Geometric phase and quantum correlations for a bipartite two-level system

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Abstract. We calculate the geometric phase of a bipartite two-level system coupled to an external environment. We compute the correction to the unitary geometric phase through a kinematic approach. To this end, we analyse the reduced density matrix of the bipartite system after tracing over the environmental degrees of freedom, for arbitrary initial states of the composite system. In all cases considered, the correction to the unitary phase has a similar structure as a function of the degree of the entanglement of the initial state. In the case of a maximally entangled state (MES), the survival phase is only the topological phase, and there is no correction induced by the environments. Further, we compute the quantum discord and concurrence of the bipartite state and analyse possible relations among these quantities and the geometric phase acquired during the non-unitary system's evolution.

1. Introduction

For a bipartite quantum system, it is important to know whether the system is entangled, separable, classically correlated or quantum correlated. It is well known that entanglement makes possible tasks in quantum information that could not be possible for its classical counterpart. As a valuable resource in quantum information processing, entanglement attracts much attention from researchers either in theory or in experiment and much progress concerning entanglement has been achieved [1]. Due to its unique property in the sense that it has no classical counterpart, entanglement has been applied to the implementation of quantum teleportation [2] and quantum cryptography [3].

A bipartite quantum state contains both classical $\mathcal{C}(\rho^{\rm AB})$ and quantum correlations $\mathcal{Q}(\rho^{\rm AB})$. These correlations are justified jointly by their "quantum mutual information" $\mathcal{I}(\rho^{\rm AB})$, since it is written as the sum of both $\mathcal{I}(\rho^{\rm AB}) = \mathcal{C}(\rho^{\rm AB}) + \mathcal{Q}(\rho^{\rm AB})$. This quantum part is known as quantum discord [4]. Even for the simplest case of two entangled qubits, the relation between quantum discord, entanglement and classical correlation is not yet clear. For pure states quantum correlation is exactly equal to entanglement, whereas classical correlation attains its maximal value 1. However, for a general two-qubit mixed state, the situation is more complicated. Qubit-qubit entanglement has been characterised completely, while quantum discord has only been quantified for particular cases [5].

From another point of view, a system can storage the information of its motion when it undergoes a cyclic evolution, in the form of a geometric phase (GP), which was first put forward by Pancharatnam in optics [6] and later studied explicitly by Berry in a general quantal system [7]. Since then, great progress has been achieved in this field. The geometric phase has been

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Journal of Physics: Conference Series **626** (2015) 012043 doi:10.1088/1742-6596/626/1/012043

extended to the case of non-adiabatic evolutions [8]. As an important evolvement, the application of the geometric phase has been proposed in many fields, such as the geometric quantum computation. Due to its global properties, the geometric phase is propitious to construct fault tolerant quantum gates. In this line of work, many physical systems have been investigated to realise geometric quantum computation, such as NMR (Nuclear Magnetic Resonance) [9], Josephson junction [10], Ion trap [11] and semiconductor quantum dots [12]. The quantum computation scheme for the geometric phase has been proposed based on the Abelian or non-Abelian geometric phase, in which geometric phase has been shown to be intrinsic against faults in the presence of some kind of external noise due to the geometric nature of Berry phase. It was therefore seen that the interactions play an important role for the realisation of some specific operations. Consequently, study of the geometric phase was soon extended to open quantum systems. Following this idea, many authors have analysed the correction to the geometric phase under the influence of an external environment using different approaches [13, 14, 15, 16, 17, 18, 19, 20].

In this context, we shall study the geometric phase acquired by a bipartite system in the presence of an external environment. We shall consider both the presence of a bosonic and spin environment. We shall also study the relation between classical and quantum correlations when the geometric phase is affected by the environment.

2. Model

We shall consider a bipartite system, that is to say, two interacting two-level systems, both coupled to an external reservoir. In terms of the Hamiltonians, the model can be mathematically described by the Hamiltonian of the free bipartite system $H_{\rm S}$ is

$$H_{\rm S} = \frac{\hbar\Omega_1}{2}\sigma_z^1 + \frac{\hbar\Omega_2}{2}\sigma_z^2 + \gamma\ \sigma_z^1 \otimes \sigma_z^2, \tag{1}$$

and the Hamiltonian of interaction between the bipartite and the external bath $H_{\rm I}$

$$H_{\rm I} = \sigma_z^1 \otimes \sum_{n=1}^{N} \lambda_n q_n + \sigma_z^2 \otimes \sum_{n=1}^{N} g_n q_n \tag{2}$$

where the constants λ_n and g_n couple the system to each oscillator in the environment, and γ is the coupling strength between both spin-1/2 particles. Here we have assumed that the coupling constant of the two-level systems with the environment is different being λ_n for the spin 1 and g_n for spin 2. The external bath $H_{\rm B}$ can be either considered as set of delocalised bosonic field modes ($H_{\rm B} = \sum_{n=1}^{N} \hbar\omega_n a_n^\dagger a_n$) or spin environments which are typically the appropriate model in the low temperature regime. In the case of having a spin environment, the interaction Hamiltonian is given by

$$H_{\rm I} = \sigma_z^1 \otimes \sum_{i=1}^{N} \epsilon_i \sigma_{zi} + \sigma_z^2 \otimes \sum_{i=1}^{N} \lambda_i \sigma_{zi}. \tag{3}$$

In the most general case, for an initial state of the system $|\Phi(0)\rangle = \alpha|00\rangle + \beta|01\rangle + \zeta|10\rangle + \delta|11\rangle$, the reduced density matrix for this model can be written as,

$$\rho_{\rm r}(t) = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-i(2\gamma+\Omega_2)t}\Gamma_{12} & \alpha\zeta^* e^{-i(2\gamma+\Omega_1)t}\Gamma_{13} & \alpha\delta^* e^{-i(\Omega_1+\Omega_2)t}\Gamma_{14} \\ \beta\alpha^* e^{i(2\gamma+\Omega_2)t}\Gamma_{21}^* & |\beta|^2 & \beta\zeta^* e^{-i(\Omega_1-\Omega_2)t}\Gamma_{23} & \beta\delta^* e^{-i(\Omega_1-2\gamma)t}\Gamma_{24} \\ \zeta\alpha^* e^{i(2\gamma+\Omega_1)t}\Gamma_{31}^* & \zeta\beta^* e^{i(\Omega_1-\Omega_2)t}\Gamma_{32}^* & |\zeta|^2 & \zeta\delta^* e^{-i(\Omega_2-2\gamma)t}\Gamma_{34} \\ \delta\alpha^* e^{i(\Omega_1+\Omega_2)t}\Gamma_{41}^* & \delta\beta^* e^{i(\Omega_1-2\gamma)t}\Gamma_{42}^* & \delta\zeta^* e^{i(\Omega_2-2\gamma)t}\Gamma_{43}^* & |\delta|^2 \end{pmatrix},$$

The effect of the environment is encoded in the $\Gamma_{ij}(t)$ functions. They are deduced for bosonic and fermionic environments in [21]. In the case of having an environment composed by an infinite set of harmonic oscillators at temperature $T = 0$, the coefficients Γ_{ij} are given by

$$\Gamma(t) = e^{-2\gamma_0 \log(1+\Lambda^2 t^2)}, \tag{4}$$

where it is supposed that all the coupling strengths are the same, therefore we have omitted subindex in the expression for the decoherence factors (detailed calculations can be found in Ref. [21]). In this case, γ_0 is a dissipation constant (related to the coupling constant between the system and the environment) and Λ is the frequency cutoff of the bath. In the case of a spin-environment, the decoherence factors can be written as

$$\Gamma(t) = \prod_{i=1}^{N} \left\{ 1 - \left(\frac{2(\varepsilon_i \pm \lambda_i)^2}{h_i^2 + (\varepsilon_i \pm \lambda_i)^2} \right) \sin^2(t\sqrt{h_i^2 + (\varepsilon_i \pm \lambda_i)^2}) \right\}, \tag{5}$$

where signs $(\pm)$ in the parenthesis, are related with the type of initial state selected. These cases will be defined in the next Section.

3. Geometric phase

In order to compute the geometric phase for the open system, we shall use a kinematic approach proposed by [13], defined as

$$\phi_G \;=\; \arg\left\{ \sum_k \sqrt{\varepsilon_k(0)\varepsilon_k(\tau)} \langle \Psi_k(0)|\Psi_k(\tau)\rangle \times e^{-\int_0^\tau dt \langle \Psi_k | \frac{\partial}{\partial t} | \Psi_k \rangle} \right\}, \tag{6}$$

where $\varepsilon_k(t)$ are the eigenvalues and $|\Psi_k\rangle$ the eigenstates of the reduced density matrix $\rho_{\rm r}$ (obtained after tracing over the reservoir degrees of freedom). In the last definition, τ denotes a time after the total system completes a cyclic evolution when it is isolated from the environment. Taking into account the effect of the environment, the system no longer undergoes a cyclic evolution. However, we shall consider a quasi cyclic path $\mathcal{P}: \; t \; \epsilon \; [0, \tau]$, with $\tau = 2\pi/\Omega$ (Ω is the system's characteristic frequency). When the system is open, the original GP, i.e. the one that would have been obtained if the system had been closed ϕ_G^U, is modified. This means, in a general case, the phase is $\phi_G = \phi_G^U + \delta\phi_G$, where $\delta\phi_G$ depends on the kind of environment coupled to the main system [14, 15, 16].

In this manuscript we shall consider different initial states to compute the correction to the unitary geometric phase when these states evolve in a no-unitary way.

- *Case 1.* We consider the initial state $\rho = a|\phi^+ >< \phi^+| + (1-a)|1,1 >< 1,1|$ ($0<a \leq 1$), where $|\phi^+ >= (|0,0> + |1,1>)/\sqrt{2}$ is a maximally entangled state. In this case, we can compute the geometric phase by extracting the eigenvalues from the reduced density matrix. In order to know how the geometric phase of this family of states is affected by the presence of an environment, we shall present the rate between the open geometric phase and the unitary geometric phase in a density plot in Fig.1, where the vertical axis corresponds to the coupling constant γ_0 for the oscillators bath, and ϵ (or λ) in the case of spin environment. The horizontal is for the parameter a which set the degree of entanglement. Therein we consider both cases: (a) a bosonic environment and (b) a fermonic environment.

Journal of Physics: Conference Series **626** (2015) 012043 doi:10.1088/1742-6596/626/1/012043

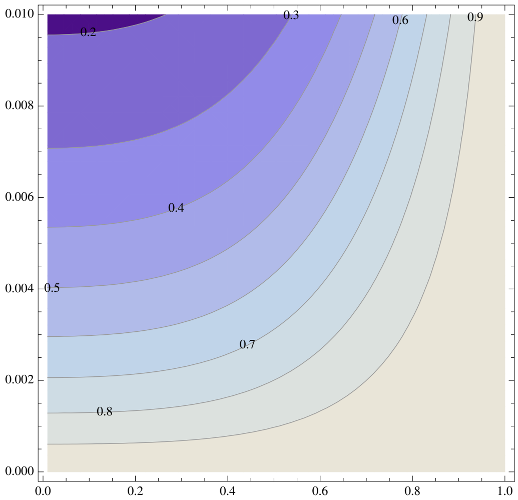

Fig1.(a) ϕ/ϕ_u for the bosonic environment. There is no correction to the GP for the MES.

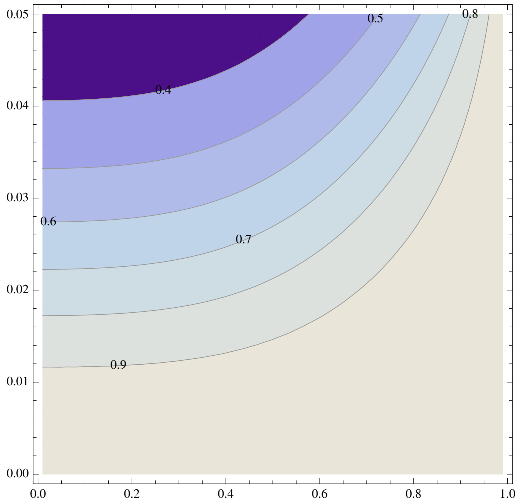

Fig1.(b) ϕ/ϕ_u in the case of a spins environment.There is no correction to the GP for the MES.

In this case, the parameters used are: $\Lambda = 100$ for a set of harmonic oscillators at zero temperature while we used and $h = 1$ for an environment composed of $N = 100$ spins. In all cases, we considered that the interaction between each spin and the environment was equivalent. In both plots of Fig.1 we can see that the correction to the geometric phase $\phi/\phi_u \approx 1$ (which means $\delta\phi_G \approx 0$) when the parameter a $\rightarrow 1$, which corresponds to a state approaching a maximally entangled state (MES). In this case, the survival phase is only the topological phase, and there is no correction induced by the environments. These results has also been showed in [21] with another set of states, which reveals that the robustness of the topological phase under the influence of decoherence is a more general result, not restricted to some states or kind of environmental influence. The topological phase, which is indeed a consequence of the geometry of the entangled two-level system, has been studied for MES and it is at the origin of singularities appearing in the phase of MES during a cyclic evolution. In Ref. [22], it is studied the phase dynamics of entangled qubits under unitary cyclic evolutions. Therein, it is shown that, after a cyclic evolution, the combination of the different phases always leads to a global phase of an entire multiple of π. This result, already known and verified experimentally for a single qubit is recovered here for an entangled qubit with maximal degree of entanglement in the presence of an environment. The total phase gained by a state in a closed evolution is a combination of not only the dynamical and geometrical phase but also the topological phase. Similarly to one qubit states, MES also gain a total phase of π (or $n\pi$) under a cyclic evolution. However, this phase is of topological origin. In the case of a MES there is no correction to the unitary phase. For other states, the correction to the phase increases with the smaller values of parameter a.

- *Case 2.* We take the class of states defined as $\rho = a|\psi^+ >< \psi^+| + (1-a)|1,1 >< 1,1|$ (0<a $\leq$ 1), where $|\psi^+ >= (|0,1> + |1,0>)/\sqrt{2}$ is a maximally entangled state.

  In Fig.2. we present the results for this case, where decoherence does not affect the system. Therefore, the geometric phase is the unitary geometric phase for all set of states given for any value of a. These states are fully robust under the influence of external conditions such the ones presented in these examples. In all cases, we considered that the interaction between each spin and the environment was equivalent. We have considered the same

Journal of Physics: Conference Series **626** (2015) 012043 doi:10.1088/1742-6596/626/1/012043

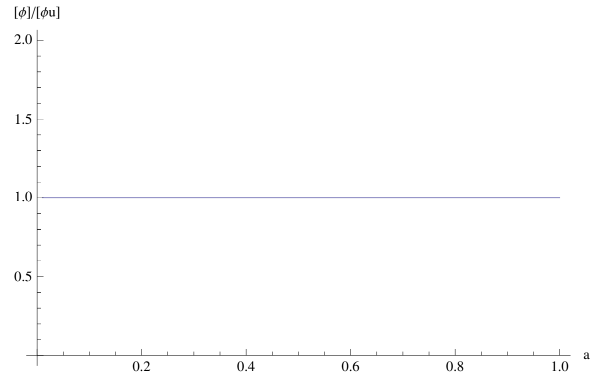

Fig2. ϕ/ϕ_u for this type of state under either a bosonic or fermonic environment. There is no correcction to the GP for these states.

parameters as in Case 1.

4. Correlations

In this section we can analyse the quantum and classical correlations that exist for both family of states considered in Case 1 and Case 2 above, for an isolated system (the effect of the environment on these quantities will be presented elsewhere). A measure of the classical relations can be written as $\mathcal{C}(\rho) := \sup_{B_k}\mathcal{I}(\rho|B_k)$ where B_k means all the possible von Neumann measurements of the qubit B. As the quantum discord is defined as $\mathcal{Q}(\rho) := \mathcal{I}(\rho) - \mathcal{C}(\rho)$, the obstacle to computing this quantity lies in this complicated maximisation procedure. For a general two-qubit X state, the quantification of quantum discord is still missing with some particular results available. Herein, we shall follow the method developed in [5] to evaluate the classical correlation and quantum discord of the two-qubit X states considered above. Classical and quantum correlations of the bipartite quantum system are quantified jointly by the mutual information. If ρ^{AB} is the density matrix (operator) of the bipartite system AB, ρ^A (ρ^B) is the density matrix associated to the part A (B). Thus, one can define the quantum mutual information as

$$\mathcal{I}(\rho^{AB}) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}), \tag{7}$$

where $S(\rho^A) = -\mathrm{tr}(\rho \log_2 \rho)$ is the von Neumann entropy.

In Fig3. we present the classical correlation ($\mathcal{C}$ dotted black line), the quantum discord ($\mathcal{Q}$ dashed red line), and the concurrence (C in blue solid line), for each family of states considered above. In these examples, the quantum correlation is measured by the concurrence (see for instance Ref. [21]).

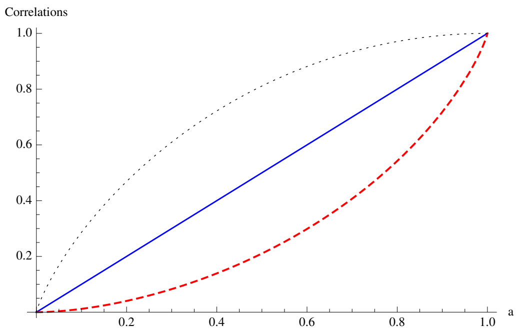

Fig.3-a) Classical correlations are bigger when we consider initial state is $\rho = a|\phi^+ >< \phi^+| + (1-a)|1,1 >< 1,1|$.

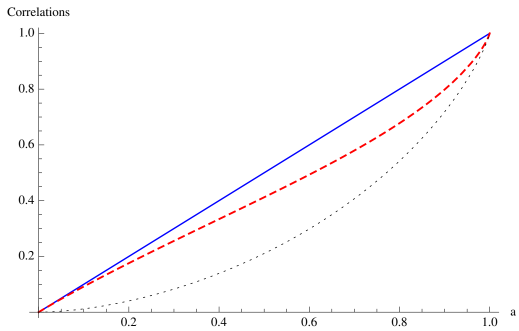

Fig.3-b) Quantum correlations are an upper bound when the initial state is $\rho = a|\psi^+ >< \psi^+| + (1-a)|1,1 >< 1,1|$.

From the plots, we can see that in *Case 1* there is a given hierarchy in the correlations. Classical correlations are always much bigger than the quantum discord, which is also smaller than the concurrence of these states. In the case of a MES, all measure of correlations coincide in the same value. On the other side, it is possible to note that the quantum correlation (concurrence in our example) results in an upper bound for all type of correlations, classical or even the quantum discord, in the *Case 2* example. In this case we have shown that there is not decoherence effects coincidentally with the fact that the quantum correlation remains bigger than the others.

5. Conclusion

As can be noticed, the environment has a stronger influence on the geometric phase when the state is less correlated, say small values of a. In those cases, the family of states proposed in *Case 1*, have a considerable correction to the unitary geometric phase. However, this correction decreases as the states become more correlated. For the family states of *Case 2*, we see that decoherence does not affect the geometric phase at all. In that case, we can see that the quantum correlations are very important being always bigger than classical correlations. Maybe a further insight into the quantum correlations of the quantum states can set light on the way the geometric phase is corrected for a bipartite quantum state under no-unitary evolution.

Acknowledgments

This work was supported by UBA, CONICET and ANPCyT - Argentina. We would like to thank H.-Thomas Elze for the organization of DICE2014. FCL acknowledges B. Lok Hu, Nick Mavromatos, and M. Blasone for the very interesting conversations about topology, decoherence and geometric phases.

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