Violation of Leggett–Garg inequalities for quantum–classical hybrids

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Abstract. Violation of Leggett–Garg inequalities can serve as a signature of a failure of (macroscopic) realism. We investigate violation of the simplest Leggett–Garg inequality for a qubit coupled to an integer j spin (angular momentum). Such a system effectively reveals quantum–classical hybrid behavior in the limit of large j values. We show that a maximal violation of the Leggett– Garg inequality is larger for quantum–classical hybrids than for fully quantum systems.

1. Introduction

Discoveries of entanglement [1] and other types of quantum correlations like quantum discord [2] play essential role in a way of seeing and understanding of the microscopic and macroscopic reality. Well known violation of Bell inequalities [3] is the best evidence of the existence of such spatial correlations which cannot be understand in terms of some "classical" local hidden variables.

Moreover, recently also temporal quantum correlations are studied. In analogy to the spatial correlations quantified by the Bell inequalities [3] one predicts and observes violations of the so called Leggett-Garg inequalities (LGI) [4]. LGI are derived under two assumptions: *macroscopic realism* and *non–invasive measurability* of the system. The second requirement is subtle and often leads to conceptual difficulties related to the clumsiness loophole [4]. The first requirement, the macroscopicity, also is often abandoned as the LGI are interesting by its own also in microscopic systems [5].

It was shown that violation of LGI by free spin particle with dichotomous measurement increases with respect to absolute value of spin [5]. It is interesting fact since systems with higher spin are naturally considered as more macroscopic (like atomic nucleus) and hence probably 'more realistic'. In this paper we consider a simple composite system consisting of a qubit coupled to an integer spin (angular momentum) exhibiting well defined classical limit. In other words, we consider quantum–classical hybrid system with a qubit as a quantum part. It is known that the spatial correlations (entanglement) in quantum–classical hybrid systems can exhibit unexpected properties [6, 7]. Here we show that it is also the case for LGIs which are more violated by hybrid systems than by those which operate in fully quantum regime.

2. Model

Leggett–Garg inequalities are constructed from time correlation functions of a series of measurements of some dichotomous variable carried on a considered system. We study the simplest among LGIs [4] when the system undergoes the measurement only thrice:

$$K_3 \quad := \quad C_{21} + C_{32} - C_{31} \leq 1 \tag{1}$$

where

$$C_{\beta\alpha} \quad = \quad \sum_{m,l} q_m q_l \mathrm{Tr}\left[\Pi_m U_{\beta\alpha} \Pi_l U_{\alpha 0} \rho_0 U^\dagger_{\alpha 0} \Pi_l U^\dagger_{\beta\alpha}\right] \tag{2}$$

is the time correlation function for some dichotomous variable Q and and $q_l, q_m = \pm 1$ represent (projective) measurement outputs given by eigenvalues of the observable corresponding to Q and associated with a set of projectors Π_l. Here ρ_0 is the initial state of the system and $U_{\beta\alpha} = U(t_\beta, t_\alpha) = e^{-i(t_\beta - t_\alpha)H}$ is the unitary time-evolution operator with the Hamiltonian H. For non–invasive measurement of Q, violation of the LGI in Eq.(1) indicates possible failure of the (macroscopic) realism.

For our purposes we assume a simple spin–orbit type of interaction:

$$H \quad = \quad \alpha\, \vec{\sigma} \cdot \vec{J} \tag{3}$$

typical for Hydrogen–like atomic objects. Here $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli matrices acting on a qubit subspace, $\vec{J} = \frac{1}{j}(J_x, J_y, J_z)$ is a vector of an integer spin operator ($j = 1, 2, \ldots$) and α is a coupling constant. In the limit of $j \to \infty$ there is a well defined classical limit of the integer spin part of the system [8]. As the qubit part, given by $\vec{\sigma}$, remains quantum, the total system effectively mimics in large j limit quantum–classical hybrid. Let us notice that there are alternative approaches for studying quantum–classical hybrid systems [9, 10].

For the quantity which is measured to obtain the correlation function Eq.(2) we set an x–component of the qubit spin (i.e. $m, l = \pm 1$ in Eq.(2)). Such an observable is represented by a pair of projectors

$$\Pi_\pm \quad = \quad |\pm x\rangle \langle \pm x| \otimes \mathcal{I} \tag{4}$$

where $|\pm x\rangle$ is an eigenstate of the x–component of σ_x and the identity $\mathcal{I}$ acts on the integer spin part of the system.

3. Results

In the absence of qubit–spin coupling ($\alpha = 0$) there is no time evolution, always $K_3 = 1$ and the LGI Eq.(1) is never violated. For $\alpha > 0$ let us consider the following initial preparation of the system:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right) \otimes |m\rangle \tag{5}$$

where the states $|0\rangle$ and $|1\rangle$ are the eigenstates of σ_z operator (with eigenvalues $+1$ and -1) and $|m\rangle$ is one of eigenstates of J_z operator, i.e. $J_z\,|m\rangle = m\,|m\rangle$.

In the following we assume that the measurements required to calculate K_3 in Eq.(1) appearing in Eq.(2) are equally separated in time i.e. $t_1 = \tau, t_2 = 2\tau$ and $t_3 = 3\tau$.

3.1. One-dimension coupling

Let us first consider a simplified analytically solvable model with interaction restricted only to one spin direction:

$$H = \frac{\alpha}{j}\,\sigma_z J_z \tag{6}$$

For the initial state (5) the K_3 function reads as follows:

$$K_3 = 2\cos(\omega\tau) - \cos(2\omega\tau) \tag{7}$$

where $\omega = 2\alpha m/j$ and the period of K_3 function is $T_1 = j\pi/\alpha m$ where the subscript '1' indicates one-dimensional interaction Eq.(6).

Any changes of coupling strength α and integer spin value j affects just the period of the K_3 function. In the (semi)classical limit of large j, the smaller are the values of m of the integer spin projection on quantization axis, the longer is the period of K_3 oscillation. Moreover, with increasing m the period of K_3 function becomes minimal (given by $T_1 = \pi/\alpha$) and independent on the dimension of the Hilbert space. It holds true for all initial preparations of the form Eq.(5) with $|m = j\rangle$. One can conclude that the interaction between a qubit and a (semi)classical object results [for small m in Eq.(5)] in stretching of time periods of the LGI violation. In particular, an amplitude of the K_3 function remain the same with an exception of the initial state Eq.(5) with $|m = 0\rangle$ when $K_3 \equiv 1$.

3.2. Three-dimension coupling

In order to examine violation of the LGI of the system evolving according to the full Hamiltonian Eq. (3) we applied numeric simulations using Python toolbox QuTiP [11]. As in simplified one-dimensional model Eq.(6), the multiplication constant in a front of Hamiltonian Eq.(3) representing a strength of an interaction affects only the period of the K_3 function (and not its amplitude). The period of K_3 reads as follows:

$$T_3 = \frac{2\pi j}{\alpha N} \tag{8}$$

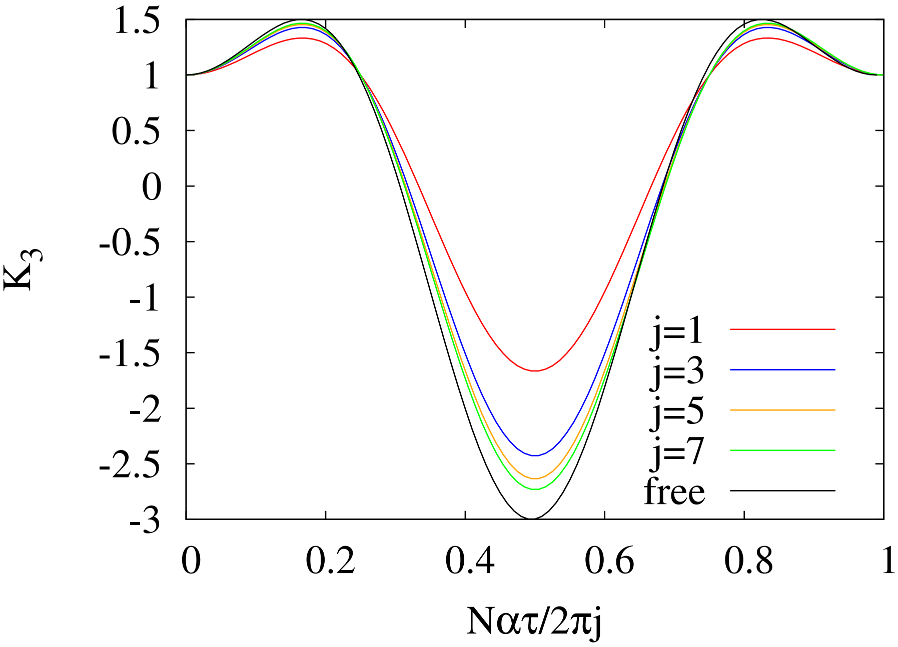
where $N = 2j + 1$ is the dimension of the spin j Hilbert subspace. In the classical limit we obtain:

$$T_3^c \;:=\; \lim_{j\to\infty} T_3 = \frac{\pi}{\alpha} \tag{9}$$

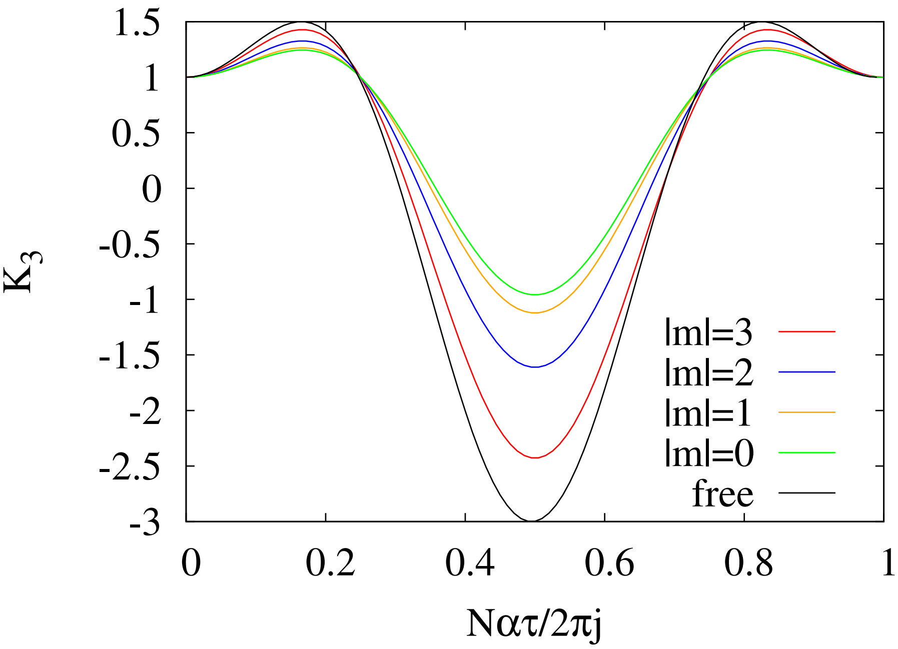
i.e. the same value as the minimal period in the case one-dimension coupling T_1. However, in the three dimensional case, the time period of the quantum–classical hybrid system Eq.(9) is maximal i.e. $T_3^c \geq T_3$. Moreover, in this case, the T_3–period does not depend on m in the initial state Eq.(5) and hardly increases with respect to the j value.

Let us notice the *qualitative* difference between one– and three–dimensional case. In the later one not only the period of K_3 function but also its amplitude changes with respect to the quantum numbers m and j in the initial state Eq.(5) as in Fig.1(a) and Fig.1(b).

Particularly interesting is of *maximal violation* of the LGI when the j–component of the system approaches classical limit i.e. when the qubit–spin system starts to be a quantum–classical hybrid. Maximum of the LGI violation for different values of j and initial preparations is presented in Fig.2. One can infer that the LGI violation is stronger in the system interacting with higher spin values i.e. for quantum–classical hybrids when the LGI violation approaches its maximal possible value $K_3 = 3/2$ [4].



(a)



(b)

Figure 1. (Color online) K_3 function for a qubit interacting with integer j spin in the initial state $|m\rangle$ in Eq.(5): *(a)* for different values of j and $m = j$; *(b)* for different values of m for fixed $j = 3$. Time interval τ between measurements is rescaled with respect to the period T_3 in Eq.(8).

4. Conclusion

Violation of the Leggett–Garg inequalities, as in the case of Bell inequalities [3], provide an insight into very counter–intuitive properties of Nature. In our work we investigate a composite

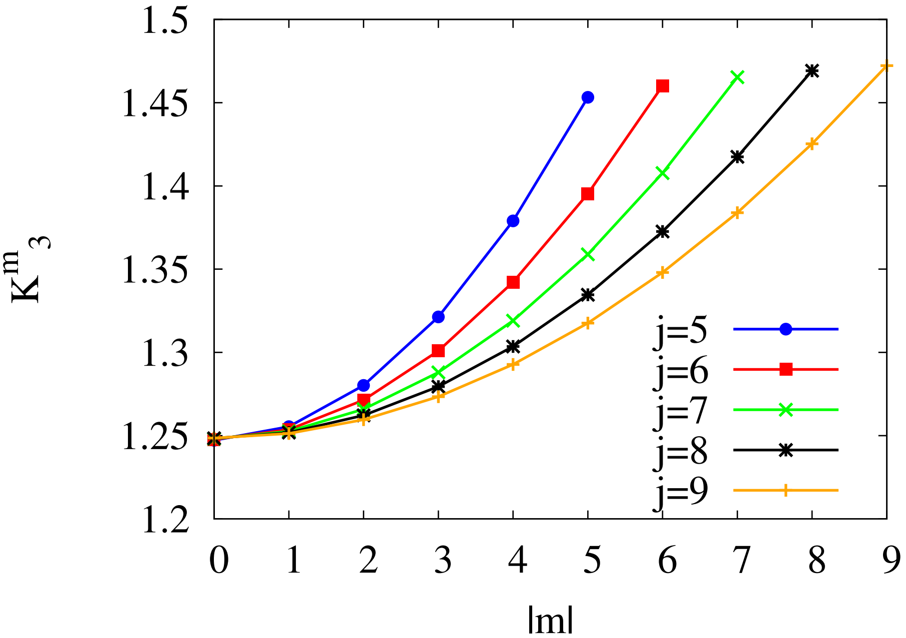


Figure 2. (Color online) Maximal value K_3^m of K_3 function of the qubit coupled to integer spin j according to Eq.(3) prepared in the state Eq.(5) versus m and for different values of j and initial state Eq.(5).

system consisting of a qubit, a basic building block for quantum information, and the integer spin, angular momentum, applicable in various models of atomic physics. We considered dichotomous measurement of the x–component of the qubit corresponding to projecting the system on eigenstates of σ_x operator tensorized with identity acting on integer spin system. Our aim was to study violation of the LGIs in the limit of large j when the composite qubit–spin system can mimic quantum–classical hybrid. For the simplest one–dimensional qubit–spin coupling we obtain analytic results for the K_3 function with a period (but not an amplitude) changing with increasing j. In general three–dimensional case, requiring numerical treatment, the amplitude of the K_3 function became j–dependent in a rather counter–intuitive fashion: maximal violation of the LGI occurring for the quantum–classical hybrid system is *larger* than in the fully quantum systems with small values of j. In particular, for $j \to \infty$ $K_3 \to 3/2$ i.e. its value approaches largest possible value.

Since systems with high spin values are naturally considered as more macroscopic we show that, counter–intuitively, microscopic qubit violates LGI more when it interacts with more macroscopic object.

Acknowledgments

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