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Isochronous spacetimes and cosmologies

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Abstract. It is shown, in the context of general relativity, how to modify any cosmological model so that its local (in time) properties are unchanged, but its global time evolution is *isochronous*, i. e. completely periodic with a fixed period independent of the initial data. In this manner the Big Bang singularity might be avoided, even if this singularity is featured by the original model.

1. Introduction

The possibility has been recently demonstrated [1] to identify (nonrelativistic, Hamiltonian) many-body problems which feature an isochronous time evolution with an arbitrarily assigned period T yet mimic with an arbitrarily good approximation, or even exactly, the time evolution of any given many-body problem (belonging to a very large, quite realistic, class: arbitrary dimensions of ambient space, arbitrary number of particles with arbitrary masses, arbitrary interparticle forces) over a time $\tilde{T}$ which may also be arbitrarily large (but of course such that $\tilde{T} < T$). This finding is valid both in a classical and quantal context; hence—rather disturbingly—it applies to most of nonrelativistic physics. Purpose and scope of this presentation is to report the possibility to extend it to a general relativity context (classical, non quantal) as described in [2, 3]. In particular we show how to modify any cosmological model so that its local (in time) properties are unchanged, but its global time evolution is *isochronous* (and might avoid, in the past and future, the Big Bang, even if this singularity is featured by the original model).

2. Synthetic recall of previous results

The Hamiltonian characterizing the standard, overall translation-invariant, nonrelativistic N-body problem reads as follows (here, merely for notational simplicity, we consider unit-mass particles in a one-dimensional space environment, but this is not a significant limitation, see below):

$$H(\underline{p},\underline{q}) = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + V(\underline{q}), \qquad V(\underline{q}+a) = V(\underline{q}). \tag{1}$$

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Here of course $\underline{p}$ respectively $\underline{q}$ denote the N-vectors the components of which, p_n respectively q_n, are the standard canonical coordinates.

Let us now review some standard related developments, trivial as they are. We hereafter denote with P the total momentum, and with Q the (canonically-conjugate) centre-of-mass coordinate:

$$P \equiv \sum_{n=1}^{N} p_n, \qquad Q \equiv \frac{1}{N} \sum_{n=1}^{N} q_n \ . \tag{2}$$

Thanks to the translation invariance of the Hamiltonian the total momentum Poisson-commutes with the Hamiltonian, $[H, P] = 0$, where—here and hereafter—the Poisson bracket of two generic functions of the canonical variables is defined as usual,

$$[F, G] \equiv \sum_{n=1}^{N} \left[\frac{\partial F}{\partial p_n} \frac{\partial G}{\partial q_n} - \frac{\partial F}{\partial q_n} \frac{\partial G}{\partial p_n} \right] \ ; \tag{3}$$

and let us recall that the evolution of any function $F(p, q)$ of the canonical coordinates is determined by the equation $F' = [F, H]$, where the appended prime denotes differentiation with respect to the "timelike" variable corresponding to the evolution induced by the Hamiltonian H.

It is now convenient to introduce the "relative coordinates" and the "relative momenta" via the standard definitions

$$x_n \equiv q_n - Q; \qquad y_n \equiv p_n - \frac{P}{N} \ . \tag{4}$$

Note that these are not canonically conjugated quantities, since $[x_n, y_n] = \delta_{nm} - 1/N$, and they are not independent since obviously their sum vanishes: $\sum_{n=1}^{N} x_n = 0$, $\sum_{n=1}^{N} y_n = 0$.

It is moreover convenient to introduce the "relative-motion" Hamiltonian $h(\underline{y}, \underline{x})$ via the formula

$$h(y, x) \equiv \frac{1}{2} \sum_{n=1}^{N} y_n^2 + V(\underline{x}) = \frac{1}{4N} \sum_{n,m=1}^{N} (p_n - p_m)^2 + V(\underline{q}) \ , \tag{5}$$

implying

$$H(\underline{p}, \underline{q}) = \frac{P^2}{2\ N} + h(\underline{y}, \underline{x}) \ . \tag{6}$$

Note that this definition of the relative-motion Hamiltonian $h(\underline{y}, \underline{x})$ entails that it Poisson-commutes with both P and Q: $[P, h] = [Q, h] = 0$.

For completeness and future reference let us also display the equations of motion implied by the original Hamiltonian $H(p, q)$:

$$q_n' = p_n \ ; \quad p_n' = -\frac{\partial V(q)}{\partial q_n} = q_n'' \ , \tag{7}$$

where (for reasons that will be clear below) we denote as τ the independent variable corresponding to this Hamiltonian flow and with appended primes the differentiations with respect to this variable:

$$q_n \equiv q_n(\tau); \qquad p_n \equiv p_n(\tau); \qquad q_n' = \frac{dq_n}{d\tau}; \qquad p_n' = \frac{dp_n}{d\tau} \ . \tag{8}$$

Hence $Q' = P/N$ and $P' = 0$, yielding

$$Q(\tau) = Q(0) + \frac{P(0)}{N}\tau; \qquad P(\tau) = P(0) \ , \tag{9}$$

as well as

$$x'_n = y_n = \frac{\partial h(y,x)}{\partial y_n}, \qquad y'_n = -\frac{\partial V(x)}{\partial x_n} = -\frac{\partial h(y,x)}{\partial x_n} \ . \tag{10}$$

Note that these equations, (10), have the standard Hamiltonian form even though, as mentioned above, the coordinates y_n and x_n are not quite canonically conjugated variables.

This ends our review of quite standard results for the classical nonrelativistic N-body problem. Let us also emphasize that, above and below, the restriction to unit-mass particles, and to one-dimensional space, is merely for simplicity: generalizations—also of the following results—to the more general case with different masses and arbitrary space dimensions is quite elementary, essentially trivial. As for the number of particles N, it is an arbitrary positive integer: interesting cases are $N = 2$ (the generally solvable two-body problem), $N \approx 10^{25}$ (the standard macroscopic system, where statistical mechanics and thermodynamics rules), $N \approx 10^{85}$ (maybe the entire Universe, the topic of cosmology; which however should be treated in a general relativity context, see below).

3. An isochronous Hamiltonian

A "Ω-modified" isochronous Hamiltonian $\tilde{H}(p,q;\Omega)$ is now defined [1] by the formula

$$\tilde{H}(\underline{p},\underline{q};\Omega) \equiv \frac{1}{2}\left[\left(P + \frac{h(\underline{y},\underline{x})}{d}\right)^2 + \Omega^2\ Q^2\right] \ , \tag{11}$$

where d is an irrelevant arbitrary constant (introduced for dimensional reasons: it has the dimensions of a momentum[1]) and Ω is hereafter a positive constant the choice of which will be our privilege. Let us emphasize that hereafter the evolution of the various quantities is that caused by this new Ω-modified Hamiltonian; the corresponding independent variable is hereafter denoted as t and interpreted as "time", and differentiations with respect to this variable will be denoted, as usual, by superimposed dots; and of course the time evolution of any function $F \equiv F(\underline{p},\underline{q})$ of the canonical variables will be determined by the standard equation $\dot{F} = [\tilde{H}, F]$.

The solution of the time-evolution yielded by the isochronous Hamiltonian $\tilde{H}(p,q;\Omega)$ is [1]

$$Q(t) = Q(0)\cos(\Omega t) + \dot{Q}(0)\frac{\sin(\Omega t)}{\Omega} = C\,d\,\frac{\sin\left(\Omega(t-\bar{t})\right)}{\Omega} \ , \tag{12a}$$

$$P(t) = P(0)\cos(\Omega t) + \dot{P}(0)\frac{\sin(\Omega t)}{\Omega} + \frac{h(y(0),x(0))}{d}\left[\cos(\Omega t) - 1\right] \ , \tag{12b}$$

where $C = \sqrt{2\tilde{H}(\underline{p},\underline{q};\omega)}/d$ is a dimensionless constant, and

$$\tilde{x}_n(t) = x_n(\tau); \qquad \tilde{y}_n(t) = y_n(\tau) \ , \tag{12c}$$

where (changing for convenience notation) we now denote as $\tilde{x}_n \equiv \tilde{x}_n(t)$ and $\tilde{y}_n \equiv \tilde{y}_n(t)$ the dependent variables whose time-evolution is determined by the Ω-modified Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ and as $x_n \equiv x_n(\tau)$ and $y_n \equiv y_n(\tau)$ the dependent variables whose τ-evolution is determined by the original, unmodified Hamiltonian $H(\underline{p},\underline{q})$. And here (most importantly)

$$\tau \equiv \tau(t) = \frac{A\ \sin(\Omega t) + B\ \left[1 - \cos(\Omega t)\right]}{\Omega} = \frac{Q(t)}{d} \ , \tag{12d}$$

where the constants A and B are given by simple explicit formulas in terms of the initial position $Q(0)$ and velocity $P(0)$ of the centre-of-mass of the system and the value of the Hamiltonian

[1] Note that (5) implies that $h(y,x)$ has the dimension of a squared momentum.

$\tilde{H}(\underline{p}, \underline{q}; \Omega)$ (which is of course a constant of motion) [1]. The crucial observation is that $\tau(t)$—hence the entire solution, see (12)—is a periodic function of t with period $T = 2\pi/\Omega$.

The solution formulas displayed above (see in particular (12c) and (12d)) show that the dynamical evolution implied by the *isochronous* Hamiltonian (11) features exactly the *same* trajectories in the (phase) space of the coordinates x_n and y_n as those yielded by the standard N-body Hamiltonian (1); but in the case of the standard N-body Hamiltonian (1) the corresponding evolution corresponds to a uniform travel along those trajectories; while in the case of the *isochronous* Hamiltonian (11) the evolution of the system corresponds to an oscillatory travel—periodic with the fixed period T—along those same trajectories, only a portion of which gets therefore explored. It also entails that the dynamics yielded by the *isochronous* Hamiltonian (11) does not differ—on a time scale much shorter than the period T—from that yielded by the original Hamiltonian (1) (up to a *constant* shift and rescaling of time). Indeed clearly $\tau \equiv \tau(t)$ on a sufficiently short time scale varies linearly in t, since in the neighborhood of any time $\bar{t}$—except when $\dot{\tau}\,(\bar{t})$ vanishes—clearly

$$\tau(t) = \bar{C} + \bar{D}\ \ (t - \bar{t}\,) + O\left[\left(\frac{t - \bar{t}}{T}\right)^2\right]\ , \tag{13a}$$

$$\bar{C} = \frac{A\ \sin(\Omega\bar{t}) + B\ \ [1 - \cos(\Omega\bar{t})]}{\Omega}\ , \tag{13b}$$

$$\bar{D} = A\ \cos(\Omega\bar{t}) + B\ \sin(\Omega\bar{t})\ . \tag{13c}$$

Remark 3.1. A different modified Hamiltonian can be introduced which also yields an *isochronous* evolution with the arbitrary period T but reproduces the dynamics of the original, standard N-body Hamiltonian (1) over an arbitrary time $\tilde{T}$ (of course such that $\tilde{T} < T$) *exactly* (rather than with arbitrary accuracy). And the property of *isochrony* remains true in a *quantal* context. [1] □

4. Extension to general relativity

The following presentation is terse, focusing on the main ideas.

The standard approach. The following space-time metric is generally introduced as appropriate to describe a homogeneous and isotropic Universe:

$$ds^2 = dt^2 - a(t)^2\ d\vec{x}^2\ . \tag{14}$$

Our approach. We generalize the class of metrics to be considered, writing instead

$$ds^2 = b(t)^2\ dt^2 - a(t)^2\ d\vec{x}^2\ , \tag{15}$$

where $b\,(t)^2$ is commonly termed the "lapse" function.

Standard objection in the general relativity context. This new metric is physically equivalent to that written above, since it can be reduced to it by a change of the time variable from t to τ with

$$d\tau = b(t)dt; \qquad \tau(t) = \int^t dt' b(t')\ , \tag{16}$$

whereby the metric (15) becomes

$$ds^2 = d\tau^2 - \alpha(\tau)^2\ d\vec{x}^2\ , \tag{17}$$

of course with $a(t) = \alpha(\tau) = \alpha(\tau(t))$. Indeed, such a redefinition of the time-variable has no physical meaning in general relativity, where a change of coordinates (in this case, from t to τ)

is *physically irrelevant*: what is relevant is the geometry of space-time, not the variables used to describe it.

Our rejoinder. Yes, in the general relativity context this is indeed the case, but only if $b(t)$ is a one-to-one invertible function (in which case the two metrics (14) and (15) are related to each other by a diffeomorphism). However, this is certainly *not* the case if $b(t)$ is a periodic function (with arbitrarily assigned period T and vanishing mean value):

$$b(t+T) = b(t) \; ; \qquad B\,(t) \equiv \int_0^t b\left(t'\right) dt' \; , \quad B\,(t+T) = B\,(t) \; . \tag{18}$$

Then the two metrics (14) and (15) are indeed *locally equivalent* (in time), but they are *not globally equivalent* (in time). And clearly with such an assignment one arrives at an *isochronous* cosmology.

This is essentially our argument: given an arbitrary cosmology, one can construct another one (indeed an infinity of other ones) which is *locally equivalent* (in time) but *not globally equivalent* (in time) to the previous one: for instance one, but not the other, might feature a Bing Bang. Clearly the corresponding space-times would be physically different; but they would not be distinguishable by any (local in time) measurement... And the new cosmology might be *isochronous*... i. e., entail a time evolution which is periodic with an arbitrary, *a priori* assigned, period T.

Let us now be a bit more specific. As usual in cosmological investigations based on the Einstein equations, we assume that the energy-momentum tensor of the Universe is that of an ideal fluid with a 4-velocity given by $U_\mu = |b(t)| \; \delta_{\mu 0}$, so that the ideal fluid has zero spatial velocity and the frame of reference is co-moving, while the zero component of the 4-velocity is always nonnegative (as it should be). The Einstein equations for the metric (15) are then drastically simplified and read

$$3 \left[\frac{\dot{a}(t)}{a(t)}\right]^2 = k \; b(t)^2 \; \rho(t) \; , \tag{19a}$$

$$\dot{\rho}(t) + 3 \left[\frac{\dot{a}(t)}{a(t)}\right] \; [\rho(t) + P(t)] = 0 \; . \tag{19b}$$

Here and hereafter superimposed dots denote differentiations with respect to the time t; ρ and P are respectively the energy density and pressure of the universe; $k = 8\pi G/c^4$; G is the gravitational constant; c the speed of light.

This system of two ODEs must be complemented by an equation of state relating the pressure P of the universe to its energy density ρ: we shall here use for simplicity the simple ideal fluid relation $P = \omega\rho$, with constant ω. But note that, after this assignment, this system of two ODEs is not quite closed: it is still possible to assign, essentially arbitrarily, the function $b(t)$. Of course this arbitrariness corresponds to the freedom in general relativity to reparametrize time; but—as discussed above—it goes beyond this if $b(t)$ is *not* invertible, hence if the relevant reparametrization is *not* a global diffeomorphism.

If we consider a perfect fluid with constant equation of state parameter $\omega > -1$, it follows from the system (19) that the energy density of the fluid is

$$\rho(t) = \rho\,(0) \left[\frac{a(0)}{a(t)}\right]^{3(\omega+1)} \; , \quad \text{if} \quad \omega > -1 \; , \tag{20}$$

while for $\omega = -1$ one has a constant energy density,

$$\rho = \Lambda \quad \text{if} \quad \omega = -1 \; . \tag{21}$$

It is now convenient [2, 3] to introduce a new variable τ by setting (see (18))

$$\tau(t) \equiv B\,(t)\ , \tag{22}$$

and new auxiliary functions $\alpha\,(\tau)\,,\ r\,(\tau)\,,\ p\,(\tau)$ by setting

$$a(t) \equiv \alpha\,(\tau(t))\ ,\quad \rho(t) \equiv r\,(\tau(t))\ ,\quad P(t) \equiv p\,(\tau(t))\ . \tag{23}$$

Then the functions $a(t)$ and $\rho(t)$ are solutions of (19) provided $\alpha(\tau)$ and $r(\tau)$ are solutions of the following system:

$$3\left[\frac{\alpha'(\tau)}{\alpha(\tau)}\right]^2 = k\ r(\tau)\ , \tag{24a}$$

$$r'(\tau) + 3\left[\frac{\alpha'(\tau)}{\alpha(\tau)}\right][r(\tau) + p(\tau)] = 0\ . \tag{24b}$$

Here the prime denotes differentiation with respect to τ, and $p(\tau)$ and $r(\tau)$ satisfy the same equation of state $p(\tau) = \omega\, r(\tau)$ as $P(t)$ and $\rho(t)$. And since the definition of $\tau(t)$ (see (22) with (18)) implies that this function is periodic with period T,

$$\tau(t+T) = \tau(t)\ , \tag{25}$$

the same property is inherited via the definitions (23) by the physical quantities $a(t)$, $\rho(t)$ and $P(t)$:

$$a(t+T) = a(t)\ ,\quad \rho(t+T) = \rho(t)\ ,\quad P(t+T) = P(t)\ . \tag{26}$$

This property holds for the solutions of the system (19) characterized by any initial conditions: the physical quantities $a(t)$, $\rho(t)$ and $P(t)$ are therefore *isochronous*. Moreover, since $b(t)$ can be assigned arbitrarily, the period T of these *isochronous* solutions is a parameter which can be freely assigned.

It is immediate to identify the system (24) with the standard Friedman-Robertson-Walker (FRW) system characterized by the metric (17) with the energy momentum tensor of a perfect fluid in co-moving coordinates (see [4] for a review on FRW equations), while the system (19) is a generalized FRW system corresponding to the metric (15). But let us re-emphasize that the new variable τ cannot be given globally the significance of "time", hence the fact that it might change sign—indeed, it certainly does so, since it depends periodically on the time t—has no unphysical connotation.

Let us now show two explicit examples of *isochronous* cosmological solutions of Einstein's equations obtained in this manner. In both cases we assign for simplicity the function $b(t)$ and $B\,(t)$ as follows:

$$b(t) \equiv \cos\,(\Omega t)\ ,\quad B\,(t) = \frac{\sin\,(\Omega t)}{\Omega}\ ,\quad \Omega = \frac{2\pi}{T}\ . \tag{27}$$

As first example, consider a universe filled with a dark energy fluid with constant density $\rho = \Lambda$ and equation of state parameter $\omega = -1$. Assuming that the metric tensor is given by (15), the system (19) has the following solution:

$$a(t) = a_0 \exp\left[\sqrt{\frac{k\Lambda}{3}}\ \frac{\sin\,(\Omega t)}{\Omega}\right]\ ,\quad a_0 \equiv a\,(0)\ . \tag{28}$$

Note that at the times $t_n = (1/2+n)\pi/\Omega$ with n integer the scale factor $a(t)$ reaches its maximum (for n even) and minimum (for n odd) values $a_\pm \equiv a_0 \exp\left[\pm\sqrt{k\Lambda/3\Omega^2}\right]$. Also note that, even if the metric (15) is degenerate since $b(t_n) = 0$, all physical quantities such as the energy density

and pressure, the Ricci scalar curvature R, etc., are *not* singular since they can be expressed as functions of the scale factor $a(t)$ (e. g. $R = k[\rho(a) - 3P(a)]$), which is finite for all time, see (28)). Therefore t_n is a fictitious singularity corresponding to the time when the universe passes from an expanding to a contracting epoch and viceversa.

As second example, consider the case of a perfect fluid with equation of state parameter $\omega > -1$. Then the scale factor is

$$a(t) = a_0 \left[1 + (1+\omega)\sqrt{\frac{3k\rho_0}{4}}\frac{\sin(\Omega t)}{\Omega}\right]^{\frac{2}{3(1+\omega)}} , \quad a_0 \equiv a(0) . \tag{29}$$

The maxima and minima of the scale factor are now again reached at $t_n = (1/2 + n)\pi/\Omega$ and have the values $a_\pm = a_0\left[1 \pm \sqrt{3k\rho_0}\frac{1+\omega}{2\Omega}\right]^{\frac{2}{3(1+\omega)}}$. Also in this case t_n corresponds to the transition from the expanding to the contracting phases and viceversa and to a fictitious singularity of the metric, since $b(t_n) = 0$ but the scale factor $a(t)$ and the relevant physical quantities may remain finite for all time. Indeed for any assignment of the parameter Ω such that

$$\Omega > \frac{\sqrt{3k\rho_0}}{2}\ (1+\omega) , \tag{30}$$

the scale factor $a(t)$ is positive for all time (see (29)) and the big bang singularity of the FRW model is avoided. Hereafter we assume that this restriction, (30), is always enforced. Of course with a different assignment of $b(t)$ one obtains different solutions, all of which yield the same dynamics locally (in time); but possibly quite different dynamics globally (in time).

Furthermore, it has been shown that, by choosing $b(t)$ so that $a(t) = \alpha(\tau(t)) > 0$ for any t, such *isochronous* metrics can be manufactured to be geodesically complete [2] and therefore singularity-free [2], so that the geodesic motion as well as all physical quantities described by scalar invariants are always well defined [2] and the Big Bang singularity may be avoided. This implies that these *isochronous* metrics may describe a *singularity-free* universe which—while featuring a time evolution which reproduces identically (up to diffeomorphic time reparameterization; *locally* in time, except at a discrete set of instants t_n) that of the FRW metric (24)—features an expansion which stops at some instant, to be followed by a period of contraction, until this phase of evolution stops and is again followed by an expanding phase, this pattern being repeated *ad infinitum*.

Let us also note that this class of metrics realize *de facto* the reversal of time's arrow, as discussed for instance in [5].

It is also evident that these *isochronous* metrics must be degenerate at a discrete set of instants t_n when the time reversals occur, say from expansion to contraction and viceversa; a phenomenology whose inclusion in general relativity might be considered problematic, because at these instants t_n—when the metric is degenerate—the "equivalence principle" corresponding to the requirement that the metric tensor have a Minkowskian signature is violated. On the other hand a *physically unobservable* violation of a "principle" can be hardly considered *physically relevant*; and the *isochronous* metrics are solutions of a version of general relativity whose dynamics, while still governed by Einstein's equations[3], does allow the violation of the equivalence principle at a discrete set of hypersurfaces.

Let us note that the consideration in the context of general relativity of degenerate metrics is not new. For instance, they were already introduced in [7, 8] and subsequently investigated in a series of papers [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25], in order to

[2] A spacetime is singularity-free if it is geodesically complete, i.e. if its geodesics can be always past- and future-extended [4].

[3] Einstein's equations by itself do not determine the signature of spacetime, see the discussion in [6, 7].

study signature-changing spacetimes which are a classical realization of the change of signature in quantum cosmology conjectured by Hartle and Hawking [26] (see also [27, 28, 29]), which has philosophical implications on the origin of the universe [30].

Let us also mention that in [3] a different realization of *isochronous* cosmologies has been considered: via *nondegenerate* metrics featuring a jump in their first derivatives at the inversion times t_n, which then implies a distributional contribution in the stress-energy tensor at t_n. These metrics are diffeomorphic to the continuous and degenerate *isochronous* metrics everywhere except on the inversion hypersurfaces $t = t_n$; hence they describe the same physics except at the infinite set of discrete times t_n; and clearly they may also be manufactured so as to agree with all cosmological observations (*locally* in time). Since these two realizations of *isochronous* cosmologies are *locally* but *not globally* diffeomorphic, they correspond to different spacetimes and may be considered to emerge from different theories: the *nondegenerate* ones are generalized (in the sense of distributions) solutions of Einsteinian general relativity (including universal validity of the equivalence principle, but allowing jumps—whose physical significance and justification is moot—at the discrete times t_n); the *degenerate* ones feature no mathematical or physical pathologies but fail to satisfy the "equivalence principle" at the discrete times t_n.

Finally let us emphasize that the argument based on the transition in the reduced Einstein equations from the time t to the auxiliary variable τ—illustrated above when the right-hand side is characterized by the simple ideal fluid relation $P = \omega\rho$—remains valid in the more general case in which the system (24) corresponds to a universe filled with a mixture of ultra-relativistic and nonrelativistic perfect fluids plus a cosmological constant and an inflaton scalar field (the currently most popular candidate to describe our Universe); then the solution of the (so-modified) system (24) gives, for any time evolution over a limited time period such that $|t - \bar{t}| \ll T$, exactly the "right" sequence of inflation / radiation domination / matter domination / late time acceleration epochs characterizing the Λ-CDM universe. Note that it shall then also display any phenomenology interpretable as observable consequence of a past Big Bang singularity implied by this evolution; in spite of the fact that, as discussed above, a suitable assignment of Ω hence of the isochrony period T entails that the evolution of the so-modified system (24) would never feature such an event.

5. Conclusions

We have shown that for any given FRW cosmology described by the metric (14) it is possible to find an infinite number of *isochronous cosmologies* described by the periodic metric (15) with (18), and that one can always assign the arbitrary function $b(t)$ so that the new metric (15) is *singularity-free* and yields an *isochronous* time evolution which coincides—locally in time—with that yielded by the FRW metric (14). One might wonder how general this finding is. To answer this question, we note that the mechanism to manufacture (theoretically) these *isochronous cosmologies* can be applied to any synchronous metric, i.e. to any metric with $g_{00} = 1$ and $g_{0j} = 0$. Therefore, as long as one can perform a coordinate transformation that makes a solution of the Einstein equations *synchronous*, one can also find an *isochronous* solution of the relevant field equations which is locally—but only locally (in time)—physically identical to the original solution up to a diffeomorphic reparametrization of the time variable. Since it is possible to write most metrics in *synchronous* form by a diffeomorphic change of coordinates, this makes our finding quite general.

Let us also note (see *Remark 3.1* above) that, while the results that are valid for the quite general (nonrelativistic) many-body problems mentioned at the beginning of this presentation can be extended from a classical to a quantal context in a (theoretically and mathematically) rigorous manner, an analogous quantal extension of the approach described above is unfeasible as long as there is no mathematically rigorous theory encompassing general relativity and quantization. And let us conclude by emphasizing that we are *not* arguing that our Universe

evolves periodically indeed *isochronously*, nor do we wish to enter the related *philosophical* issues (not our cup of tea!); we merely point out a (disturbing indeed unpleasant!) fact: in the context of theoretical and mathematical physics such a possibility does not seem to be excluded.

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