

Noncommutative spectral geometry, Bogoliubov transformations and neutrino oscillations

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Abstract. In this report we show that neutrino mixing is intrinsically contained in Connes’ noncommutative spectral geometry construction, thanks to the introduction of the doubling of algebra, which is connected to the Bogoliubov transformation. It is known indeed that these transformations are responsible for the mixing, turning the mass vacuum state into the flavor vacuum state, in such a way that mass and flavor vacuum states are not unitary equivalent. There is thus a red thread that binds the doubling of algebra of Connes’ model to the neutrino mixing.

1. Introduction

In the noncommutative spectral geometry (NCSG) construction [1]–[17], introduced by Alain Connes, non commutative geometry is combined with a spectral triples so to obtain the full Lagrangian of the Standard Model (SM), minimally coupled with gravity and compatible with neutrino mixing. The coupling with gravity is obtained thanks to the fact that this construction uses a group of symmetry which encodes both, the diffeomorphism which control general relativity, and the local gauge invariance which rules the gauge theories the SM is based on.

The key point in Connes’ construction is the doubling of the space and of the algebra. It has been shown [18] that this doubling is related to the gauge field structure, to dissipation and that the NCSG classical construction carries in itself the seeds of quantization. Closely following ref. [19], in this report we show that the doubling of the algebra is also the main mathematical tool on which neutrino mixing is based.

A brief summary of the NCSG construction is presented in Section 2. The doubling of the algebra and the Bogoliubov transformations are discussed in Sections 3 and 4 and the neutrino mixing is studied in Section 5. Section 6 is devoted to the conclusions.

2. A brief summary of NCSG construction

The skeleton of the NCSG construction for the Standard Model is presented below.

Considering the action functional S at low energy scale, $S = S_{\mathcal{E}-\mathcal{H}} + S_{\mathcal{SM}}$, which is the sum of the Einstein-Hilbert action ($S_{\mathcal{E}-\mathcal{H}}$) and SM action ($S_{\mathcal{SM}}$), it can be noticed that the two parts do not share the same symmetries. The former is ruled by outer



automorphism invariance (diffeomorphism) the latter by inner automorphism (local gauge transformation). Connes considers then a model of a two-sheeted space, realized as the product of a four dimensional smooth compact Riemannian manifold \mathcal{M} with a fixed spin structure by a discrete noncommutative space \mathcal{F} composed by only two points. The geometric space is thus defined as the tensor product of the continuous geometry of \mathcal{M} by an internal geometry \mathcal{F} .

In this approach, the SM is seen as a phenomenological model with the geometry of space-time such that the Maxwell-Dirac action functional leads to the SM action. The noncommutative nature of the discrete space \mathcal{F} is expressed by the real spectral triple $\mathcal{F} = (\mathcal{A}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}, D_{\mathcal{F}})$, where $\mathcal{A}_{\mathcal{F}}$ is an involution of operators on the finite-dimensional Hilbert space $\mathcal{H}_{\mathcal{F}}$ of Euclidean fermions, and $D_{\mathcal{F}}$ is a self-adjoint unbounded operator in $\mathcal{H}_{\mathcal{F}}$. The spectral nature of the triple implies that the Dirac operator $D_{\mathcal{F}}$ of the internal space is the fermionic mass matrix. The Dirac operator is the inverse of the Euclidean propagator of fermions. The information carried by the metric are contained in the algebra $\mathcal{A}_{\mathcal{F}}$. The spectral nature approach is intrinsic to the the noncommutative spectral geometry. The $\mathcal{M} \times \mathcal{F}$ product geometry is specified by the rules:

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\mathcal{F}} = C^\infty(\mathcal{M}, \mathcal{A}_{\mathcal{F}}), \quad (1)$$

$$\mathcal{H} = \mathcal{L}^2(\mathcal{M}, S) \otimes \mathcal{H}_{\mathcal{F}} = \mathcal{L}^2(\mathcal{M}, S \otimes \mathcal{H}_{\mathcal{F}}) \quad (2)$$

$$D = \not{D}_{\mathcal{M}} \otimes 1 + \gamma_5 \otimes D_{\mathcal{F}}, \quad (3)$$

where $\mathcal{L}^2(\mathcal{M}, S)$ is the space of square integrable Dirac spinors over \mathcal{M} , the Dirac operator on \mathcal{M} is denoted by $\not{D}_{\mathcal{M}} = i\gamma^\mu \nabla_\mu^s$ (with ∇_μ^s is the spin connection $\nabla_\mu^s = \partial_\mu + \frac{1}{2}\omega_\mu^{ab}\gamma_{ab}$), γ_5 is the chirality operator in the four-dimensional case, $C^\infty(\mathcal{M})$ is the algebra of smooth functions on \mathcal{M} acting on $\mathcal{H}_{\mathcal{F}}$ as simple multiplication operators. Within the noncommutative spectral geometry D plays the role of the inverse of the line element ds .

The main results of the model are obtained by using the spectral action principle, which is of the form

$$Tr(f(\frac{\mathcal{D}}{\Lambda})), \quad (4)$$

where \mathcal{D} denotes the inverse of the Dirac operator, Λ fixes the energy scale and f is a cut-off positive even function of the real variable; it falls to zero for large values of its argument. The function f only plays a role through its momenta f_0 , f_2 and f_4 , where $f_k = \int_0^\infty f(u)u^{k-1} du$ for $k > 0$ and $f_0 = f(0)$, which are three parameters of the model and are related to the coupling constant at unification, the gravitational constant and the cosmological constant, respectively. Its asymptotic expansion is:

$$Tr(f(\frac{\mathcal{D}}{\Lambda})) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4. \quad (5)$$

The action functional, applied to inner fluctuations, only accounts for the bosonic part of the model. The term in Λ^4 gives the cosmological term, the term in Λ^2 gives the Einstein-Hilbert action functional, and the Λ -independent term yields the Yang-Mills action for the gauge fields. The coupling with fermions is obtained by adding the term $\frac{1}{2} \langle J\psi, \mathcal{D}\psi \rangle$, where J is a real structure on the spectral triple and ψ is a spinor in the Hilbert space $\mathcal{H}_{\mathcal{F}}$ of the quarks and leptons. The computation of the spectral action functional gives the full Lagrangian for the Standard Model minimally coupled with gravity, with neutrino mixing and Majorana mass terms. The vacuum expectation value of the Higgs field is related to the non commutative distance between the two sheets. In the model neutrino mixing is obtained in analogy with the quarks case.

3. Doubling the Algebra and the deformed Hopf algebra coproduct

In this Section we show that Bogoliubov transformations are “built in” in the algebra doubling in the NCSG. This implies in turn that the the NCSG construction insists on a collection of

unitarily inequivalent Hilbert spaces which are related among themselves (phase transitions) by Bogoliubov transformations. Since these also characterize the mixing transformations of neutrinos, the NCSG construction also contains implicitly the neutrino mixing phenomenon.

Consider the Dirac spectral triple $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1) = (C^\infty(\mathcal{M}, \mathcal{L}^2(\mathcal{M}, S)), \not{D}_{\mathcal{M}})$ and its product with the finite geometry $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2) = (\mathcal{A}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}})$. The product geometry $\mathcal{M} \times \mathcal{F}$ is given by

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_1 \otimes \mathcal{A}_2, \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \\ \mathcal{D} &= \mathcal{D}_1 \otimes 1 + \gamma_1 \otimes \mathcal{D}_2, \\ \gamma &= \gamma_1 \otimes \gamma_2 J = J_1 \otimes J_2,\end{aligned}\tag{6}$$

with

$$J^2 = -1, \quad [J, \mathcal{D}] = 0, \quad [J_1, \gamma_1] = 0, \quad \{J, \gamma\} = 0,\tag{7}$$

where J is an antilinear isometry and commutators and anticommutators are denoted by square and curl brackets, respectively. In the formalism of the algebra doubling an important rôle is played by the noncommutative q -deformed Hopf algebra, pointing to a deep physical meaning of the noncommutativity in this construction. Indeed, the map $\mathcal{A} \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$ is just the Hopf coproduct map $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{A}$. Actually for a noncommutative algebra the coproduct is defined by

$$\Delta a_q = a_q \otimes q^H + q^{-H} \otimes a_q\tag{8}$$

$$\Delta a_q^\dagger = a_q^\dagger \otimes q^H + q^{-H} \otimes a_q^\dagger.\tag{9}$$

The fermionic algebra $h(1|1)$ is generated by a set of operators $\{a, a^\dagger, H, N\}$ with anticommutation relations:

$$\{a, a^\dagger\} = 2H, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [H, \bullet] = 0.\tag{10}$$

The deformed algebra $h_q(1 | 1)$ relations are

$$\{a_q, a_q^\dagger\} = [2H]_q, \quad [N, a_q] = -a_q, \quad [N, a_q^\dagger] = a_q^\dagger, \quad [H, \bullet] = 0,\tag{11}$$

where $N_q \equiv N$ and $H_q \equiv H$. The Casimir operator \mathcal{C}_q is given by $\mathcal{C}_q = N[2H]_q - a_q^\dagger a_q$, where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

and together with the coproduct defined in Eqs.(8) we have

$$\Delta H = H \otimes \mathbf{1} + \mathbf{1} \otimes H,\tag{12}$$

$$\Delta N = N \otimes \mathbf{1} + \mathbf{1} \otimes N.\tag{13}$$

Requiring that a and a^\dagger , and a_q and a_q^\dagger are adjoint operator implies that q can only be real or of modulus one; for fermion, in fact, $q \sim e^{i\theta}$. Considering a two-mode Fock space $F_2 = F_1 \otimes F_1$ it is possible to write

$$\Delta a_q = a_1 q^{\frac{1}{2}} + q^{-\frac{1}{2}} a_2,\tag{14}$$

$$\Delta a_q^\dagger = a_1^\dagger q^{\frac{1}{2}} + q^{-\frac{1}{2}} a_2^\dagger,\tag{15}$$

$$\Delta N = N_1 + N_2, \quad \Delta H = 1.\tag{16}$$

One key point is that the noncommutative Hopf coproduct turns out to be strictly related to the Bogoliubov transformations [20]. It is indeed possible to identify $a_1 \equiv a$, $a_1^\dagger \equiv a^\dagger$, and $a_2 \equiv \tilde{a}$, $a_2^\dagger \equiv \tilde{a}^\dagger$, where \tilde{O} , with O a generic operator, denotes the "tilde-coniugation" operation defined by the "tilde-coniugation" rules [21]:

$$\widetilde{(OO')} = \tilde{O}\tilde{O'}, \quad (17)$$

$$\widetilde{(\alpha O + \beta O')} = \alpha^* \tilde{O} + \beta^* \tilde{O'}, \quad (18)$$

$$\widetilde{(\tilde{O}^\dagger)} = \tilde{O}^\dagger, \quad (19)$$

$$\widetilde{(\tilde{O})} = O, \quad (20)$$

$$\{O, \tilde{O}'\} = \{O, \tilde{O}'^\dagger\} = 0, \quad (21)$$

$$\widetilde{|\tilde{O}\rangle} = |O\rangle. \quad (22)$$

It is now possible to define the new operators A_q and B_q as follow

$$A_q \equiv \frac{\Delta a_q}{\sqrt{[2]_q}} = \frac{1}{\sqrt{[2]_q}}(e^{i\theta}a + e^{-i\theta}\tilde{a}), \quad (23)$$

$$B_q \equiv \frac{1}{i\sqrt{[2]_q}}\frac{\delta}{\delta\theta}\Delta a_q = \frac{2q}{\sqrt{[2]_q}}\frac{\delta}{\delta q}\Delta a_q = \frac{1}{\sqrt{[2]_q}}(e^{i\theta}a - e^{-i\theta}\tilde{a}). \quad (24)$$

and h.c., with $q = q(\theta) \equiv e^{i2\theta}$. The anti-commutation relation are:

$$\{A_q, A_q^\dagger\} = 1, \quad \{B_q, B_q^\dagger\} = 1, \quad \{A_q, B_q\} = 0, \quad (25)$$

$$\{A_q, B_q^\dagger\} = -i \tanh i2\theta, \quad (26)$$

whereas all other anti-commutators are equal to zero. A set of operators A and B with canonical commutation relations and commuting among themselves is given by

$$A(\theta) \equiv \frac{\sqrt{[2]_q}}{2\sqrt{2}}[A_{q(\theta)} + A_{q(-\theta)} + A_{q(\theta)}^\dagger - A_{q(-\theta)}^\dagger], \quad (27)$$

$$B(\theta) \equiv \frac{\sqrt{[2]_q}}{2\sqrt{2}}[B_{q(\theta)} + B_{q(-\theta)} - B_{q(\theta)}^\dagger + B_{q(-\theta)}^\dagger]. \quad (28)$$

and h.c., so that

$$\{A(\theta), A^\dagger(\theta)\} = 1, \quad \{B(\theta), B^\dagger(\theta)\} = 1, \quad \{A(\theta), B^\dagger(\theta)\} = 0. \quad (29)$$

and all other anti-commutators equal to zero. It's also possible to write:

$$A(\theta) = \frac{1}{\sqrt{2}}(a(\theta) + \tilde{a}(\theta)), \quad B(\theta) = \frac{1}{\sqrt{2}}(a(\theta) - \tilde{a}(\theta)). \quad (30)$$

with

$$a(\theta) = \frac{1}{\sqrt{2}}(A(\theta) + B(\theta)) = a \cosh i\theta - \tilde{a}^\dagger \sinh i\theta, \quad (31)$$

$$\tilde{a}(\theta) = \frac{1}{\sqrt{2}}(A(\theta) - B(\theta)) = \tilde{a} \cosh i\theta - ia^\dagger \sinh i\theta, \quad (32)$$

and

$$\{a(\theta), a^\dagger(\theta)\} = 1, \quad \{\tilde{a}(\theta), \tilde{a}^\dagger(\theta)\} = 1. \quad (33)$$

All others anti-commutators are equal to zero and $a(\theta)$ and $\tilde{a}(\theta)$ anti-commute among themselves. Eqs. (31) and (32) are nothing but the Bogoliubov transformations of the pair of creation and annihilation operators (a, \tilde{a}) into a new pair $(a(\theta), \tilde{a}(\theta))$. In other words, Eqs. (31)-(33) show that the Bogoliubov-transformed operators $a(\theta)$ and $\tilde{a}(\theta)$ are linear combinations of the coproduct operators defined in terms of the deformation parameter $q(\theta)$ and their θ -derivatives.

4. Neutrino mixing

In the context on NCSG, neutrinos appear as Majorana spinors. Therefore, we refer below to Majorana neutrinos; however, provided that convenient changes are introduced, our formalism can be readily extended to Dirac neutrinos and to other particle mixing.

Majorana fermions are self-conjugate particles. The charge-conjugation operator C satisfies the relations

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T, \quad C^\dagger = C^{-1}, \quad C^T = -C. \quad (34)$$

The charge conjugate ψ^c of ψ is defined by

$$\psi^c(x) \equiv \gamma_0 C \psi^*(x). \quad (35)$$

A Majorana fermion is a field that satisfies both, the Dirac equation and the self-conjugation relation:

$$(i\cancel{\partial} - m)\psi = 0, \quad (36a)$$

$$\psi = \psi^c. \quad (36b)$$

In NCSG, the neutrino mass terms in the Lagrangian is written as

$$\frac{1}{2} \sum_{\lambda\kappa} \bar{\psi}_{\lambda L} \mathcal{S}_{\lambda\kappa} \hat{\psi}_{\kappa R} + \frac{1}{2} \sum_{\lambda\kappa} \overline{\bar{\psi}_{\lambda L} \mathcal{S}_{\lambda\kappa} \hat{\psi}_{\kappa R}}. \quad (37)$$

The subscripts $_{L,R}$ denote left-handed and right-handed states, respectively. The off-diagonal elements of the symmetric matrix $\mathcal{S}_{\lambda\kappa}$ are the Dirac mass terms, the diagonal ones are the Majorana mass terms.

From the equations of motion one has that the largest eigenvalue of the Majorana mass matrix M_R is of the order of the unification scale, while the Dirac mass M_ν is of the order of the Fermi energy and thus much smaller. Neutrino mixing and the seesaw mechanism are thus built in in the NCSG construction. We show below that neutrino mixing is implied by the doubling of the algebra, which is the core of Connes construction.

In order to proceed in our discussion, we first summarize briefly the QFT formalism for the neutrino mixing [22]-[26]. We present a number of relations which we need in order to show the role played in such a phenomenon by the Bogoliubov transformations. We will closely follow ref. [19], where a more detailed presentation is reported.

We start by introducing the Lagrangian:

$$L(x) = \bar{\psi}_f(x)(i\cancel{\partial} - M)\psi_f(x) = \bar{\psi}_m(x)(i\cancel{\partial} - M_d)\psi_m(x), \quad (38)$$

with $\psi_f^T = (\nu_e, \nu_\mu)$ being the flavor fields and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

The flavor fields are connected to the free fields $\psi_m^T = (\nu_1, \nu_2)$, with $M_d = \text{diag}(m_1, m_2)$, by the Pontecorvo mixing transformation

$$\nu_e(x) = \nu_1(x) \cos \theta + \nu_2(x) \sin \theta, \quad (39)$$

$$\nu_\mu(x) = -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta. \quad (40)$$

The free fields are given by

$$\nu_i(x) = \sum_{r=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} [u_{\mathbf{k},i}^r(t) \alpha_{\mathbf{k},i}^r + v_{-\mathbf{k},i}^r(t) \alpha_{-\mathbf{k},i}^{r\dagger}] \quad , \quad i = 1, 2 \quad (41)$$

where $u_{\mathbf{k},\mathbf{i}}^r(t) = e^{-i\omega_{\mathbf{k},\mathbf{i}}t} u_{\mathbf{k},\mathbf{i}}^r$, $v_{\mathbf{k},\mathbf{i}}^r(t) = e^{i\omega_{\mathbf{k},\mathbf{i}}t} v_{\mathbf{k},\mathbf{i}}^r$ and $\omega_{\mathbf{k},\mathbf{i}} = \sqrt{\mathbf{k}^2 + \mathbf{m}_1^2}$. In order for the Majorana condition (36b) to be satisfied, the four spinors must also satisfy the following condition:

$$u_{\mathbf{k},\mathbf{i}}^s = \gamma_0 C(v_{\mathbf{k},\mathbf{i}}^s)^*, \quad v_{\mathbf{k},\mathbf{i}}^s = \gamma_0 C(u_{\mathbf{k},\mathbf{i}}^s)^*. \quad (42)$$

The equal-time anti-commutation relations are

$$\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(\mathbf{x} - \mathbf{y}) \delta_{\alpha\beta} \delta_{\mathbf{ij}}, \quad (43)$$

$$\{\nu_i^\alpha(x), \nu_j^\beta(y)\}_{t=t'} = \delta^3(\mathbf{x} - \mathbf{y}) (\gamma_0 \mathbf{C})_{\alpha\beta} \delta_{\mathbf{ij}}, \quad (44)$$

with $\alpha, \beta = 1, \dots, 4$. And

$$\{\alpha_{-\mathbf{k},\mathbf{i}}^r, \alpha_{-\mathbf{q},\mathbf{i}}^{s\dagger}\}_{t=t'} = \delta^3(\mathbf{k}_\mathbf{q}) \delta_{\beta\mathbf{rs}} \delta_{\mathbf{ij}}, \quad \mathbf{i} = \mathbf{1}, \mathbf{2}. \quad (45)$$

All the other anti-commutators are zero. The orthonormality and completeness relations are :

$$u_{\mathbf{k},\mathbf{i}}^{r\dagger} u_{\mathbf{k},\mathbf{i}}^s = v_{\mathbf{k},\mathbf{i}}^{r\dagger} v_{\mathbf{k},\mathbf{i}}^s = \delta_{rs}, \quad u_{\mathbf{k},\mathbf{i}}^{r\dagger} v_{-\mathbf{k},\mathbf{i}}^s = v_{-\mathbf{k},\mathbf{i}}^{r\dagger} u_{\mathbf{k},\mathbf{i}}^s = 0, \quad (46)$$

$$\sum_r = 1, 2 u_{\mathbf{k},\mathbf{i}}^r u_{\mathbf{k},\mathbf{i}}^{r\dagger} + v_{\mathbf{k},\mathbf{i}}^r v_{\mathbf{k},\mathbf{i}}^{r\dagger} = \mathbf{1}. \quad (47)$$

Eqs. (39) and (40) can be recast in the form :

$$\nu_e^\alpha(x) = G_\theta^{-1}(t) \nu_1^\alpha(x) G_\alpha(t), \quad (48)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t) \nu_2^\alpha(x) G_\alpha(t), \quad (49)$$

where $G_\theta(t)$ is given by

$$G_\theta(t) = \exp \left[\frac{\theta}{2} \int d^3\mathbf{x} (\nu_1^\dagger(\mathbf{x}) \nu_2(\mathbf{x}) - \nu_2^\dagger(\mathbf{x}) \nu_1(\mathbf{x})) \right]. \quad (50)$$

One has $G_\theta(t) = \prod_{\mathbf{k}} G_\theta^{\mathbf{k}}(t)$. For a given \mathbf{k} , in the reference frame where $\mathbf{k} = (\mathbf{0}, \mathbf{0}, |\mathbf{k}|)$, the spins decouple and one has $G_\theta^{\mathbf{k}}(t) = \prod_r G_\theta^{\mathbf{k},r}(t)$ with

$$G_\theta^{\mathbf{k}}(t) = \exp \left\{ \theta \left[U_{\mathbf{k}}^*(t) \alpha_{\mathbf{k},\mathbf{1}}^{r\dagger} \alpha_{\mathbf{k},\mathbf{2}}^r - U_{\mathbf{k}}(t) \alpha_{-\mathbf{k},\mathbf{2}}^{r\dagger} \alpha_{-\mathbf{k},\mathbf{1}}^r - \epsilon^r V_{\mathbf{k}}^*(t) \alpha_{-\mathbf{k},\mathbf{1}}^r \alpha_{\mathbf{k},\mathbf{2}}^r + \epsilon^r V_{\mathbf{k}}(t) \alpha_{\mathbf{k},\mathbf{1}}^{r\dagger} \alpha_{-\mathbf{k},\mathbf{2}}^{r\dagger} \right] \right\}, \quad (51)$$

where $U_{\mathbf{k}}(t)$ e $V_{\mathbf{k}}(t)$ are Bogoliubov coefficients given by

$$U_{\mathbf{k}}(t) \equiv |U_{\mathbf{k}}| e^{i(\omega_{\mathbf{k},\mathbf{2}} - \omega_{\mathbf{k},\mathbf{1}})t}, \quad V_{\mathbf{k}}(t) \equiv |V_{\mathbf{k}}| e^{i(\omega_{\mathbf{k},\mathbf{2}} + \omega_{\mathbf{k},\mathbf{1}})t}, \quad (52)$$

$$|U_{\mathbf{k}}| \equiv \left(\frac{\omega_{\mathbf{k},\mathbf{1}} + m_1}{2\omega_{\mathbf{k},\mathbf{1}}} \right)^{\frac{1}{2}} \left(\frac{\omega_{\mathbf{k},\mathbf{2}} + m_2}{2\omega_{\mathbf{k},\mathbf{2}}} \right)^{\frac{1}{2}} \left(1 + \frac{|\mathbf{k}|^2}{(\omega_{\mathbf{k},\mathbf{1}} + m_1)(\omega_{\mathbf{k},\mathbf{2}} + m_2)} \right), \quad (53)$$

$$|V_{\mathbf{k}}| \equiv \left(\frac{\omega_{\mathbf{k},\mathbf{1}} + m_1}{2\omega_{\mathbf{k},\mathbf{1}}} \right)^{\frac{1}{2}} \left(\frac{\omega_{\mathbf{k},\mathbf{2}} + m_2}{2\omega_{\mathbf{k},\mathbf{2}}} \right)^{\frac{1}{2}} \left(\frac{|\mathbf{k}|}{(\omega_{\mathbf{k},\mathbf{2}} + m_2)} - \frac{|\mathbf{k}|}{(\omega_{\mathbf{k},\mathbf{1}} + m_1)} \right), \quad (54)$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1. \quad (55)$$

The flavor fields can be thus expanded as :

$$\nu_\sigma(x) = \sum_{r=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k} \cdot \mathbf{x}} [u_{\mathbf{k},\mathbf{j}}^r(t) \alpha_{\mathbf{k},\sigma}^r + v_{-\mathbf{k},\mathbf{j}}^r(t) \alpha_{-\mathbf{k},\sigma}^{r\dagger}] \quad , \quad (56)$$

with $\sigma, j = (e, 1), (\mu, 2)$ and the flavor annihilation operators given by (for $\mathbf{k} = (\mathbf{0}, \mathbf{0}, |\mathbf{k}|)$):

$$\alpha_{\mathbf{k},e}^r \equiv G_{\theta}^{-1}(t) \alpha_{-\mathbf{k},1}^r G_{\theta}(t) = \cos \theta \alpha_{\mathbf{k},1}^r + \sin \theta (U_{\mathbf{k}}^*(t) \alpha_{\mathbf{k},2}^r + \epsilon^r V_{\mathbf{k}}^*(t) \alpha_{-\mathbf{k},2}^{r\dagger}), \quad (57)$$

$$\alpha_{\mathbf{k},\mu}^r \equiv G_{\theta}^{-1}(t) \alpha_{-\mathbf{k},2}^r G_{\theta}(t) = \cos \theta \alpha_{\mathbf{k},2}^r - \sin \theta (U_{\mathbf{k}}(t) \alpha_{\mathbf{k},1}^r + \epsilon^r V_{\mathbf{k}}(t) \alpha_{-\mathbf{k},1}^{r\dagger}). \quad (58)$$

Notice the presence of the Bogoliubov coefficients $U_{\mathbf{k}}(t)$ and $V_{\mathbf{k}}(t)$ in Eqs. (57) and (58). *These relations clearly show how Bogoliubov transformations enter the neutrino mixing transformations.*

Next, we consider the action of the mixing generator on the vacuum $|0\rangle_{1,2}$. The flavor vacuum is defined as:

$$|0(\theta, t)\rangle_{e,\mu} \equiv G_{\theta}^{-1}(t) |0\rangle_{1,2} \quad (59)$$

The state for a mixed particle with definite flavor, spin and momentum is given by:

$$|\alpha_{\mathbf{k},e}^r\rangle \equiv \alpha_{\mathbf{k},e}^{r\dagger} |0(\theta, t)\rangle_{e,\mu} = G_{\theta}^{-1}(t) \alpha_{\mathbf{k},1}^{r\dagger} |0\rangle_{1,2} \quad . \quad (60)$$

The anti-commutators of the flavor ladder operators at different times are then computed and the following quantity is found to be constant in time:

$$|\{\alpha_{\mathbf{k},e}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(t')\}|^2 + |\{\alpha_{-\mathbf{k},e}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(t')\}|^2 + |\{\alpha_{\mathbf{k},\mu}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(t')\}|^2 + |\{\alpha_{-\mathbf{k},\mu}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(t')\}|^2 = 1 \quad (61)$$

The energy-momentum tensor for the fermion field is defined by $\mathcal{J}^{\mu\nu} \equiv i\bar{\psi}\gamma^{\nu}\partial_{\mu}\psi$ and the momentum operator is given by $P^j \equiv \int d^3\mathbf{x} \mathcal{J}^{0j}(\mathbf{x})$. For the free fields we have:

$$\mathbf{P}_i = \int d^3\mathbf{x} \psi_i^{\dagger}(\mathbf{x})(-i\nabla)\psi_i(\mathbf{x}) = \int d^3\mathbf{k} \sum_{r=1,2} \mathbf{k} \left(\alpha_{\mathbf{k},i}^{r\dagger} \alpha_{\mathbf{k},i}^r - \alpha_{-\mathbf{k},i}^{r\dagger} \alpha_{-\mathbf{k},i}^r \right), \quad (62)$$

for $i = 1, 2$. The momentum operator for mixed fields is given by:

$$\mathbf{P}_{\sigma}(\mathbf{t}) = \int d^3\mathbf{x} \psi_{\sigma}^{\dagger}(\mathbf{x})(-i\nabla)\psi_{\sigma}(\mathbf{x}) = \int d^3\mathbf{k} \sum_{r=1,2} \mathbf{k} \left(\alpha_{\mathbf{k},\sigma}^{r\dagger}(\mathbf{t}) \alpha_{\mathbf{k},\sigma}^r(\mathbf{t}) - \alpha_{-\mathbf{k},\sigma}^{r\dagger}(\mathbf{t}) \alpha_{-\mathbf{k},\sigma}^r(\mathbf{t}) \right), \quad (63)$$

for $\sigma = e, \mu$. We have $\mathbf{P}_{\sigma}(\mathbf{t}) = \mathbf{G}_{\theta}^{-1}(\mathbf{t}) \mathbf{P}_i \mathbf{G}_{\theta}(\mathbf{t})$ and the conservation of total momentum:

$$\mathbf{P}_e(\mathbf{t}) + \mathbf{P}_{\mu}(\mathbf{t}) = \mathbf{P}_1 + \mathbf{P}_2 \equiv \mathbf{P} \quad , \quad [\mathbf{P}, \mathbf{G}_{\theta}(\mathbf{t})] = \mathbf{0} \quad , \quad [\mathbf{P}, \mathbf{H}] = \mathbf{0}. \quad (64)$$

At time $t = 0$, the flavor state $|\alpha_{\mathbf{k},e}^r\rangle \equiv |\alpha_{\mathbf{k},e}^r(0)\rangle$ is an eigenstate of the momentum operator $\mathbf{P}_e(\mathbf{0})$:

$$\mathbf{P}_e(\mathbf{0}) |\alpha_{\mathbf{k},e}^r\rangle \equiv \mathbf{k} |\alpha_{\mathbf{k},e}^r\rangle. \quad (65)$$

At time $t \neq 0$ the expectation value of the momentum in such a state (normalized to initial time value) gives [25]:

$$\mathcal{P}_{\mathbf{k},\sigma}^e(t) \equiv \frac{\langle \alpha_{\mathbf{k},e}^r | \mathbf{P}_{\sigma}(\mathbf{t}) | \alpha_{\mathbf{k},e}^r \rangle}{\langle \alpha_{\mathbf{k},e}^r | \mathbf{P}_{\sigma}(\mathbf{0}) | \alpha_{\mathbf{k},e}^r \rangle} = |\{\alpha_{\mathbf{k},e}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(t')\}|^2 + |\{\alpha_{-\mathbf{k},e}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(t')\}|^2, \quad (66)$$

for $\sigma = e, \mu$, and the flavor vacuum expectation value of the momentum operator $\mathbf{P}_{\sigma}(\mathbf{t})$ vanishes at all time :

$${}_{e,\mu}\langle 0 | \mathbf{P}_{\sigma}(\mathbf{t}) | 0_{e,\mu} \rangle \quad , \quad \sigma = e, \mu. \quad (67)$$

The explicit calculation of the oscillating quantities $\mathcal{P}_{\mathbf{k},\sigma}^e(t)$ is found to be [25]:

$$\mathcal{P}_{\mathbf{k},e}^e(t) = 1 - \sin^2 2\theta \left[|U_{\mathbf{k}}| \sin^2 \frac{\omega_{k,2} - \omega_{k,1}}{2} t + |V_{\mathbf{k}}| \sin^2 \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right], \quad (68)$$

$$\mathcal{P}_{\mathbf{k},\mu}^e(t) = \sin^2 2\theta \left[|U_{\mathbf{k}}| \sin^2 \frac{\omega_{k,2} - \omega_{k,1}}{2} t + |V_{\mathbf{k}}| \sin^2 \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right]. \quad (69)$$

These formulas give the neutrino oscillation formulas in the QFT formalism and are different from the quantum mechanical approximation provided by the Pontecorvo formulas, to which they reduce in the large $|\mathbf{k}|$ limit, namely in $|U_{\mathbf{k}}| \rightarrow 1$ limit. In such a limit the Bogoliubov transformations and the algebra doubling are washed away. But then, as well, there is no NCSG construction anymore.

5. Conclusion

The study of the neutrino mixing carried on in the QFT formalism has shown that the mixing transformation is not just a simple rotation, but a transformation made of a rotation “nested” with a Bogoliubov transformation. This makes the mass vacuum and the flavor vacuum unitarily inequivalent. The Bogoliubov transformation is in turn the result of the algebra doubling $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, as we have discussed in Section 2. Then we have shown that Bogoliubov transformations are built in in the algebra doubling in the NCSG construction. In fact the doubling of the algebra is a characterizing feature at the basis of the NCSG construction. The neutrino mixing mechanism appears thus naturally incorporated in the NCSG construction. Since Majorana neutrinos appear in the NCSG construction, we consider their mixing and show how the neutrino oscillations emerge. We also show that in the limit of large $|\mathbf{k}|$ the quantum mechanical approximation of the Pontecorvo mixing is obtained and the Bogoliubov transformation reduces to the identity transformation. In such a limit the doubling of the algebra, characterizing feature of the NCSG construction, is also lost.

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