

Optimal sampling in damage detection of flexural beams by continuous wavelet transform

B Basu¹, B M Broderick¹, L Montanari² and A Spagnoli²

¹Dept. Civil, Structural and Environmental Engineering, Trinity College, Dublin 2, Ireland

²DICATeA, University of Parma, Parco Area delle Scienze 181/A, Parma, Italy

E-mail: basub@tcd.ie

Abstract. Modern measurement techniques are improving in capability to capture spatial displacement fields occurring in deformed structures with high precision and in a quasi-continuous manner. This in turn has made the use of vibration-based damage identification methods more effective and reliable for real applications. However, practical measurement and data processing issues still present barriers to the application of these methods in identifying several types of structural damage. This paper deals with spatial Continuous Wavelet Transform (CWT) damage identification methods in beam structures with the aim of addressing the following key questions: (i) can the cost of damage detection be reduced by down-sampling? (ii) what is the minimum number of sampling intervals required for optimal damage detection? The first three free vibration modes of a cantilever and a simple supported beam with an edge open crack are numerically simulated. A thorough parametric study is carried out by taking into account the key parameters governing the problem, including level of noise, crack depth and location, mechanical and geometrical parameters of the beam. The results are employed to assess the optimal number of sampling intervals for effective damage detection.

1. Modelling of the cracked beam

A cracked Euler-Bernoulli beam characterized by an open edge crack under different boundary conditions at the two ends, i.e. clamped-free (cantilever beam) and supported-supported (simply supported beam), is considered (Figure 1). The free vibration responses are analytically evaluated by solving the free vibration equations of the two uncracked sub-beams connected by a rotational spring (representing the local stiffness k_c of the cracked cross-section of the beam) at the crack location x_c . The beam has a rectangular cross-section with height h and width b ; the crack depth is a and L is the beam length. The symbols I , A , E and ρ represent, respectively, moment of inertia and area of the cross-section and Young's modulus and density of the material.



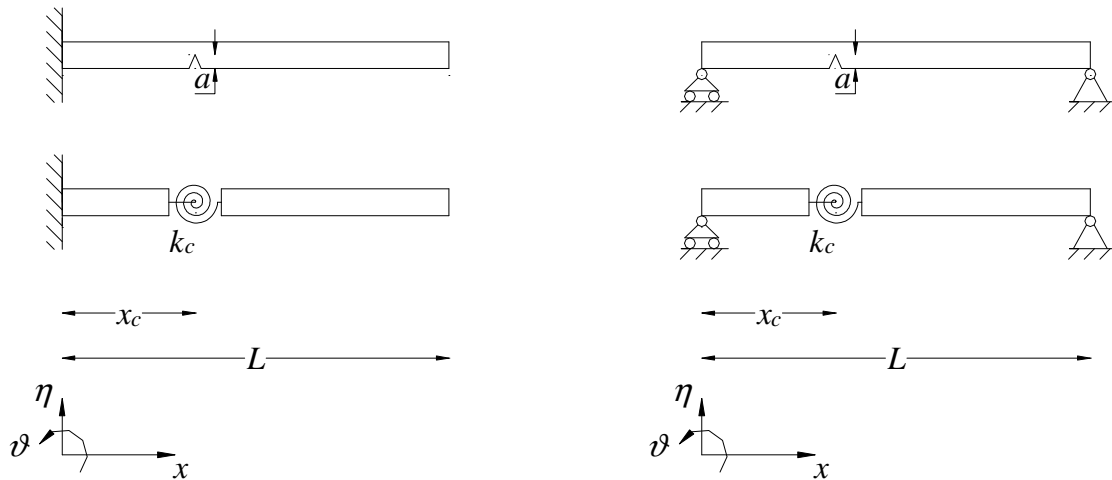


Figure 1. Sketch of the two cracked beams under study.

The free vibration equation related to the two uncracked parts of the beam can be written as

$$EI \frac{\partial^4 v(x,t)}{\partial x^4} + \rho A \frac{\partial^2 v(x,t)}{\partial t^2} = 0, \quad (1)$$

where $v(x, t)$ is the transversal displacement of the beam from its static equilibrium position at a distance x from the left end at the time t . Separating the variables in Eq. (1) ($v(x,t) = \eta(x) g(t)$) and solving the characteristic equation function of x , the mode shapes η_L and η_R of the left and right sub-beam, respectively, are as follows

$$\eta_L = C_{1L} \sin(\alpha x) + C_{2L} \cos(\alpha x) + C_{3L} \sinh(\alpha x) + C_{4L} \cosh(\alpha x) \quad 0 \leq x \leq x_c \quad (2.1)$$

and

$$\eta_R = C_{1R} \sin(\alpha x) + C_{2R} \cos(\alpha x) + C_{3R} \sinh(\alpha x) + C_{4R} \cosh(\alpha x) \quad x_c \leq x \leq L, \quad (2.2)$$

where $\alpha = \left(\frac{\rho A \omega^2}{EI} \right)^{1/4}$ and ω is the natural frequency of the cracked beam.

The C_{ij} terms are integration constants arising from the solution of the fourth order differential equation in space. By imposing the boundary conditions (see Eq. (3) below), a system of eight linear equations is formed. The natural frequencies of the cracked beam are found by setting the determinant of the matrix of the linear system to zero and solving it numerically for the roots of α . The coefficient C_{1L} is imposed to be equal to unity.

The boundary conditions of the cantilever beam at the clamped and at the free end are, respectively:

$$\eta_L(0) = 0 \quad \text{and} \quad \eta'_L(0) = 0, \quad \eta''_R(L) = 0 \quad \text{and} \quad \eta'''_R(L) = 0. \quad (3.1)$$

The boundary conditions of the simple supported beam at the two ends are:

$$\eta_L(0) = 0 \quad \text{and} \quad \eta''_L(0) = 0, \quad \eta_R(L) = 0 \quad \text{and} \quad \eta''_R(L) = 0 \quad (3.2)$$

For both the beams, the conditions of continuity of displacement, moment and shear at the crack location can be expressed as

$$\eta_L(x_c) = \eta_R(x_c), \quad \eta''_L(x_c) = \eta''_R(x_c) \quad \text{and} \quad \eta'''_L(x_c) = \eta'''_R(x_c). \quad (3.3)$$

The rotational spring at the cracked section introduces a discontinuity of the rotation which can be written as

$$\eta'_R(x_c) - \eta'_L(x_c) = \frac{EI}{k_c} \eta''_R(x_c). \quad (3.4)$$

The local stiffness k_c is evaluated, according to linear elastic fracture mechanics concepts, through the following expression [1]

$$k_c = \frac{bh^2 E}{24(\delta/(1-\delta))^2 (5.93 - 19.69\delta + 37.14\delta^2 - 35.84\delta^3 + 13.12\delta^4)} \quad (4)$$

where $\delta = a/h$ is the relative crack depth.

2. Damage detection by spatial CWT

Due to its multi-resolution properties, wavelet analysis, acting as a signal microscope, appears to have a better ability to analyze the details of non-stationary signals in comparison to traditional analysis tools, such as Fourier Transform or Short-Time Fourier Transform.

A wavelet function $\psi(x)$ is a zero mean local wave-like function which decays rapidly and satisfies particular conditions [2]. A family of wavelet functions can be obtained by considering:

$$\psi_{k,s}(x) = \frac{1}{\sqrt{s}} \psi\left(\frac{x-k}{s}\right), \quad (5)$$

where s and k are, respectively, the scale and the translation parameter. The continuous wavelet transform of a signal $\eta(x)$ with respect to the wavelet function $\psi(x)$ is defined as

$$W(k,s) = \int_{-\infty}^{+\infty} \eta(x) \frac{1}{\sqrt{s}} \psi^*\left(\frac{x-k}{s}\right) dx \quad (6)$$

where ψ^* is the complex conjugate of ψ .

The issue of the sampling interval impact in CWT damage detection is dealt with considering the 4th order Coiflets wavelet ('Coif4'), with 8 vanishing moments, which, exhibit better performances in comparison to other wavelet functions [3].

Furthermore, a MATLAB routine to perform the CWT for the aforementioned wavelets is implemented by the authors to improve the accuracy of the existing built-in one (the original routine approximates the signal through a constant piecewise function, while the implemented one considers a linear piecewise trend) [4].

Realistic situations are simulated by superimposing synthetic Gaussian white noise to the signal corresponding to the beam deflection obtained from the mechanical model described in section 1. In the following the noise level is quantified through the signal to noise ratio (SNR), defined in decibels as

$$SNR = 10 \log_{10} \left(\frac{P_{signal}}{P_{noise}} \right). \quad (7)$$

The term P with the subscripts in Eq. (7) denotes power and is computed as

$$P_{(.)} = \frac{1}{N_z} \sum_{i=1}^{N_z} |z(x_i)|^2, \quad (8)$$

where N_z denotes the number of discrete points of a generic sampled function $z(x)$.

As above-mentioned, the CWT is defined by the convolution of the input signal $\eta(x)$ with a wavelet function generated by scaling and translating its mother wavelet $\psi(x)$. For a finite-length signal, when the convolution operation is executed close to the signal ends, the wavelet window extends into a region with no available data, so that the transform values close to the borders of the signal are tainted by the nonexistence of data. Consequently, the values of the CWT coefficients very close to the signal extrema arise abnormally (border distortions) and the real signal features of that region are consequently corrupted by the transform. As a result, edge effects can provoke the masking of the damage and yield false indicators. To handle edge effects is used the polynomial extension method suggested by Montanari et al. [5], which pads the original signal by means of two high degree

polynomial functions obtained through a fitting procedure. The extension functions satisfy continuity conditions and extend the trend of the noisy signal and its derivatives in an average sense.

3. Parametric study

The effect of sampling interval of the cracked beam mode shape in damage detection through the spatial CWT is numerically analysed varying the relevant parameters of the problem, that is: the normalized sampling interval $\Delta x/L$ ($\Delta x/L$ is considered equal to 0.025, 0.01, 0.005, 0.0025, 0.001, 0.0005 and 0.00025), the noise level (SNR value is assumed equal to 130 dB, 100 dB or 70 dB), the deformed shape of the beam (i.e. the first three mode shapes of the cantilever and of the simply supported beam), the relative crack position along the beam (x_c/L is considered equal to 0.1, 0.3, 0.5, 0.7 or 0.9) and the beam parameters (ρ , h , b , L and E). Initially the parametric study is carried out assuming fixed values of the beam parameters; subsequently, their influence is analyzed to generalize the results.

In order to investigate the impact of sampling interval in CWT based damage detection, the minimum detectable (threshold) crack size for a given spatial sampling interval is obtained according to the following criterion based on an iterative procedure. While increasing gradually the relative crack depth δ from a value of 0.0001 to 0.95, the wavelet transform is executed at fixed values of the scale s ($s = 2, 4, 6, K, 200$) and the maximum absolute value of the transform is determined for each scale. If for a given relative crack depth δ this maximum is attained for all scales at the crack position for each of an arbitrary number of 20 different random Gaussian white noise distributions, this value of δ can be regarded to be detectable, otherwise a larger value of δ has to be assessed. To allow for the numerical approximation of the CWT, the crack depth is also regarded as detectable even if the CWT absolute maximum is attained in either of the two nearest points (i.e. the preceding or the following point) to the actual damage location. For very small sampling intervals ($\Delta x/L < 0.005$), the criterion is relaxed to any of the four nearest points to the damage location.

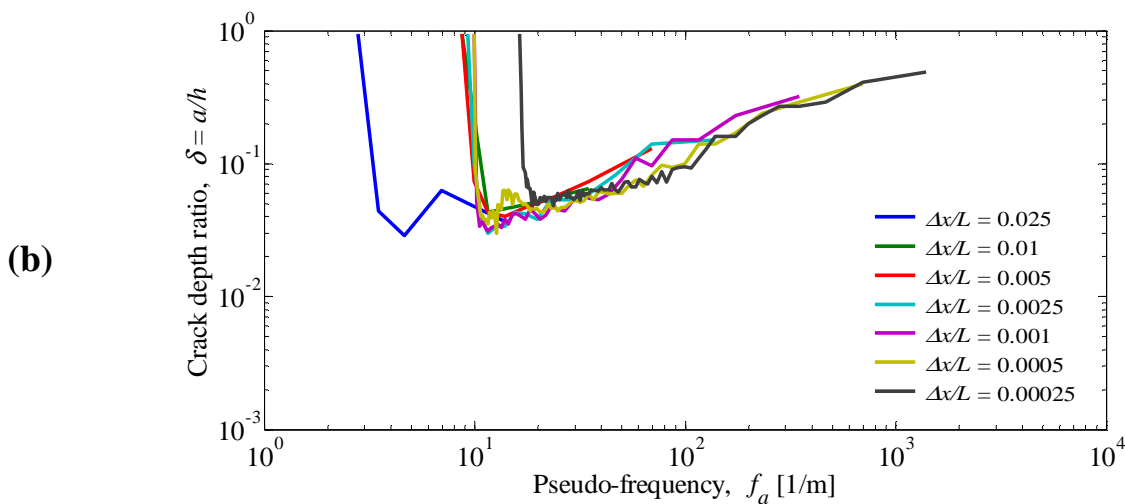
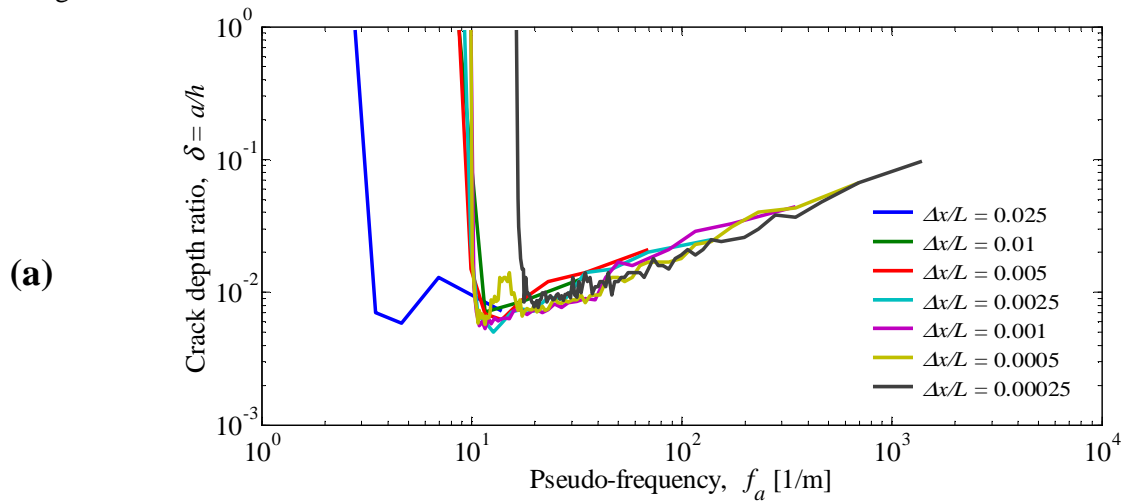
A cracked beam of length $L = 1$ m with a rectangular cross-section of height $h = 0.05L$ and width $b = 0.5h$, constituted by an elastic linear isotropic material of Young modulus $E = 200$ GPa and density $\rho = 7850$ kg/m³, is considered.

Figure 2 shows in bilogarithmic graphs the minimum detectable crack size as a function of the pseudo-frequency, $f_a = f_c / (\Delta x \cdot s)$, where f_c is the center frequency of the mother wavelet (i.e. the frequency maximizing the Fourier transform of the mother wavelet modulus [6]); for ‘Coif4’ $f_c = 0.6957$. The damage identification turns out to be, with good approximation, a function of the pseudo-frequency only (in the simulations of figure 2 the fitting parameters of the polynomial padding method are assumed to be $\beta_1 = \beta_2 = 1$, $H_1 = \bar{H}_1$ and $H_2 = \bar{H}_2$ [5]; in the other simulations, whereas the values of β_1 and β_2 vary, in order to reduce the computational cost $H_1 = \bar{H}_1$ and $H_2 = \bar{H}_2$ is assumed). The curves of figure 2 decrease monotonically from high to low pseudo-frequencies until, at a specific frequency, a sudden jump occurs toward the upper limit of δ (say, equal to 0.95). This specific value of pseudo-frequency represents, with good approximation, the *lowest* and *optimal* value of pseudo-frequency for damage identification (the optimal value is that allowing the minimum crack size to be detected as the pseudo-frequency is made to vary). It needs to be underlined that this lowest/optimal value of f_a is the same for all levels of noise.

The jump in the curves of figure 2 at the lowest/optimal value of f_a occurs because, by analyzing the cracked beam shape with wavelet functions characterized by lower pseudo-frequencies, even large cracks cannot be localized due to the influence of the edge effects. It can be noted that the curves for $\Delta x/L = 0.00025$ exhibit jumps to the upper limit of δ at a higher value of pseudo-frequency than those for larger sampling intervals. This trend is due to the fact that, when the crack is near the signal extremum, the wavelet analysis is tainted by edge distortions, and, particularly for very low sampling intervals, the maximum CWT coefficient value falls not exactly at the crack location (or at its two four

nearest points, as permitted by the detection criterion), but at sampling points other from that of the crack location.

Hereafter, the $f_{a,opt}$ is used to indicate the optimal value of pseudo-frequency f_a that gives the minimum detectable crack size (in figure 2 $f_{a,opt}$ is equal to about 12 m^{-1}). The value $f_{a,opt}$ of pseudo-frequency approximately coincides, as shown above, with the lowest pseudo-frequency for performing damage detection.



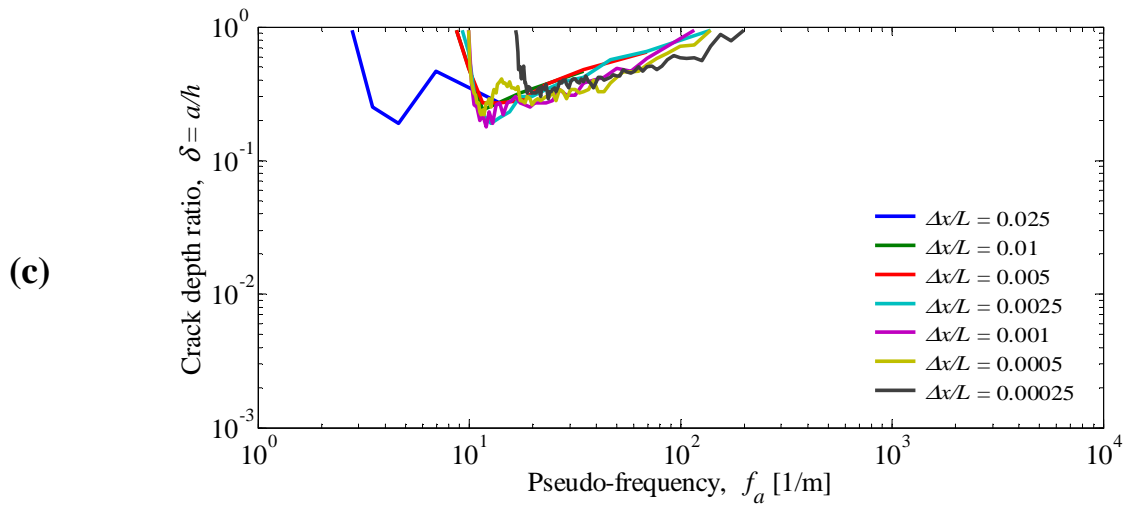


Figure 2. Impact of the sampling interval in damage detection by spatial CWT with ‘Coif4’. The polynomial padding method is used. The first mode shape of the cantilever beam with crack at $x_c/L = 0.1$ is analysed. (a) SNR = 130 dB; (b) SNR = 100 dB; (c) SNR = 70 dB.

Considering that CWT based damage detection depends on the pseudo-frequency only, it is numerically illustrated how the use of the proper scale range is essential in detecting damage when different sampling intervals are considered [3]. The first mode shape of the cantilever beam with a crack of $\delta = 0.2$ at $x_c/L = 0.1$ is analysed. The noise level is assumed to be equal to 70 dB. The same mode shape is sampled at $\Delta x/L = 0.01$ and $\Delta x/L = 0.0004$.

Figures 3(a,b) report the contour plots of the CWT (for the polynomial padding method $\beta_1 = \beta_2 = 1$ are used) when the signal is sampled at $\Delta x/L = 0.01$, and the scale ranges $s = [4 - 24]$ (i.e. $s = 4, 5, 6, K, 24$) and $s = [4 - 6]$ are respectively used. The CWT coefficients, considering the broad scale range, completely mask the crack location (figure 3a), which on the contrary can be identified using the narrow scale range (figure 3b).

Figures 3(c-d) highlight that even if the signal is sampled at an extremely small sampling interval ($\Delta x/L = 0.0004$), the CWT damage detection can be achieved only considering a proper scale range. The damage localization at the scale range $s = [2 - 40]$ fails (figure 3c), while it succeeds for $s = [96 - 158]$ (figure 3d).

In this Section, the influence of these beam parameters ρ, h, b, L and E on the optimal pseudo-frequency in CWT based damage detection is analysed and a further generalization of the parametric study is obtained.

According to Eq. (2), the functions describing the general mode shape depend on the boundary conditions and on the parameter α , and in turns the corresponding natural frequency is a function of the material density ρ and of α . Hence, the material density ρ affects the natural frequencies of the cracked beam but not its mode shapes and it is not considered in the following.

Then, by substituting the LEFM expression of the local rotational stiffness k_c due to the crack (Eq. (4)) in the boundary condition representing the rotation discontinuity at the crack section (Eq. (3.4)), we obtain:

$$\eta'_2(x_c) - \eta'_1(x_c) = 2h \tilde{f}(\delta) \eta''_2(x_c), \quad (9)$$

where $\tilde{f}(\delta) = (\delta/(1-\delta))^2 (5.93 - 19.69\delta + 37.14\delta^2 - 35.84\delta^3 + 13.12\delta^4)$ is a function of the relative crack depth δ only. For a given beam deflection shape (and, hence, for a given value of the curvature in x_c , $\eta''_1(x_c) = \eta''_2(x_c)$), Eq. (9) demonstrates that the rotation discontinuity due to the crack is a

function of h and δ , but not of the other beam parameters b and E . Therefore, the CWT based damage detection results are invariant for constant values of h , δ , and L .

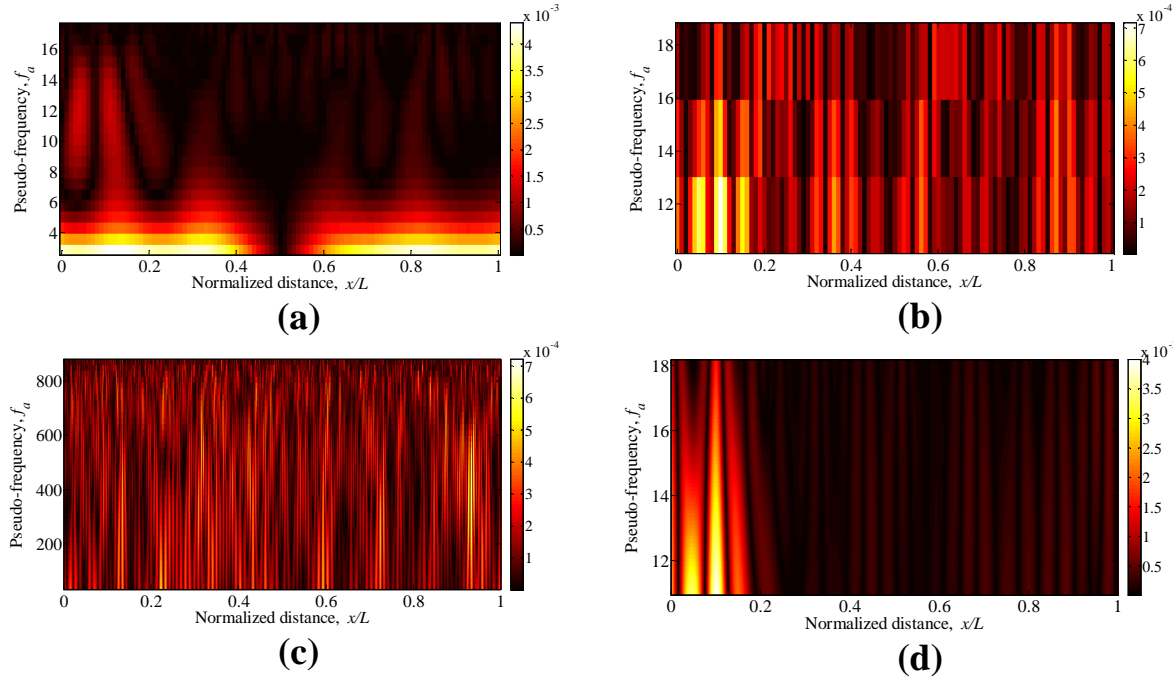


Figure 3. Contour plots of the spatial CWT using ‘Coif4’ and the polynomial padding method (SNR = 70 dB). The first mode shape of the cantilever beam with crack of $\delta = 0.2$ at $x_c/L = 0.1$ is analysed. Sampling interval $\Delta x/L$ and scale range are varied: (a) $\Delta x/L = 0.01$, $s = [4 - 24]$; (b) $\Delta x/L = 0.01$, $s = [4 - 6]$; (c) $\Delta x/L = 0.0004$, $s = [2 - 40]$; (d) $\Delta x/L = 0.0004$, $s = [96 - 158]$.

As for the beam parameter h , it can be demonstrated [4] that its variation influences the detectable crack depth but not the optimal pseudo-frequency value of CWT damage detection. In summary, the optimal pseudo-frequency $f_{a,opt}$ is independent of the beam parameters b , h , E and ρ and hence, using the polynomial padding method and a given wavelet function, it is a function of the deformed beam shape and the crack position x_c . Formally, this can be written as:

$$f_{a,opt} = f(\text{beam deflection shape}, x_c, L). \quad (10)$$

Through the dimensional analysis and Buckingham’s theorem [7], Eq. (10) can be expressed in the following dimensionless form:

$$f_{a,opt}^* = f_{a,opt} L = f(\text{beam deflection}, x_c/L), \quad (11)$$

Equation (11) states that the optimal pseudo-frequency $f_{a,opt}$ multiplied by the beam length L is a function of the beam deflection shape and the relative crack position x_c/L only.

Figure 4 illustrates the $f_{a,opt}^*$ against x_c/L curves pertaining to the first three mode shapes of the cantilever and simply supported beams. It can be noted that, mainly due to the edge effects, $f_{a,opt}^*$ turns out to be higher near the beam ends, especially for the third mode shapes. According to the curves of figure 4, for a given wavelet scale, it is possible to calculate the optimal number of sampling intervals needed to detect the smallest crack located at x_c/L using a given beam vibration mode shape.

Finally, from the definition of $f_{a,opt}^*$, the optimal number of sampling intervals yielding the best CWT damage detection at the scale s for a given crack location x_c/L , is equal to $\frac{L}{\Delta x}|_{opt} = \frac{f_{a,opt}^* s}{f_c}$.

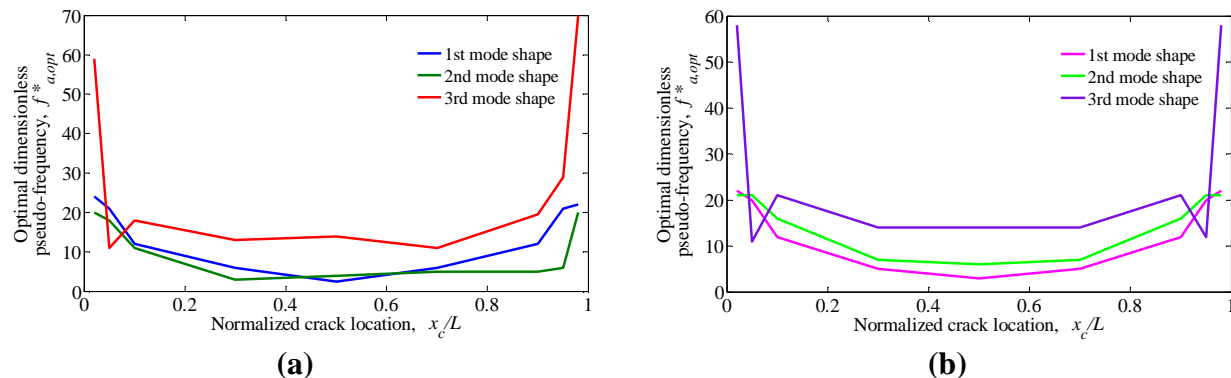


Figure 4. Variation of $f_{a,opt}^*$ for the CWT damage detection with relative crack location for the first three mode shapes of: (a) cantilever beams; (b) simply supported beams.

4. Conclusions

The present investigation examined the effect of the spatial sampling interval in damage detection by CWT, with the aim of answering the following key questions: Can the cost of damage detection be reduced by down-sampling? What is the minimum number of sampling intervals required for optimal damage detection? An in-depth parametric study has been carried out by analyzing the first three mode shapes of a cantilever and a simply-supported beam varying the relevant parameters of the problem, that is: the sampling interval, the noise level, the crack depth and position along the beam, and the mechanical and geometrical beam parameters. The effect of the sampling interval in CWT based damage detection is investigated in terms of the minimum detectable crack size (which is defined on the basis of an ad-hoc criterion) as a function of the key parameter, pseudo-frequency. The curves of minimum detectable crack size against pseudo-frequency for different relative sampling intervals coincide and decrease monotonically with decreasing pseudo-frequency down to a lower bound value of pseudo-frequency f_a below which no damage detection is possible. This lower bound of f_a can be regarded as the optimal pseudo-frequency $f_{a,opt}$ capable of detecting the smallest crack size by CWT. The parametric study shows that such an optimal pseudo-frequency depends on the mode shape and on the crack location along the beam. In addition, it has been observed that, for a given mode shape and crack position, different beam structures, characterized by the same value of x_c/L , attain the same optimal value of the dimensionless pseudo-frequency $f_{a,opt} L$.

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