

Twisted spectral geometry for the standard model

Pierre Martinetti

Università di Trieste, via Valerio 12/1, I-34127

E-mail: pmartinetti@units.it

Abstract. In noncommutative geometry, the spectral triple of a manifold does not generate bosonic fields, for fluctuations of the Dirac operator vanish. A Connes-Moscovici twist forces the commutative algebra to be multiplied by matrices. Keeping the space of spinors untouched, twisted-fluctuations then yield perturbations of the spin connection. Applied to the spectral triple of the Standard Model, a similar twist yields the scalar field needed to stabilize the vacuum and to make the computation of the Higgs mass compatible with its experimental value.

1. Introduction

Noncommutative geometry [1] provides a description of the standard model of elementary particles [SM] as a purely gravitational theory [2]. By this, one means that assuming space (time) is described by a slightly non-commutative generalization of a manifold, then Einstein-Hilbert action (in Euclidean signature) together with the bosonic action of the SM, including the Higgs sector, are obtained from one single action formula, the *spectral action* [3]. The bosonic Lagrangian is the noncommutative counterpart of the Einstein-Hilbert action, and the Higgs field comes out on the same footing as the other bosons as a connection 1-form, but associated to the noncommutative part of the geometry.

More precisely, the Einstein-Hilbert action together with the various pieces of the SM are obtained as the asymptotic expansion $\Lambda \rightarrow \infty$ of the spectral action, where Λ is a cut-off parameter. The spectral action thus provides some boundary conditions between the parameters of the SM at a putative energy of unification, and physical predictions are obtained by running these parameters under the renormalization group flow down to the electroweak breaking scale [4]. The mass of the Higgs boson is then function of the inputs of the theory (mainly the Yukawa coupling of the fermions and the mixing angles for quarks and neutrinos). Assuming there is no new physics between the electroweak and the unification scales, it is computed around 170 GeV, a value ruled out by Tevatron in 2008.

Since then the Higgs boson has been discovered around 125 GeV, which is below the threshold of stability of the electroweak vacuum: the later is a metastable state. It is not clear whether this is a problem or not, since the life-time of this metastable state is far larger than the age of the universe. However the estimation of the probability that somewhere in our past light-cone the Higgs field has tunneled down to its true vacuum - liberating a quantity of energy that should have destroyed the whole universe - depends on the model of Inflation at the tip of the light-cone. So even if not problematic, it is at least intriguing that the electroweak vacuum is metastable, but on the edge of stability (see [5] for a recent update on these issues).

To cure this instability, a long known solution proposed by particle physicists is to postulate another heavy scalar field, say σ , suitably coupled to the Higgs. In noncommutative geometry,



bosonic fields are obtained by so called *fluctuations of the metric*, roughly speaking a way to turn the constant parameters of the theory (the Yukawa couplings) into fields (there is in reality a intricate re-parametrization and the correspondance Yukawa couplings/bosonic fields is more subtle). Chamseddine and Connes noticed in [6] that by turning into a field one of the constant entry of the generalized Dirac operator describing the SM, namely the Majorana mass k_R of the neutrino, then one gets exactly the field σ suitably coupled to the Higgs. As a bonus, σ modifies the flow of the renormalisation group and makes the computation of the Higgs mass compatible with its experimental value. The question is then to understand how to turn the constant parameter k_R into a field respecting the framework of noncommutative geometry. The problem is that unlike the other bosonic fields, σ cannot be obtained by a fluctuation of the metric because the constant parameter k_R only fluctuates to a constant field. This impossibility has its origin in of one mathematical requirements of noncommutative geometry, namely the condition asking that the generalized Dirac operator is a first-order differential operator.

Various models have been proposed to justify the turning of k_R into a field. In [7, 8] the first order condition is relaxed, yielding to a Pati-Salam generalization of the standard model. The later is retrieved dynamically, as a minimum of the spectral action. Earlier models pre-2012 had already shown how to lower the Higgs mass thanks to extra-scalar fields, but they required also new fermions [9, 10]. Recently, in [11] a variation on the notion of symmetry in NCG yields a model with extra bosonic and scalar fields carrying a $B - L$ charge; in [12] new fields are obtained as a consequence of a non-standard grading.

In [13] we proposed to generate the field σ in a way satisfying, at least partially, the first order condition. The key idea is to allow the commutative algebra $C^\infty(\mathcal{M})$ of smooth functions on a compact spin manifold \mathcal{M} to act non-trivially on the space of spinors $L^2(\mathcal{M}, S)$. The drawback is that the commutator

$$[\not{D}, f] \quad f \in C^\infty(\mathcal{M}) \quad (1)$$

with the free Dirac operator \not{D} is no longer bounded [14], in contradiction with one of the primary requirements of noncommutative geometry. There exists however a variation of these requirements, introduced in [15] to deal precisely with this kind of problem. Given an algebra \mathcal{A} and a generalized Dirac operator, rather than the boundedness of $[D, a]$ one asks that there exists an automorphism ρ of \mathcal{A} such that the *twisted commutator*

$$[D, a]_\rho := Da - \rho(a)D \quad (2)$$

is bounded. Such twists have mathematical motivations that have nothing to do with physics, but we showed in [16] how a very simple twist of the model proposed in [13] - in fact a chiral transformation - permits to obtain a coherent picture of the SM in which the field σ is generated by a fluctuation of k_R that satisfies a twisted version of the first-order condition.

In this note we give a non-technical account of these results. Rather than the up-bottom approach developed in [16] (i.e. twists as solution to the unboundeness of the commutator coming from the non-trivial action of $C^\infty(\mathcal{M})$ on spinors), we propose a bottom-up approach: after some generalities in section 2, we show in section 3 how requiring a non-trivial twist forces the manifold to be multiplied by a matrix geometry. We discuss the physical consequences, in particular the generation of new fields. Finally in section 4 we state the results of [16] on the standard model of elementary particles.

2. Twisted almost-commutative geometry

2.1. Almost commutative geometry in a nutshell

By “slightly noncommutative generalization of a manifold”, one intends a “space” such that the set of functions defined on it is no longer commutative, but is of the kind

$$C^\infty(\mathcal{M}) \otimes \mathcal{A}_F \quad (3)$$

where \mathcal{A}_F is a finite dimensional algebra. This is called an *almost-commutative geometry* because the center $C^\infty(\mathcal{M})$ of the algebra (3) is infinite dimensional (as an algebra). It is represented on

$$L^2(\mathcal{M}, S) \otimes \mathcal{H}_F \quad (4)$$

where \mathcal{H}_F is a finite dimensional space (whose basis are the fermions of the model) carrying a representation of \mathcal{A}_F , while $L^2(\mathcal{M}, S)$ is the space of spinors on which functions act by multiplication: $(\pi(f)\psi)(x) = f(x)\psi(x)$ for any $x \in \mathcal{M}$. That is f is represented as the operator

$$\pi(f) := f\mathbb{I} \quad (5)$$

where \mathbb{I} is the identity matrix of dimension n the numbers of components of a spinor.

For the standard model, $\mathcal{H}_F = \mathcal{H}_{sm}$ has dimension the number of fermions (2 colored quarks +1 electron +1 neutrino = 8 multiplied by 2 (chirality), 2 (antiparticles) and 3 (generations) = 96). The space (4) is then the space of fermionic fields of the SM. Notice the overcounting of degrees of freedom: the distinction between chirality and anti/particles is taken into account both in the finite dimensional space \mathcal{H}_F and by the number of components of the spinors in $L^2(\mathcal{M}, S)$. This *fermion doubling* is projected out on the fermionic action thanks to a Pfaffian [4], thus it is not a real nuisance, except maybe from an aesthetic point of view. From our perspective, it provides in fact a solution for generating the field σ .

The key idea of noncommutative geometry is that all the geometrical information of the manifold \mathcal{M} is encoded within the Dirac operator $\not{D} = -i\gamma^\mu \nabla_\mu$, where γ^μ 's are the Dirac matrices and $\nabla_\mu := \partial_\mu + \omega_\mu$ is the covariant derivative associated to the spin connection ω_μ . Connes worked out a purely algebraic characterization of the Dirac operator, that he then exported to the noncommutative setting [2]. Hence the notion of *spectral triple*, that is an algebra \mathcal{A} (non necessarily commutative), acting through a representation π on a Hilbert space \mathcal{H} , together with an operator D with compact resolvent (or a generalization of this condition in case the algebra is not unital) called (generalized) Dirac operator such that

$$|[D, \pi(a)]| \text{ is bounded for any } a \in \mathcal{A}. \quad (6)$$

One also asks that \mathcal{H} is a graded Hilbert space, that is there exists an operator Γ such that $\Gamma^2 = 1$ which, furthermore, anticommutes with D and commutes with \mathcal{A} . For the standard model one has

$$\Gamma = \gamma^5 \otimes \Gamma_{sm} \quad (7)$$

where, writing

$$\mathcal{H}_{sm} = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^c, \quad (8)$$

as the sum of four copies of \mathbb{C}^{24} labelled by left/right and anti/particles indices, one has $\Gamma_{sm} = \text{diag}(\mathbb{I}_{24}, -\mathbb{I}_{24}, -\mathbb{I}_{24}, \mathbb{I}_{24})$ while γ^5 is the product of the Dirac matrices.

Adding other conditions, one shows that given any spectral triple with unital commutative algebra \mathcal{A} , then there exists a compact spin manifold \mathcal{M} such that $\mathcal{A} = C^\infty(\mathcal{M})$ [17]. The conditions on the analytic properties of the operator D are automatically satisfied in the finite dimensional case. As well, we shall not take into account here the orientability condition and Poincaré duality, but will focus on the *real structure* and the already mentioned *first order condition*. The real structure J is an antilinear operator ($J(\lambda\psi) = \bar{\lambda}J\psi$ for $\lambda \in \mathbb{C}$, $\psi \in \mathcal{H}$) whose square is $\pm\mathbb{I}$, and which commutes or anticommutes with the gradation Γ and the operator D . The three signs

$$J^2 = \epsilon\mathbb{I}, \quad JD = \epsilon'DJ, \quad J\Gamma = \epsilon''\Gamma J \quad (9)$$

determines the so called *KO*-dimension of the spectral triple.

All these conditions are satisfied by the triple $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \not{D})$, with real structure the charge conjugation \mathcal{J} , grading the chirality γ^5 . The KO -dimension is then the dimension of the manifold \mathcal{M} (modulo 8). A classification of finite dimensional spectral triples satisfying these conditions as been made in [4, 18] and yields an (almost) unique choice $(\mathcal{A}_{sm}, \mathcal{H}_{sm}, D_{sm})$ relevant for the standard model. The choice of the algebra \mathcal{A}_{sm} is discussed in §4, the Hilbert space is $\mathcal{H}_{sm} = \mathbb{C}^{96}$ described below (5), the constant entries of 96×96 matrix D_{sm} are the Yukawa coupling of fermions and the mixing matrices for quarks and neutrinos. A general formula of products of spectral triples yields the almost-commutative spectral triple

$$C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}, \quad L^2(\mathcal{M}, S) \otimes \mathcal{H}_{sm}, \quad D := \not{D} \otimes \mathbb{I} + \gamma^5 \otimes D_{sm}. \quad (10)$$

The graduation is given in (7) and the real structure is $\mathcal{J} \otimes J_{sm}$ where J_{sm} is the operators that exchange particles with antiparticles in \mathcal{H}_{sm} .

2.2. Bosonic fields and spectral action

Bosonic fields are generated by the so-called *fluctuations of the metric*. Given an arbitrary spectral triple $(\mathcal{A}, \mathcal{H}, D)$, those are defined as the substitution of the Dirac operator with a *covariant operator*

$$D_A := D + A + JAJ^{-1} \quad (11)$$

where A is a selfadjoint element of the set of (generalized) 1-forms

$$\Omega_D^1(\mathcal{A}) := \{ \pi(a^i)[D, \pi(b_i)], \quad a^i, b_i \in \mathcal{A} \}. \quad (12)$$

The name is justified because for $\mathcal{A}_F = M_N(\mathbb{C})$ and $D_F = 0$, then D_A is nothing but the covariant Dirac operator of a $U(n)$ gauge theory on \mathcal{M} . For the almost-commutative geometry (10) of the standard model, these fluctuations generate the bosonic fields and the Higgs.

The spectral actions consists in counting the eigenvalue of D_A smaller than an energy scale Λ ,

$$S = \text{Tr} f \left(\frac{D_A^2}{\Lambda^2} \right) \quad (13)$$

where f is a smooth approximation of the characteristic function of the interval $[0, 1]$. As explained in introduction, for the spectral triple (10) fluctuated as in (11), the asymptotic expansion of S yields Einstein-Hilbert action and the SM bosonic action, including the Higgs.

2.3. Twisted spectral triple

In the definition (11) of the covariant Dirac operator, it is important that the commutators $[D, \pi(a)]$ are bounded, otherwise bosons would be described by unbounded operators. Whatever the finite dimensional spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F)$, the commutator $[D_F, \pi(a)]$ is automatically bounded. The same is true for the commutative part $[\not{D}, \pi(f)] = (\not{D}f)\mathbb{I}$. Hence as long as the almost-commutative algebra (3) acts on (4) with the trivial action (5) on spinors, the commutator $[D, \pi(a)]$ is always bounded.

Nevertheless, as explained in [15], there are situations where the requirement (6) is too strong, like the lift \not{D}' to \not{D} of a conformal map. Then $[\not{D}', \pi(f)]$ is no longer bounded, but there exists an automorphism ρ of $C^\infty(\mathcal{M})$ such that the twisted commutator $[\not{D}', \pi(f)]_\rho$ is bounded. More generally, requiring the boundedness of the twisted commutator makes sense mathematically and allows to treat cases (type III spectral triple) where the usual commutator is never bounded [15]. This yields the definition of a *twisted spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \rho)$, similar to a spectral triple except that $[D, \pi(a)]$ bounded is replaced by $[D, \pi(a)]_\rho$ bounded for some automorphism ρ .

To the best of our knowledge, the conditions for the reconstruction theorem [17] (in particular the real structure) have not been adapted to the twisted case yet. This does not matter because in

the commutative case $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \not{D})$ with representation (5) (which is the one addressed by the reconstruction theorem), twists are not relevant. Indeed, whatever the automorphism ρ of $C^\infty(\mathcal{M})$,

$$[\not{D}, \pi(f)]_\rho = (\not{D} \pi(f)) - i[\gamma^\mu \pi(f) - \pi(\rho(f)) \gamma^\mu] \nabla_\mu \quad (14)$$

is bounded if and only if the differential part vanishes, that is

$$[\gamma^\mu, \pi(f)]_\rho = 0 \quad \forall f \in C^\infty(\mathcal{M}). \quad (15)$$

This is equivalent to $f = \rho(f)$ for any f , that is ρ is the trivial automorphism.

3. Need for twist

The precedent section shows that for the usual spectral triple of a manifold, not only there is no need for a twist because the usual commutator is bounded, but there is no space for it. One may wonder what the minimal modifications are, so that to authorize a non-trivial twist. We already know one answer [15]: the lift to the Dirac operator of a conformal map. But sticking to the idea that physics is contained within the Dirac operator, we want to keep the usual Dirac operator \not{D} and rather play with the other parameters: the algebra and its representation.

3.1. Degenerate representation and almost-commutative algebra

We first try a degenerate representation $\pi(f) = fp$ for some projection $p \neq \mathbb{I}$. Condition (15) becomes

$$\gamma^\mu fp - \rho(f)p\gamma^\mu = 0 \quad \forall f \in C^\infty(\mathcal{M}). \quad (16)$$

For $f = \rho(f) = 1$, this implies $[\gamma^\mu, p] = 0 \forall \mu$. Only the multiple of the identity commute with all Dirac matrices, hence $p = \lambda \mathbb{I}$ for some $\lambda \in \mathbb{C}$. $\pi(1) = \pi(1)^2$ fixes $\lambda = 1$, and one is back to (5).

A possibility to have a non-trivial twist would be to act with the automorphism on ρ on p rather than on f in (16). To understand this better, let us assume \mathcal{M} has dimension 4 to fix notations. In the chiral basis, the Euclidean Dirac matrices are

$$\gamma^\mu = \begin{pmatrix} 0_2 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0_2 \end{pmatrix} \quad (17)$$

where $\sigma^\mu = \{\mathbb{I}, -i\sigma_i\}$, $\tilde{\sigma}^\mu = \{\mathbb{I}, i\sigma_i\}$ with $\sigma_{i=1,2,3}$, the Pauli matrices. Consider the representation of the algebra \mathbb{C} on the Hilbert space \mathbb{C}^2 as $\mathbb{C} \ni \lambda \rightarrow \lambda \mathbb{I}_2$. Let ρ be the automorphism of \mathbb{C}^2 that permutes the two terms,

$$\rho(\lambda_1, \lambda_2) = (\lambda_2, \lambda_1) \quad \forall (\lambda_1, \lambda_2) \in \mathbb{C}^2. \quad (18)$$

For π the representation of \mathbb{C}^2 on \mathbb{C}^4 given by $\pi(\lambda_1, \lambda_2) = \lambda_1 \mathbb{I}_2 \oplus \lambda_2 \mathbb{I}_2$, one has

$$[\gamma^\mu, \pi(\lambda_1, \lambda_2)]_\rho = \begin{pmatrix} 0_2 & [\sigma^\mu, \lambda_2 \mathbb{I}_2] \\ [\tilde{\sigma}^\mu, \lambda_1 \mathbb{I}_2] & 0_2 \end{pmatrix} = 0. \quad (19)$$

If one could work with two representations $\pi(f) = fp$, $\pi'(f) := fp'$ of $C^\infty(\mathcal{M})$ where p, p' are two orthogonal projections in $L^2(\mathcal{M}, S)$, then the algebra isomorphism $\tau : \pi(f) \rightarrow \pi'(f)$ would define a modified-commutator

$$\not{D}\pi(f) - \tau(\pi(f))\not{D} = -i\gamma^\mu \nabla_\mu(pf) - if(\gamma^\mu p - p'\gamma^\mu) \nabla_\mu \quad (20)$$

where $\nabla_\mu(pf) := \partial_\mu(pf) + [\omega_\mu, pf]$. This is bounded iff the second term vanishes, for instance when $p = \text{diag}(\mathbb{I}_2, 0_2)$, $p' = \text{diag}(0_2, \mathbb{I}_2)$. The point is that τ is not an automorphism of $C^\infty(\mathcal{M})$, since the algebra generated by $\pi(f)$ and $\pi'(f)$ for $f \in C^\infty(\mathcal{M})$ is not $C^\infty(\mathcal{M})$, but two copies of it, that is $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$.

In other terms, to have a non-trivial twist one needs to multiply the manifold by a matrix geometry. In this sense, a “raison d’être” of almost-commutative algebra is to allow non-trivial twists, which are forbidden in the case $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \not{D})$.

3.2. Twisted fluctuation of the metric

Consider $\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$ acting on $L^2(\mathcal{M}, S)$ in agreement with §3.1, that is such that there exists an automorphism ρ of \mathcal{A} guaranteeing that (omiting the symbol of representation)

$$[\gamma^\mu, a]_\rho := \gamma^\mu a - \rho(a)\gamma^\mu = 0 \quad \forall \mu. \quad (21)$$

We define the *twisted-covariant free Dirac operator*

$$\not{D}_{A_\rho} := \not{D} + A_\rho + \mathcal{J}A_\rho\mathcal{J}^{-1} \quad (22)$$

where A_ρ is a element of the set of twisted 1-forms

$$\Omega_{\not{D}, \rho}^1(\mathcal{A}) := \{a^i[\not{D}, b_i]_\rho, a^i, b_i \in \mathcal{A}\}. \quad (23)$$

By (21) one has $[\not{D}, a]_\rho = -i\gamma^\mu \nabla_\mu a$ where $\nabla_\mu a := (\partial_\mu a) + [\omega_\mu, a]$. So a twisted 1-form is

$$A_\rho = -ia^i \gamma^\mu \nabla_\mu b_i = -i\gamma^\mu X_\mu \quad \text{where} \quad X_\mu := \rho^{-1}(a^i) \nabla_\mu b_i. \quad (24)$$

Fluctuating \not{D} by $C^\infty(\mathcal{M})$ acting as in (5) has no interest, because [19] $\not{D}_A = \not{D}$. This is no longer true for a twisted-fluctuation (22). To see it, consider $a^i = (f^i, g^i)$, $b_i = (f'_i, g'_i)$ in $\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ acting as in (19) with $\rho = \rho^{-1}$ as in (18), that is

$$\rho(a^i) = \begin{pmatrix} g^i \mathbb{I}_2 & 0_2 \\ 0_2 & f^i \mathbb{I}_2 \end{pmatrix}, \quad b_i = \begin{pmatrix} f'_i \mathbb{I}_2 & 0_2 \\ 0_2 & g'_i \mathbb{I}_2 \end{pmatrix}, \quad X_\mu = \begin{pmatrix} g^i \partial_\mu f'_i \mathbb{I}_2 & 0_2 \\ 0_2 & f^i \partial_\mu g'_i \mathbb{I}_2 \end{pmatrix} \quad (25)$$

where we use that $\omega_\mu = -\frac{1}{4}\Gamma_{\mu\rho}^\nu \gamma^\rho \gamma_\nu$ (Γ the Christoffel symbol in the orthonormal basis) commutes with b_i . Since $[\mathcal{J}, i\gamma^\mu] = 0$, one has $\mathcal{J}A_\rho\mathcal{J}^{-1} = -i\gamma^\mu \mathcal{J}X_\mu\mathcal{J}^{-1}$. Furthermore,

$$\mathcal{J}X_\mu\mathcal{J}^{-1} = \mathcal{J}\rho(a^i)\mathcal{J}^{-1}\mathcal{J}\partial_\mu b_i\mathcal{J}^{-1} = \rho(a^i)^* \partial_\mu b_i^* = X_\mu^* \quad (26)$$

because $[\mathcal{J}, \partial_\mu] = 0$ and $\mathcal{J}a\mathcal{J}^{-1} = a^*$ commutes with $\mathcal{J}\rho(b)\mathcal{J}^{-1} = \rho(b)^*$. Therefore

$$A_\rho + \mathcal{J}A_\rho\mathcal{J}^{-1} = -i\gamma^\mu (X_\mu + X_\mu^*). \quad (27)$$

In the non-twisted case, X_μ, X_μ^* commute with γ^μ so (27) is selfadjoint iff $X_\mu = -X_\mu^*$, that is $A_\rho + \mathcal{J}A_\rho\mathcal{J}^{-1} = 0$ as announced. In the twisted case, X_μ, X_μ^* twisted-commute with γ^μ : the adjoint of (27) is $i\gamma^\mu \rho(X_\mu + X_\mu^*)$ and \not{D}_{A_ρ} is selfadjoint as soon as $X_\mu + X_\mu^* = -\rho(X_\mu^* + X_\mu)$.

This condition holds for instance when $g^i \partial_\mu f'_i = -f^i \partial_\mu g'_i := f_\mu$ is a real function. Then

$$\not{D}_{A_\rho} = \not{D} - 2i\gamma^\mu \begin{pmatrix} f_\mu \mathbb{I}_2 & 0_2 \\ 0_2 & -f_\mu \mathbb{I}_2 \end{pmatrix}. \quad (28)$$

This simple example shows that a non-trivial twist may have interesting physical consequences: while fluctuations of the of the free Dirac operator by $C^\infty(\mathcal{M})$ are trivial, twisted fluctuations by $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$ generate a vector field X_μ . Its physical interpretation is delicate: by making functions acting non-trivially on spinors, one breaks the invariance of the representation of $C^\infty(\mathcal{M})$ under the spin group. In this sense, these models are “pre-geometric”: the spin structure is not explicit in the representation, but is somehow “hidden” in the Dirac operator.

One may be puzzled by our lack of care in viewing $L^2(\mathcal{M}, S)$ as $L^2(\mathcal{M}) \otimes \mathbb{C}^4$. We argue in [13] that this makes sense in a local trivialization. Eventual non-local effect should be studied.

Also, a peculiarity of the simple model presented here is that the almost-commutative algebra acts on $L^2(\mathcal{M}, S)$, and not on $L^2(\mathcal{M}, S) \otimes \mathcal{H}_F$. The field X_μ is a perturbation of the spin connection, whereas in case there is an action of \mathcal{A}_F on a finite dimensional Hilbert space \mathcal{H}_F (like for the SM, in next section), then a twisted-fluctuation of $\not{D} \otimes \mathbb{I}$ yields a perturbation of the $U(\mathcal{A}_F)$ connection 1-form induced by a non-twisted fluctuation. One may wonder if the spin connection ω_μ itself could be generated by a twisted fluctuation of the flat Dirac operator.

4. Twist for the standard model

Let us conclude by the applications to the standard model. We work with one generation only, so that the finite dimensional algebra \mathcal{A}_{sm} (discussed below) acts on the finite dimensional space $\mathcal{H}_{sm} = \mathbb{C}^{96/3=32}$.

4.1. Grand symmetry

Independent considerations on the signature of the metric [20], the mass of neutrinos and the fermion doubling [21] indicates that the KO -dimension of the finite dimensional part of the spectral triple of the SM should be 6. Under natural hypothesis (irreducible actions of the algebra and of the real structure, existence of a separating vector) and an explicit ad-hoc “symplectic hypothesis”, it has been shown in [18] that in order to accomodate a real structure J and a non-trivial grading Γ , the finite dimensional algebra of the almost-commutative geometry of the SM has to be

$$M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C}) \quad (29)$$

for a an integer greater than 1, acting on a space with dimension $d = 2(2a)^2$. For $a = 2$ the dimension $d = 32$ is precisely the number of particles per generation of the SM. By further imposing the grading condition $([\Gamma, a] = 0)$ and the first order condition $([[D, a], Jb^*J^{-1}] = 0)$, one arrives to the algebra of the standard model

$$\mathcal{A}_{sm} := \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}). \quad (30)$$

In [13] we noticed that for $a = 4$, the dimension $d = 128$ was precisely 4 times the number of particles per generation. Viewing 4 as the dimension of spinors on a four-dimensional space (time) \mathcal{M} , one identifies locally $L^2(\mathcal{M}, S) \otimes \mathcal{H}_{sm}$ as $L^2(\mathcal{M}) \otimes (\mathcal{H}_{sm} \otimes \mathbb{C}^4)$, which provides precisely the space needed to represent the *grand algebra*

$$\mathcal{A}_G := M_4(\mathbb{H}) \oplus M_8(\mathbb{C}). \quad (31)$$

Any element of $M_4(\mathbb{H})$ and $M_8(\mathbb{C})$ are viewed as 2×2 matrices with entry in $M_2(\mathbb{H})$ and $M_4(\mathbb{C})$. These entries act on \mathcal{H}_{sm} as does (29) for $a = 2$, so that at the end of the game one retrieves the action of \mathcal{A}_{sm} . The novelty is that the 2×2 block have a non trivial action on the remaining \mathbb{C}^4 . In this way, one obtains a representation of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ with a non-trivial action on spinors.

The grading condition breaks \mathcal{A}_G to

$$\mathbb{H}_L^l \oplus \mathbb{H}_R^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^r \oplus M_4(\mathbb{C}) \quad (32)$$

where L, R are the left-right indices in \mathcal{H}_F and l, r the left-right indices of spinors. Because of the non-trivial action on spinors, the commutator $[\not{\partial} \otimes \mathbb{I}, a]$ is never bounded. But there exists a twist ρ such that $[\not{\partial} \otimes \mathbb{I}, a]_\rho$ is bounded, this is simply the exchange of the spinorial left-right indices:

$$\mathbb{H}_L^l \Longleftrightarrow \mathbb{H}_L^r, \quad \mathbb{H}_R^l \Longleftrightarrow \mathbb{H}_R^r. \quad (33)$$

By further considering the the twisted version of the first order condition

$$[[D, a]_\rho, Jb^*J^{-1}]_\rho = 0 \quad (34)$$

for the Dirac operator of the SM

$$D = \not{\partial} \otimes \mathbb{I} + \gamma_5 \otimes D_{sm}, \quad (35)$$

one works out a sub-algebra of (32) acting on $L^2(\mathcal{M}, S) \otimes (\mathbb{C}^4 \otimes \mathcal{H}_{sm})$, namely

$$\mathcal{B} := \mathbb{H}_L^l \oplus \mathbb{C}_R^l \oplus \mathbb{H}_L^r \oplus \mathbb{C}_R^r \oplus M_3(\mathbb{C}) \quad (36)$$

which, together with D , defines a twisted spectral triple [16].

4.2. Extra scalar field and additional vector fields

Twisted fluctuations of $\not{D} \otimes \mathbb{I}$ by \mathcal{B} yields a vector field X_μ as in (25). Twisted fluctuations of $\gamma^5 \otimes D_R$, where D_R is the part of D_{sm} containing the Majorana mass k_R of the neutrino, generates a scalar field σ which coincides with the field σ studied in [18] up to a global γ^5 factor.

The spectral action yields a potential for these two fields, including an interaction term. This potential is minimum precisely when D is fluctuated by the sub-algebra of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ invariant under the twist, that is by $C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}$ [16].

5. Conclusion

The twist (33) allows to build a twisted spectral triple for the standard model with a non-trivial action on spinors. The extra scalar field σ is generated by a twisted-fluctuation of the Majorana part of the Dirac operator, while twisted-fluctuations of the free Dirac operator generate an additional-vector field. All these fluctuations satisfy a twisted version of the first order condition. Furthermore, as in [8] the breaking to the standard model (in our case: the un-twisting) is obtained dynamically by minimizing the spectral action.

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