger T_{ij}^a c_j)\alpha_0^a \tag{24}$$

For a magnetic field pointing along the z direction, the momentum of a charged particle is restricted to discrete Landau levels [6, 12] and hence:

$$\frac{S}{(2\pi)^2}\int\int dk_x\, dk_y = \frac{S|Q_i|B}{2\pi}$$

with S being the area in the $x-y$ plane. From this last expression we have:

$$\int\int dk_x\, dk_y = 2\pi|Q_i|B \tag{25}$$

and the statistical sum becomes:

$$\frac{1}{V}\sum_{\vec{k},s,n} \longrightarrow \frac{\gamma_f}{(2\pi)^3}\sum_n \int d^3k = \frac{\gamma_f|Q_i|B}{2\pi^2}\sum_n \int_0^\infty dk_z \tag{26}$$

where γ_f is the statistical degeneracy factor of the quark f. The parallel pressure ($p_\parallel$) to the magnetic field, the magnetization (M) and perpendicular pressure ($p_\perp$) are given respectively by [6, 12]:

$$p_\parallel = -\frac{\Omega}{V} \quad , \quad M = -\frac{1}{V}\frac{\partial\Omega}{\partial B} = \frac{\partial p_\parallel}{\partial B} \quad \text{and} \quad p_\perp = p_\parallel - MB \tag{27}$$

The quark density ρ and the entropy density s read [10, 11]:

$$\rho = -\frac{1}{V}\frac{\partial\Omega}{\partial\mu_f} \qquad \text{and} \qquad s = -\frac{1}{V}\frac{\partial\Omega}{\partial T} \tag{28}$$

The energy density ε is calculated from the Gibbs relation [10, 11]:

$$\varepsilon = -p_\parallel + Ts + \sum_f \mu_f\rho_f \tag{29}$$

The evaluation of (27) to (29) with the potential (21) gives the following results:

$$p_\parallel = \frac{3{g_h}^2}{16{m_G}^2}\rho^2 - \mathcal{B}_{QCD} - \frac{B^2}{2} + \sum_f \frac{\gamma_f|Q_f|B}{2\pi^2}\sum_n \int_0^\infty dk_z \frac{{k_z}^2}{\mathcal{E}_n^f}\left(d_f + \bar{d}_f\right) \tag{30}$$

$$M = -B - T\sum_f \frac{\gamma_f|Q_f|}{2\pi^2}\sum_n \int_0^\infty dk_z \left[ln(1-d_f) + ln(1-\bar{d}_f)\right]$$

$$-\sum_f \frac{\gamma_f |Q_f| B}{2\pi^2} \sum_n \int_0^\infty dk_z \left[\frac{d_f\, n|Q_f|}{\mathcal{E}_n^f} + \frac{\bar{d}_f\, n|Q_f|}{\mathcal{E}_n^f} \right] \tag{31}$$

$$p_\perp = \frac{3{g_h}^2}{16{m_G}^2}\rho^2 - \mathcal{B}_{QCD} + \frac{B^2}{2} + \sum_f \frac{\gamma_f |Q_f| B^2}{2\pi^2} \sum_n \int_0^\infty dk_z \left[\frac{d_f\, n|Q_f|}{\mathcal{E}_n^f} + \frac{\bar{d}_f\, n|Q_f|}{\mathcal{E}_n^f} \right] \tag{32}$$

$$\rho = \sum_f \frac{\gamma_f |Q_f| B}{2\pi^2} \sum_n \int_0^\infty dk_z \Big(d_f - \bar{d}_f \Big) \tag{33}$$

$$s = -\sum_f \frac{\gamma_f |Q_f| B}{2\pi^2} \sum_n \int_0^\infty dk_z \Big\{ d_f ln(d_f) + (1-d_f)\, ln(1-d_f) + \bar{d}_f\, ln(\bar{d}_f) + (1-\bar{d}_f)\, ln(1-\bar{d}_f) \Big\} \tag{34}$$

$$\varepsilon = \frac{3{g_h}^2}{16{m_G}^2}\rho^2 + \mathcal{B}_{QCD} + \frac{B^2}{2} + \sum_f \frac{\gamma_f |Q_f| B}{2\pi^2} \sum_n \int dk_z\, \mathcal{E}_n^f (d_f + \bar{d}_f) \tag{35}$$

2.4. Zero temperature

In the zero temperature limit [6, 10, 12], applied to astrophysics, we have the distributions (22) given by:

$$d_f = \Theta(\nu_f - \mathcal{E}_n^f) \qquad \text{and} \qquad \bar{d}_f = 0 \tag{36}$$

and also [10]:

$$\lim_{T\to 0} T\, ln\Big(1 - d_f\Big) = (\mathcal{E}_n^f - \nu_f) \qquad \text{and} \qquad \lim_{T\to 0} T\, ln\Big(1 - \bar{d}_f\Big) = 0 \tag{37}$$

The quark density at zero temperature is obtained by inserting (36) into (33) [6]:

$$\rho = \sum_f \frac{\gamma_f |Q_f| B}{2\pi^2} \sum_{n=0}^{n_{max}^f} k_{z,F}^f(n) \tag{38}$$

From $\Theta(\nu_f - \mathcal{E}_n^f)$ and using (23) we have the Fermi momentum for the quark f:

$$k_{z,F}^f(n) = \sqrt{{\nu_f}^2 - m_f^2 - 2n|Q_f|B} \tag{39}$$

The summation over the Landau levels is calculated on the condition that the expression under the square root in (39) is positive, i.e., ${\nu_f}^2 \geq m_f^2 + 2n|Q_f|B$ [6]. Thus

$$n \leq n_{max}^f = int\Big[\, ({\nu_f}^2 - m_f^2)/(2|Q_f|B) \Big] \tag{40}$$

where $int[a]$ denotes the integer part of a. Inserting the results (36) to (39) into (30), (31), (32) and (35) we find the energy density and the pressure components:

$$\varepsilon = \frac{3{g_h}^2}{16{m_G}^2}\rho^2 + \mathcal{B}_{QCD} + \frac{B^2}{2}$$

$$+ \sum_f \frac{\gamma_f |Q_f| B}{2\pi^2} \sum_{n=0}^{n_{max}^f} \int_0^{k_{z,F}^f(n)} dk_z \sqrt{m_f^2 + k_z^2 + 2n|Q_f|B} \tag{41}$$

$$p_{\parallel} = \frac{3{g_h}^2}{16{m_G}^2}\rho^2 - \mathcal{B}_{QCD} - \frac{B^2}{2}$$
$$+\sum_f \frac{\gamma_f|Q_f|B}{2\pi^2}\sum_{n=0}^{n^f_{max}}\int_0^{k^f_{z,F}(n)} dk_z \, \frac{{k_z}^2}{\sqrt{m_f^2+k_z^2+2n|Q_f|B}} \tag{42}$$

and

$$p_{\perp} = \frac{3{g_h}^2}{16{m_G}^2}\rho^2 - \mathcal{B}_{QCD} + \frac{B^2}{2}$$
$$+\sum_f \frac{\gamma_f|Q_f|^2B^2}{2\pi^2}\sum_{n=0}^{n^f_{max}} n\int_0^{k^f_{z,F}(n)} \frac{dk_z}{\sqrt{m_f^2+k_z^2+2n|Q_f|B}} \tag{43}$$

As in [6], the pressure components $p_{\parallel}$ and $p_{\perp}$ are different for any value of the magnetic field B. In the limit $B >> ({\nu_f}^2 - m_f^2)/2|Q_f|$ expression (40) may be rewritten as:

$$\lim_{B\to\infty} n \leq \lim_{B\to\infty} n^f_{max} = \lim_{B\to\infty} int\left[\frac{\left({\nu_f}^2 - m_f^2\right)}{2|Q_f|B}\right] = 0 \tag{44}$$

which implies that only the lowest Landau level ($n = 0$) contributes. In this limit (39) becomes:

$$k^{f\,(B\to\infty)}_{z,F} = \sqrt{{\nu_f}^2 - m_f^2} \tag{45}$$

In the large magnetic field limit (41) to (43) are finally written as:

$$\varepsilon_{\ (B\to\infty)} = \frac{3{g_h}^2}{16{m_G}^2}\rho^2 + \mathcal{B}_{QCD} + \frac{B^2}{2} + \sum_f \frac{\gamma_f|Q_f|B}{2\pi^2}\int_0^{k^{f\,(B\to\infty)}_{z,F}} dk_z\sqrt{m_f^2+k_z^2} \tag{46}$$

$$p_{\parallel\ (B\to\infty)} = \frac{3{g_h}^2}{16{m_G}^2}\rho^2 - \mathcal{B}_{QCD} - \frac{B^2}{2} + \sum_f \frac{\gamma_f|Q_f|B}{2\pi^2}\int_0^{k^{f\,(B\to\infty)}_{z,F}} dk_z \, \frac{{k_z}^2}{\sqrt{m_f^2+k_z^2}} \tag{47}$$

$$p_{\perp\ (B\to\infty)} = \frac{3{g_h}^2}{16{m_G}^2}\rho^2 - \mathcal{B}_{QCD} + \frac{B^2}{2} \tag{48}$$

and the quark density (38) reads:

$$\rho_{\ (B\to\infty)} = \sum_f \frac{\gamma_f|Q_f|B}{2\pi^2}\sqrt{{\nu_f}^2 - m_f^2} \tag{49}$$

3. Conclusions

We have derived the equation of state of the strongly interacting QGP at finite temperature under uniform magnetic field. In all expressions the term proportional to ${g_h}^2/{m_G}^2$ is new with respect to other works in the field and comes from the EOS [1]. Special attention was given to the zero temperature case, which will be applied to stellar structure calculations.

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5. References

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