Multi-objective shape optimization of plate structure under stress criteria based on sub-structured mixed FEM and genetic algorithms

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Abstract.
This paper presents a methodology for the multi-objective (MO) shape optimization of plate structure under stress criteria, based on a mixed Finite Element Model (FEM) enhanced with a sub-structuring method. The optimization is performed with a classical Genetic Algorithm (GA) method based on Pareto-optimal solutions and considers thickness distributions parameters and antagonist objectives among them stress criteria. We implement a displacement-stress Dynamic Mixed FEM (DM-FEM) for plate structure vibrations analysis. Such a model gives a privileged access to the stress within the plate structure compared to primal classical FEM, and features a linear dependence to the thickness parameters. A sub-structuring reduction method is also computed in order to reduce the size of the mixed FEM and split the given structure into smaller ones with their own thickness parameters. Those methods combined enable a fast and stress-wise efficient structure analysis, and improve the performance of the repetitive GA. A few cases of minimizing the mass and the maximum Von Mises stress within a plate structure under a dynamic load put forward the relevance of our method with promising results. It is able to satisfy multiple damage criteria with different thickness distributions, and use a smaller FEM.

1. Introduction

The goal of this paper is to implement a DM-FEM for plate structure enhanced with a sub-structuring reduction method for MO structural optimizations with thickness repartition parameters and mass/stress criteria. This reduced model provides a fast and efficient analysis of complex plate structures vibrations and facilitates the work of the GA used for the optimization.

The optimization of a structure's parameters in order to prevent damages is a major concern in mechanical engineering but can only be achieved at the expense of others essential criteria. Even though often carried out manually, MO-optimization methods also naturally appear in the litterature, such as gradient-based method [1, 7] and MO-GA-based methods [6, 8]. In this study, the optimization is performed with a classical GA-based NSGA-II method [4], as it permits to dispense with any weighting of criteria. This kind of method enables to find a set of Pareto-optimal solutions, which are all optimal compared to each other, for at least one criterion. Nevertheless, an inconvenient lies in the repetitions of the criteria's evaluation, depending on the chosen method and theory.

In the literature, some of the previous works on thin structures optimization focus on laminated composites [1, 8] with thickness and orientation layer parameters, as well as

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piezoelectric smart structures [6] with input energy and deflection variables. Shell structures optimization has also been studied [7, 3] and make the integration between FEM and CAD modeling softwares. In all these cases, the calculation of stresses is often necessary as a constraint of the optimization, and more rarely as a proper objective. Furthermore, they use a primal FEM that discretizes the displacements and requires extra calculation to get to the stress, and eventually needs rebuilding of the meshing, which increase the CPU time. The originality of the present work is the use of a different FEM formulation giving stresses as a primary result of the structural analysis, and avoid rebuilding as well, in order to improve the repetitive GA.

Many different formulations for mechanical problems permit to access to various parameters in the same model. Washizu's book [10] gives a good insight of all the variational methods with different fields. Among these, the Hellinger-Reissner's [9] describes both displacements and stresses, and the application to FEM is well explained by Wriggers' book [11]. In this work, we program a mixed displacement-generalized stress dynamic mixed FEM (DM-FEM), based on Kirchoff-Love thin plate theory, which allows a direct access to the stress field within the plate, and save operations, each iteration. Moreover, the plate theory implementation permits to act on the thickness of the plate without building a whole new assembly. This also improves the performance of the optimization techniques as the thicknesses of the different zone of the structure constitute parameters for the optimization.

In addition to the DM-EM, a sub-structuring reduction method imagined by the authors [5] and adapted to mixed models is used in order to reduce the numerical size of the structure. That method allows to compute the stress criteria on a mixed reduced FEM, even smaller than the primal corresponding FEM (that would only give the displacement). Moreover, that method also provides the advantage of being linearly dependent from the thickness, and turn the sub-structures into small super-elements defined by its thicknesses. Thus, it allows to quickly work on a smaller model and improves the technique all the more.

A few investigations are made, using a GA-based MO optimization method improved by the DM-FEM and the sub-structuring method. We consider stress criteria and thickness parameters. The idea is to split the test structure into sub-structures with their own thicknesses (parameters) so as to minimize the mass and the Von Mises (VM) stress within the whole plate (objectives) under a dynamic load. We put forward the principal differences between the regular method and ours, and show interesting results satisfying multiple damage criteria with different thickness distributions.

In the two first parts, we present the optimization problem and give a quick description of the GA. We continue with the main point of the paper: the implementation of the mixed FEM and the associated sub-structuring method. Finally we present the principal relevant examples.

2. Optimization problem

The "example" structure we want to optimize is built up with 768 thin plate elements (see figure 1). It is composed of 3 flat sub-structures, with their own thicknesses, which are the parameters of the optimization (range defined by the user).

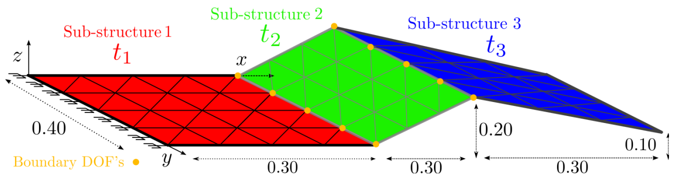

Figure 1: Plate structure composed of 3 flat sub-structures with their own thicknesses

The problem we try to solve is the following: we want to minimize both the maximum VM stress within the whole plate structure for a given n^{th} mode, and the mass in a bid of costs. These two objectives being antagonistic, the goal is to find some thickness distributions that are good compromises between the mass and the maximum stress undergone by the structure. As such, the MO-GA-based method quite naturally appears.

3. Genetic Algorithm method

The GA methods are based on the evolution of species in their natural environment. It consists in making evolve a population, whose individuals, solutions of the problem, tend to improve for the purpose of our objectives as the generations follow. In this context, the word used to explain its functioning are taken from biology, namely:

- an "individual" is a solution to the problem (here, a combination of 3 thicknesses (one each sub-structure) that gives a compromises between the objectives)
- a "population" is a group of "individuals" (here, a set of thickness combinations, that forms a Pareto front with one compromises per thickness distribution)
- a "generation" is an iteration of the algorithm and correspond to a population

The functioning of the algorithm is summarized in figure 2). It is composed of 5 different stages ("evaluation" stages 2 being the most costly).

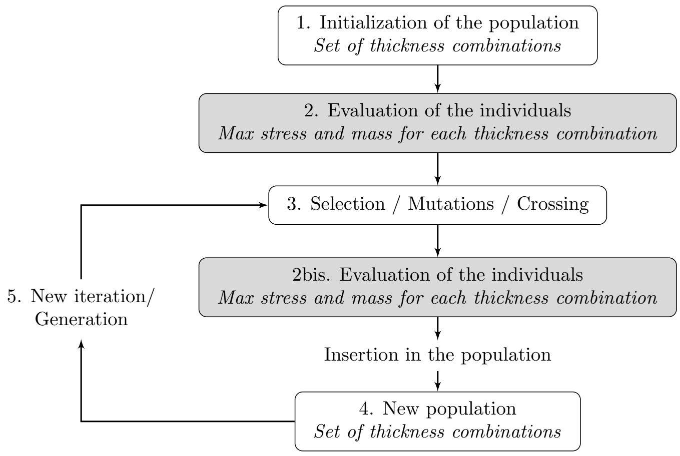

Figure 2: Principle of the genetic algorithm

4. Mixed Finite Element Model for thin Kirchhoff-Love plate

The "evaluation step" of the GA is by far the most costly because we need to evaluate the stress within the whole plate structure, and identify the maximum. The classical method that consists in building a primal FEM to access the displacement and use extra calculations to reach the stress is the most common and intuitive method, but it appears to be heavy going because of the extra calculations. The originality of our method consists in the construction of a displacement-generalized stress DM plate FEM that gives a direct access to both fields in the response of the structure. Another important benefit of this method, when choosing a plate theory, is the possibility of modifying the thickness as a parameters without rebuilding a whole assembly.

This type of DM-FEM is based on the HR mixed functional [9, 11] expressed for dynamics problems. It may correspond to the regular Lagrangian used in dynamics, but computed with mixed component, both functions of displacements and stresses, as follows:

$$\Pi_{HRD} \;=\; \iiint_V -\boldsymbol{\sigma}_{ij}\boldsymbol{e}_{ij}(\boldsymbol{u}_i) \;+\; \frac{1}{2}\boldsymbol{\sigma}_{ij}\boldsymbol{S}_{ijkl}\boldsymbol{\sigma}_{kl} \;+\; \boldsymbol{b}_i\boldsymbol{u}_i \;+\; \frac{1}{2}\rho\dot{\boldsymbol{u}}_i^2 \,\mathrm{dV} \tag{1}$$

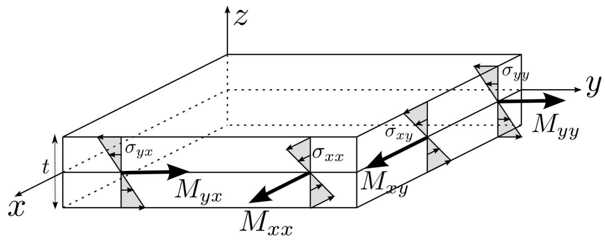

Figure 3: Bending and twisting moment and transverse shear force

considering $\boldsymbol{\sigma}_{ij}$ the stress, $\boldsymbol{u}_i$ the displacement, $\boldsymbol{e}_{ij}(\boldsymbol{u}_i)$ the strain function of the displacement $\boldsymbol{u}_i$, $\boldsymbol{b}_i$ the body force, ρ the density and $\boldsymbol{S}_{ijkl}$ the elastic compliance matrix. The stationary condition or Euler-Lagrange equations can then be applied to the functional so as to conventionally solve a dynamic structure problem.

The discretization of the generalized stresses and the displacements (according to figure 3) with plate elements using Kirchhoff-Love (KL) thin plate is as follows:

$$\boldsymbol{\sigma}_{ij} = \left\{M_x, M_y, M_{xy}\right\}^T = \boldsymbol{P}\boldsymbol{\beta} \tag{2}$$

$$\boldsymbol{u}_i = \left\{w, \theta_x, \theta_y\right\}^T = \boldsymbol{N}\boldsymbol{U} \tag{3}$$

$$\boldsymbol{e}_{ij} = \left\{\epsilon_{xx}, \epsilon_{yy}, \gamma_{xy}\right\}^T = \boldsymbol{D}\,\boldsymbol{u}_i = \boldsymbol{D}\boldsymbol{N}\boldsymbol{U} \tag{4}$$

with $\left\{M_x, M_y, M_{xy}\right\}^T$ respectively the bending moments in the x, and y direction, and the twisting moment within the plate (thickness t), $\left\{w, \theta_x, \theta_y\right\}^T$ respectively the transverse displacement of the plate, and the section rotations around x and y. $\boldsymbol{P}$ is the generalized stresses shape function matrix, $\boldsymbol{\beta}$ the generalized stress parameters vector within the plate, $\boldsymbol{N} = \left\{\boldsymbol{N}_w, \boldsymbol{N}_\theta\right\}^T$ the displacements shape function matrix, $\boldsymbol{U} = \left\{\boldsymbol{U}_w, \boldsymbol{U}_\theta\right\}^T$ the displacements parameters vector et $\boldsymbol{D}$ the displacement-strain tensor-operator.

When carefully splitting the matrix in order to separate the matrices dependent on t, t^3 and $\frac{1}{t^3}$, the application of the Euler-Lagrange equations leads us to the following matrix formulation of the dynamic mixed FEM (for one plate element of surface S):

$$\begin{Bmatrix} t\tilde{\boldsymbol{M}}_{F_w} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & t^3\tilde{\boldsymbol{M}}_{F_\theta} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{Bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{U}}_{F_w} \\ \ddot{\boldsymbol{U}}_{F_\theta} \\ \ddot{\boldsymbol{\beta}} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{G}_w^T \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{G}_\theta^T \\ \boldsymbol{G}_w & \boldsymbol{G}_\theta & \frac{1}{t^3}\tilde{\boldsymbol{H}} \end{Bmatrix} \begin{Bmatrix} \boldsymbol{U}_{F_w} \\ \boldsymbol{U}_{F_\theta} \\ \boldsymbol{\beta} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F}_w \\ \boldsymbol{F}_\theta \\ \boldsymbol{0} \end{Bmatrix} \tag{5}$$

with $\tilde{\boldsymbol{M}}_{F_w} = \iint_S \boldsymbol{N_w}^T \rho \boldsymbol{N_w} \mathrm{dS}$, $\tilde{\boldsymbol{M}}_{F_\theta} = \iint_S \boldsymbol{N_\theta}^T \frac{\rho}{12} \boldsymbol{N_\theta} \mathrm{dS}$, $\boldsymbol{G_w} = \iint_S \boldsymbol{P}^T \boldsymbol{D} \boldsymbol{N_w} \mathrm{dS}$, $\boldsymbol{G_\theta} = \iint_S \boldsymbol{P}^T \boldsymbol{D} \boldsymbol{N_\theta} \mathrm{dS}$, $\boldsymbol{G} = \left\{\boldsymbol{G}_w, \boldsymbol{G}_\theta\right\}$ $\tilde{\boldsymbol{H}} = \iint_S -\boldsymbol{P}^T \tilde{\boldsymbol{S}}_{ijkl} \boldsymbol{P} \mathrm{dS}$, $\tilde{\boldsymbol{S}}_{ijkl} = t^3 \boldsymbol{S}_{ijkl}$ and $\boldsymbol{F} = \left\{\boldsymbol{F}_w, \boldsymbol{F}_\theta\right\}^T$ being the nodal forces (splitted for w and θ parameters).

The advantages of this method are the direct access to the stress (doesn't requires to rebuild the stress field from the displacements) and the thickness parameters easily mutable for the same meshing (doesn't require to rebuild $\tilde{\boldsymbol{M}}_{F_w}$, $\tilde{\boldsymbol{M}}_{F_\theta}$, $\boldsymbol{G}$ and $\tilde{\boldsymbol{H}}$). Nevertheless, it has the inconvenient of being numerically bigger than a primal FEM (see Table 1). A sub-structuring reduction method is thus implemented, in order to decrease the size of the DM-FEM, taking care of the thickness independence of the method for each sub-structure.

5. Sub-structuring method

The method used in this paper has been explained in [5]. The idea is to split the structure into few sub-structures, and reduce each of them separately using the primal corresponding FEM. The initial Degrees Of Freedom (DOF) of a given sub-strucures a are separated in internal DOF $\boldsymbol{U}_i^a$, junction DOF $\boldsymbol{U}_j^a$ (permitting to link sub-structures between them) and stress DOF $\boldsymbol{\beta}^a$. The displacements are projected on a basis composed of truncated primal "fixed" modes (see Craig & Bampton method [2]) and the stresses on a basis composed of truncated primal "fixed" modes as well, but projected on the stresses. Thus, the new reduced parameters are: modal components $\boldsymbol{\eta}^a_{FI\,(U,\beta)}$ representing both the displacements and the stresses, and the junction DOF $\boldsymbol{U}_j^a$ remaining unchanged. The reduction of the whole sub-structure a is given by:

$$\begin{Bmatrix} \boldsymbol{U}_i^a \\ \boldsymbol{U}_j^a \\ \boldsymbol{\beta}^a \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\Phi}^a_{FIi\,(U,\beta)} & \boldsymbol{\Psi}_i^a \\ \boldsymbol{0} & \boldsymbol{I}_{ij} \\ t^3 \tilde{\boldsymbol{P}}^a \begin{Bmatrix} \boldsymbol{\Phi}^a_{FIi\,(U,\beta)} \\ \boldsymbol{0} \end{Bmatrix} & t^3 \tilde{\boldsymbol{P}}^a \begin{Bmatrix} \boldsymbol{\Psi}_i^a \\ \boldsymbol{I}_{ij} \end{Bmatrix} \end{Bmatrix} \begin{Bmatrix} \boldsymbol{\eta}^a_{FI\,(U,\beta)} \\ \boldsymbol{U}_j^a \end{Bmatrix} \tag{6}$$

with $\tilde{\boldsymbol{P}}^a = -(\tilde{\boldsymbol{H}}^a)^{-1}\boldsymbol{G}^a$, $\boldsymbol{\Phi}^a_{FIi\,(U,\beta)}$ being a truncated basis of the "fixed" modes of the sub-structures a (with the junction considered fixed) and $\boldsymbol{\Psi}_i^a = -\boldsymbol{K_{ii}}^{-1}\boldsymbol{K_{ij}}$ being the matrix of the constraint static modes, taken from the primal FEM. We assemble two sub-structures a and b considering:

$$\boldsymbol{U}_j^a = \boldsymbol{U}_j^b = \boldsymbol{U}_j \tag{7}$$

The formulation of the reduction basis gives us another advantage: in fact, the "fixed" modes of the sub-structures a contained in the basis $\boldsymbol{\Phi}^a_{FIi\,(U,\beta)}$ remains the same for a flat sub-structure, even when the thickness change. In the same way, the basis $\left\{\boldsymbol{\Psi}_i^a, \boldsymbol{I}_{ij}\right\}^T$ representing the reaction of the sub-structure when its junction is moving doesn't depend on the thickness of the plate, as long as it remains flat. In this way, our reduction method is all the more robust that it is easily transposable to different thickness distributions (doesn't requires to rebuild $\tilde{\boldsymbol{P}}^a$, $\boldsymbol{\Psi}_i^a$ and $\tilde{\boldsymbol{P}}^a$).

Basically, each sub-structure works as a super-element, defined by its thickness only. Each individual of each generation, an almost instantaneous simple operation is made to modify the thickness (colored in red) in the matrices of equation 5 and 6. Thus, the "evaluation step" (see figure 2) to reach the stress simply comes down to the computation of the structure response, using mixed reduced matrices smaller than the primal ones classically used (see Table 1).

Table 1: Characteristics of the FEMs and DOFs for a 768 elements structure

FEM	DOFs	Stress field
Primal (displacement)	2448	Reconstruction needed
Mixed (displacement and stress)	9360	Direct
Mixed reduced (20 truncated modes each sub-structure)	168	Direct

6. Example: optimization of the thickness combination to minimize the mass & the maximum stress for one mode

The example we use is schematized in figure 1 in section 2. We minimize the maximum VM stress within the whole structure for a given mode n (see shape of mode 1 in figure 4c and mode 7 in figure 5c), and the mass. The parameters are the thicknesses of each "single plate" or "sub-structure" (thickness zones in figures 4b and 5b). Thickness parameters vary in a range between 1 and 2 mm. The plate is made of $S210$ steel. The load applied is a harmonic force in the z direction, whose pulsation corresponds to the n mode, and is considered normalized. A hysteretic damping is also added to the model for each mode to keep a realistic system. The FEM is composed of 768 elements. Table 1 shows the improvement DOF-wise for this example.

The results are presented as "Pareto Front" (black points in 4a and 5a). These points are defined by the right ordinate axis and the abscissa axis representing respectively the mass of the structure and the maximum VM stress. Each of these optimal points are better than the others in the sense of at least one of the two objectives. In that way they are all compromises we can select to design our structure, depending on the objective we want privilege. Once a Pareto point chosen, the colored curves plotted in addition to the Pareto Front in 4a and 5a leads to the parameters corresponding to the point. They are defined by the left ordinate axis and the abscissa axis representing respectively the thickness of each of the sub-structures for the chosen Pareto point (with the same abscissa) and the maximum VM stress in the whole structure for this point. Basically, when selecting a point of the Pareto Front, the corresponding thicknesses to design the structure correspond to ordinate of the colored three points with the same abscissa.

Those results were obtained with NSGA-II with a population of 100 individuals and 1000 generations. We used the DM-FEM reduced with the sub-structuring method to improve the CPU time, and easily changed the t thicknesses of each sub-structure for each evaluation step. The GA has been compiled several times so as to check the convergence of the results.

The first subsection deals with the optimization for the mode 1, a simple example whose sketch of results could be intuitive. The second one treats of the optimization for the mode 7 and shows interesting results with "parametric typing".

6.1. Simple example: mode 1

The results are presented in figure 4. This example shows some thickness combinations that allow to have a VM stress between 7 and 140 MPa, with a mass of the structure between 2.9 and 4.7 Kgs. In this case, more than the Pareto-Front itself, the most relevant is the evolution of the thicknesses in function of VM stress (colored curves in figure 4a). In fact, we observe that the most important zone to "reinforce" so as to decrease damages is the first one, and then the second one. When taking a look at the stress distribution for the first mode (figure 4c), this evolution of thicknesses in function of the stress seems coherent. Intuitively, we would decrease the thickness where the stress is low, and increase it where it is high. This test puts forward the validity of the method qualitatively, and brings a quantitative point in term of stress.

6.2. Test example: mode 7

The results are presented in figure 5. Considering the stress distribution for the mode 7 (figure 5c), the results are not intuitive. This example shows some thickness combinations that allows us to have a VM stress between 7 and 150 MPa, with a mass of the structure between 2.9 and 4.7 Kgs. The evolution of the 3 parameters in function of VM stress (colored curves in figure 5a) is very specific. In fact, we distinguish 3 main parametric typing:

- from 7 until 56 MPa: it is the lowest stress range. It appears that the thickest sub-structures of this range are the first one near the housing, and the second one in the middle of the whole structure. The evolution of each of them is not identical, but the first zone remains thicker than the second one. On the other hand, that range contains a particularity: the thickness of the third sub-structure remains equal to the range minimum 1 mm.
- from 57 until 72 MPa: it is the middle stress range. It presents some significant differences with the first one. In fact, the second sub-structure becomes the thickest, and the third one is no longer minimum. Plus, that range still contains a particularity: the first sub-structure which was the thickest in the first stress range is now the thinest with a minimal thickness.

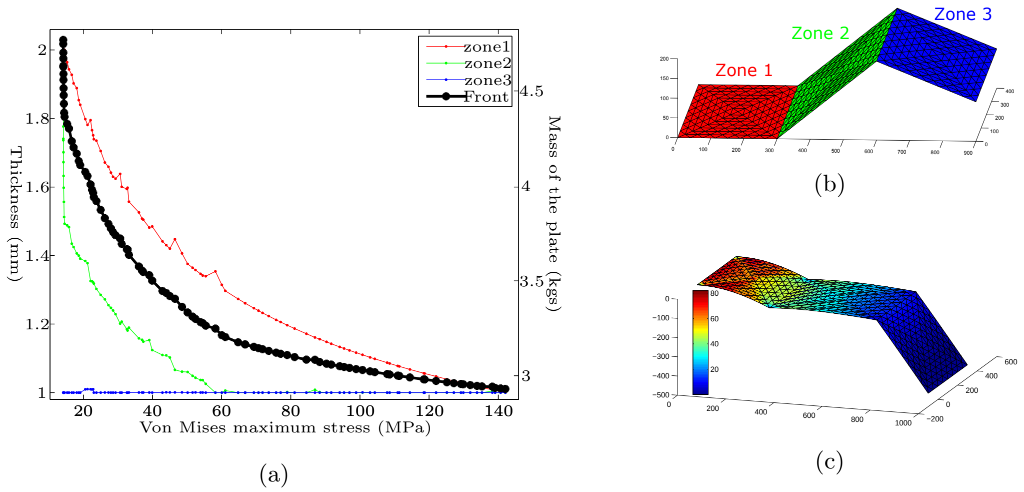

Figure 4: Optimization results for mode 1: a/ Pareto Front (black) & thicknesses for each point (colored), b/ Independent thickness zones, c/ Stress distribution on mode 1 (Mpa)

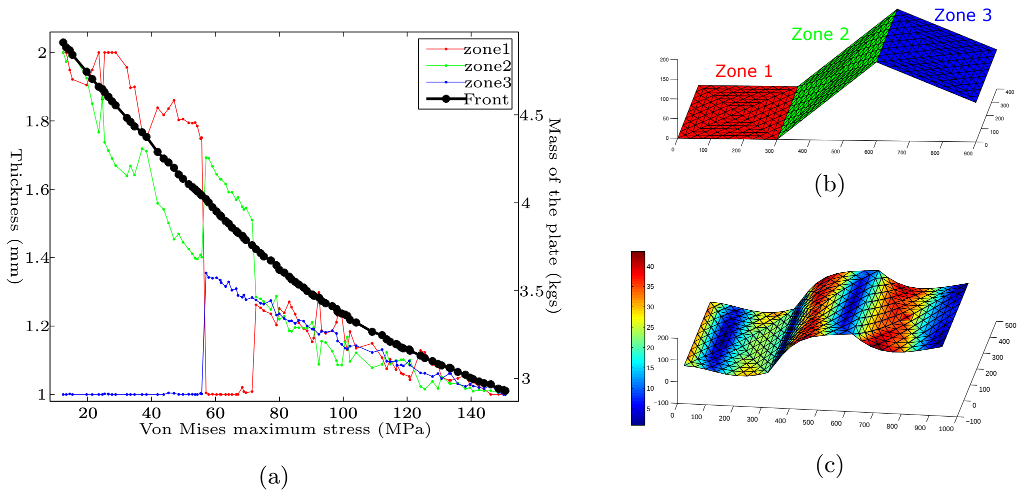

Figure 5: Optimization results for mode 7: a/ Pareto Front (black) & thicknesses for each point (colored), b/ Independent thickness zones, c/ Stress distribution on mode 7 (Mpa)

- from 73 until 150 MPa: it is the highest stress range. The evolution of the thicknesses and their respective order of magnitude is completely different. In fact, there is no "minimal value" zone compared to the 2 first ranges. The three parameters have close values (between 1 and 1.3 mm) and their evolution is close as well, since they all decrease the same from 1.3 till 1 mm as the VM stress increase until its maximum.

Those results define a phenomenon we call "parametric typing". Although the Pareto Front has no singularities, the parameters' evolution do not follow any specific or logical rule. Nevertheless, they can be divided into three main stress domains which correspond to significantly different thickness distributions. That presents an interesting advantage in terms of design possibilities: it means you can find two different thickness distributions (two designs) that leads to almost the same Pareto point (same property in terms of the objectives we minimize). That example shows promising results that could be interesting in a more complex industrial case, and puts forward the interest of such a method, in addition to the efficiency previously highlighted.

7. Conclusion

This paper introduces a new methodology for MO optimization technique of structural dynamic plate problems with stress criteria and thickness parameters. The optimization is made with a classical NSGA-II method that finds a set of Pareto-optimal solutions, but presents the originality of using a displacement-stress DM plate FEM. Not only the latter permits a direct access to the stress field compared to a classical FEM, but it is also featured by a linearly dependence to the thickness parameters which dispense with rebuilding the meshing each iteration. Furthermore, in order to offset the inconvenient of a bigger numerical size, a fast sub-structuring reduction method is implemented, each sub-structure with its own thickness. It turns the DM-FEM into a mixed reduced DM-FEM, smaller than the classical FEM and requires less calculation to get the stress criterion. These two methods combined permits to improve the costly "evaluation step" of the GA and make this MO optimization method more efficient.

This study shows two practical cases, using a plate structure composed of 3 sub-structures and 768 elements. Their goals are to minimize the maximum VM stress and the mass of the plate structure in function of the proper thickness of each sub-structure, under a dynamic load (harmonic modal pulsation). Our optimization technique permits compute the response on a reduced FEM and save CPU time, especially with high density meshing. We find a whole group of compromises between the maximum VM stress and the mass for different thickness distributions. The first example shows intuitive results and puts forward the relevance of the techniques. The second example, less obvious, presents specific results such as parametric typing, namely some zone in the Pareto front characterized by significantly different thickness parameters evolution in function of the maximum stress. This feature leads to interesting design possibility as two antagonist thickness distributions prove a close stress response. Beyond those findings, we have implemented an efficient MO optimization method for dynamic plate structural problems, that gives the user an opportunity to have a whole set of design possibilities in order to satisfy multiple antagonist damage criteria. We could now consider the perspective of testing industrial structures and study the response on a wide frequency band.

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