

Wave propagation in the presence of a dielectric slab: The paraxial approximation

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Abstract. Light propagation processes in a homogeneous dielectric slab are considered from the point of view of the optical analog of the Schrödinger equation. The reflection and transmission coefficients and longitudinal shifts of the wavefront are determined by considering the analytical properties of the external field amplitudes as functions of the wave number. An analytic approach to obtain the resonances in the “long lifetime” limit is also presented.

1. Introduction

The analogies between light propagation processes and mechanical systems have been a matter of intense study over time. Many problems of geometrical optics in graded-index media can be easily solved [1] and a Hamiltonian (phase space) description of geometrical optics can be, as well, developed [2] by taking full advantage of the similarities between ray description of light and the Newtonian dynamics. The resemblance extends also to the regime of wave optics and quantum mechanical descriptions due to the formal equivalence between the scalar Helmholtz equation describing the propagation, in space, of light beams in the paraxial regime and the Schrödinger equation governing the time-evolution of a point particle in an interaction potential [3–6]. This equivalence has been used as a tool to simulate and test multiple processes involving quantum mechanical particles, such as Rabi oscillations, population transfer and atom optics [6–8], by means of optical-wave systems. Particular interest has been paid to the theoretical and experimental study of the transmission times in barrier and well potentials [9–11]. In this context, the similarities between the Helmholtz and Schrödinger equations have been convenient to design experiments that allow to measure the negative tunneling times by using optical systems in the microwave regime [12–15]. In the same way, many quantum formalisms have been also adapted for the description of light propagation in free space as well as in the presence of optical devices [3, 5–7, 16–19]. In particular, quantum-optical analogies have been useful to determine the bound modal fields propagating in waveguides and optical fibers [6, 7, 20] by modeling these devices as short range interaction potentials. Yet, it is well known that, under some circumstances, this kind of potentials may host bound as well as scattering and resonant states. Very recently the spectrum properties of particles subject to one dimensional short range potentials have been considered [21–29]. In those works the bound and resonant states, as well as the corresponding reflection and transmission coefficients and delay times can be obtained by inspecting the analytical properties of the reflection and transmission amplitudes as functions of the moment k . In this work we present a description of wave propagation through



a dielectric homogeneous slab in the same scheme. Thus, the spectral properties of the system will be stated and some light guiding and scattering processes will be described. In Section 2 the paraxial approximation for linear inhomogeneous media is derived starting from the Maxwell's equations. In Section 3 the scattering of a light beam by a dielectric homogeneous slab is considered. The reflection and transmission coefficients are determined as well as the longitudinal shifts by means of the stationary phase approximation. Section 4 is devoted to determine the bound and resonant modes as the solutions to the paraxial Helmholtz equation, corresponding to complex eigenvalues, fulfilling the purely outgoing wave condition. Some concluding remarks are presented in Section 5.

2. The paraxial wave equation

We depart from the Maxwell equations without sources for wave propagation in the presence of an inhomogeneous dielectric material

$$\nabla \cdot \mathbf{D}(\mathbf{q}, t) = 0, \quad \nabla \cdot \mathbf{B}(\mathbf{q}, t) = 0, \quad (1)$$

$$\frac{\partial}{\partial t} \mathbf{D}(\mathbf{q}, t) = \nabla \times \mathbf{H}(\mathbf{q}, t), \quad -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{q}, t) = \nabla \times \mathbf{E}(\mathbf{q}, t). \quad (2)$$

For linear media the constitutive relations read

$$\mathbf{D}(\mathbf{q}, t) = \epsilon_0 \epsilon(\mathbf{q}) \mathbf{E}(\mathbf{q}, t), \quad \mathbf{B}(\mathbf{q}, t) = \mu_0 \mathbf{H}(\mathbf{q}, t), \quad (3)$$

where $\epsilon(\mathbf{q})$ is the relative permittivity connected to the refractive index through $n^2(\mathbf{q}) = \epsilon(\mathbf{q})$. We are assuming that the magnetic polarizability and the dispersive properties of the medium are negligible. For one monochromatic component of the electromagnetic field the corresponding wave equations now contain terms bearing information about the inhomogeneity of the optical medium

$$\nabla^2 \mathbf{E}(\mathbf{q}) + k_0^2 n^2 \mathbf{E}(\mathbf{q}) - \nabla \ln \epsilon(\mathbf{q}) \cdot \mathbf{E}(\mathbf{q}) = 0, \quad (4)$$

$$\nabla^2 \mathbf{H}(\mathbf{q}) + k_0^2 n^2 \mathbf{H}(\mathbf{q}) - \nabla \ln \epsilon(\mathbf{q}) \times (\nabla \times \mathbf{H}(\mathbf{q})) = 0, \quad (5)$$

with $k_0 = \omega/c$ the wave number in vacuum. The solution of this problem is, in general, non trivial since the presence of these terms implies a coupling between the components of each field. Though, a simple inspection of equation (4) allows us to observe that it is possible to reduce it to a scalar wave equation for a representative component of the electric field whenever

$$\nabla \epsilon(\mathbf{q}) \cdot \mathbf{E} = 0. \quad (6)$$

In particular, we are interested in wave packets consisting of superpositions of transversal electric waves traveling in a guiding stratified medium, for which the refractive index depends only on one transversal coordinate, say x . If we choose the Z -axis as the optical axis, then the incident plane corresponds to the XZ -plane, meaning that the transversal electric field must have the form $\mathbf{E}(\mathbf{q}) = E(\mathbf{q}) \mathbf{e}_y$, with \mathbf{e}_y a unit vector in the Y -direction. In this case, the condition (6) is fulfilled automatically. The Maxwell equations (1-2) lead to

$$E = E(x, z), \quad H_x(x, z) = \frac{i}{\mu_0 \omega} \frac{\partial}{\partial z} E(x, z), \quad H_y(x, z) = 0, \quad H_z(x, z) = -\frac{i}{\mu_0 \omega} \frac{\partial}{\partial x} E(x, z) \quad (7)$$

with $E(x, z)$ the electric field amplitude that must be determined by solving the scalar Helmholtz equation

$$\frac{\partial^2}{\partial x^2} E(x, z) + \frac{\partial^2}{\partial z^2} E(x, z) + k_0^2 n^2(x) E(x, z) = 0. \quad (8)$$

In the paraxial regime, we expect a nearly harmonic behavior in the longitudinal coordinate. Then, we may write

$$E(x, z) = F(x, z) e^{ik_0 n_0 z} \quad (9)$$

where $F(x, z)$ must be a function almost constant in z and n_0 is a reference refractive index. Then, it is natural to make the approximation [4–6, 30]

$$\left| \frac{\partial^2 F}{\partial z^2} \right| \ll k_0 \left| \frac{\partial F}{\partial z} \right|, \quad (10)$$

in which case equation (8) is reduced to the time-dependent Schrödinger-type equation

$$-\frac{1}{2k_0^2 n_0} \frac{\partial^2 F}{\partial x^2} - \frac{n^2 - n_0^2}{2n_0} F = \frac{i}{k_0} \frac{\partial F}{\partial z}. \quad (11)$$

It is well known that the paraxial approximation (10) is valid in the weakly guiding regime, where the refractive index is nearly constant [6], *i.e.*, $n = n_0 + \Delta n$, $\Delta n \ll 1$, $(\Delta n)^2 \approx 0$. Thus, to first order in Δn

$$\frac{n^2 - n_0^2}{2n_0} \approx \Delta n = n - n_0. \quad (12)$$

Furthermore, since n does not depend on the longitudinal coordinate, the amplitude F can be written in the form

$$F(x, z) = e^{-i\beta k_0 z} \psi(x), \quad (13)$$

where $\psi(x)$ fulfills the stationary Schrödinger-type equation

$$-\frac{1}{2k_0^2 n_0} \frac{d^2 \psi}{dx^2} - n\psi = \varepsilon \psi, \quad \varepsilon = \beta - n_0, \quad (14)$$

known as the paraxial Helmholtz equation, for a point particle of mass n_0 exposed to an interaction potential $-n(x)$. Observe that the inverse of the wave number takes the place of the Planck's constant, implying that the “semiclassical limit”, which in the electromagnetic case correspond to the ray description of light, appears for very small wavelengths.

Introducing the operators [3]

$$\hat{x} = x, \quad \hat{p}_x = -\frac{i}{k_0} \frac{\partial}{\partial x}, \quad \hat{p}_z = -\frac{i}{k_0} \frac{\partial}{\partial z}, \quad \hat{H} = -\frac{1}{2n_0} \hat{p}_x^2 - n(\hat{x}), \quad (15)$$

we are led to the eigenvalue problem

$$\hat{H} \mathbf{E} = -\hat{p}_z \mathbf{E} = \varepsilon \mathbf{E}. \quad (16)$$

The parameter ε is then related to the z -component of the generalized momentum \mathbf{p} (see, *e.g.* [2, 3]) defining the slope of the light beam

$$\varepsilon = -p_z = -n(x) \cos \theta(x) \quad (17)$$

with θ the angle between the propagation direction and the optical axis. The eigenvalue equation (14) then defines the modal fields propagating inside and through the optical medium as the eigenstates of \hat{H} (electromagnetic modes with well defined values of θ). It is worthwhile to note that, depending on the profile $n(x)$, the spectrum may include a set of isolated points (bound modes) and a continuous interval (scattering modes) with, possibly, an internal discrete structure (resonant modes) [21], in a complete analogy to the Schrödinger problem.

3. Wave propagation through a dielectric homogeneous slab: the scattering modes

In order to present some of the consequences of the analogies between the Schrödinger and the paraxial Helmholtz equations we will consider the simple case of electromagnetic propagation through a finite, dielectric homogeneous material in this framework. This problem is equivalent to the square well potential in Quantum Mechanics which has been studied in diverse contexts and has been used as a basis of models of a number of interactions (see *e.g.* [31–33]). Let us rewrite (14) in the form

$$-\frac{1}{2k_0^2 n_0} \frac{d^2 \psi}{dx^2} + (1 - n)\psi = (\varepsilon + 1)\psi. \quad (18)$$

We are interested on wave propagation through a finite medium characterized by a real, constant refractive index n extending on the x -axis from $x = -a$ to $x = a$ tagged as region *II* (see Figure 1). Since n_0 is the reference refractive index characteristic of the optical

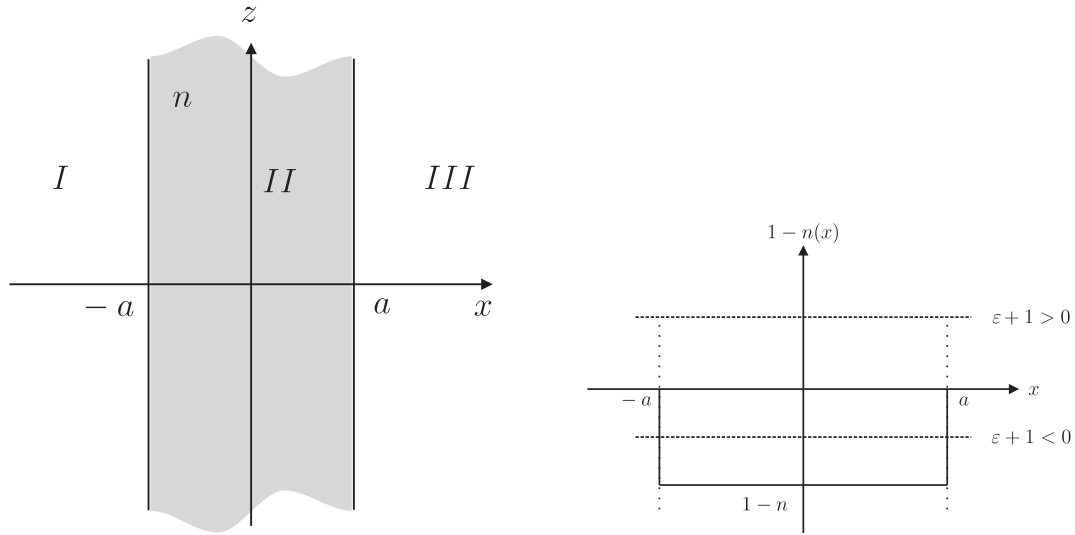


Figure 1. Left: Finite medium with constant refractive index n of width $2a$. Right: The system is analogous to a particle in a square well potential in Quantum Mechanics.

medium, equation (14) is valid within each region. However, the boundary conditions of the electromagnetic field state that the tangential components of \mathbf{E} and \mathbf{H} must be continuous at any interface separating two regions. This indicates that $E(x, z)$ and $H_z(x, z) \propto \partial E(x, z)/\partial x$, and thus the function ψ and its first derivative, must be continuous at $x = \pm a$. Then, for the internal and external fields we have respectively the equations

$$-\frac{1}{2k_0^2} \frac{d^2 \psi_i}{dx^2} = (\varepsilon + 1)\psi_i, \quad i = I, III \quad (19)$$

$$-\frac{1}{2k_0^2 n_0} \frac{d^2 \psi_{II}}{dx^2} + (1 - n)\psi_{II} = (\varepsilon + 1)\psi_{II}. \quad (20)$$

The electric field amplitude has the form

$$\psi(x) = \begin{cases} I e^{ikx} + A_1(k) e^{-ikx} & x < -a \\ A_2(k) \cos qx + B_2(k) \sin qx & -a \leq x \leq a \\ A_3(k) e^{-ikx} + B_3(k) e^{ikx} & x > a \end{cases} \quad (21)$$

with

$$k^2 = 2k_0^2(\varepsilon + 1), \quad q^2 = 2k_0^2n(\varepsilon + n), \quad (22)$$

and A_1, A_2, B_2, A_3, B_3 – functions of the wave number k to be determined by applying the proper set of boundary conditions.

3.1. The reflection and transmission coefficients

For positive values of the parameter $\varepsilon + 1$, we have real wave numbers $k = k_0\sqrt{2(\varepsilon + 1)}$. The external field amplitude ψ_I and ψ_{III} remain finite as $|x| \rightarrow \infty$. In those regions $\varepsilon = -p_z = -\cos\theta$, meaning that ε ranges in the interval $[-1, 0]$. Solution (21) describes the scattering of plane waves by the slab. If we assume that there exists a source of electromagnetic radiation at the left of the optical medium pumping waves into the interaction zone, then $I = 1$ stand for the incident amplitude, A_1 and B_3 for the reflection and transmission ones, and the coefficient A_3 is set to zero as we expect no plane waves propagating to the left in region *III*. Imposing the continuity conditions to ψ and ψ' we find

$$A_1(k) = i \frac{q^2 - k^2}{2\Delta(k)} e^{-2ika} \sin 2qa, \quad B_3(k) = \frac{kq}{\Delta(k)} e^{-2ika}, \quad (23)$$

$$A_2(k) = \frac{k}{\Delta(k)} e^{-ika} (q \cos qa - ik \sin qa), \quad B_2(k) = \frac{k}{\Delta(k)} e^{-ika} (k \cos qa - iq \sin qa), \quad (24)$$

$$\Delta(k) = (k \cos qa - iq \sin qa) (q \cos qa - ik \sin qa). \quad (25)$$

By writing the coefficients A_1 and B_3 in polar form we can obtain information about the reflection and transmission coefficients and longitudinal shifts of the scattered wave packet [9–11, 15, 26–28]. Thus we write

$$A_1(k) = r(k) e^{i(\phi(k) - 2ka)}, \quad B_3(k) = t(k) e^{i(\phi(k) - 2ka + \pi/2)}$$

with

$$r(k) = \frac{(k^2 - q^2) \sin 2qa}{\left[(k^2 + q^2)^2 \sin^2 2qa + 4k^2 q^2 \cos^2 2qa\right]^{1/2}}, \quad t(k) = \frac{2kq}{\left[(k^2 + q^2)^2 \sin^2 2qa + 4k^2 q^2 \cos^2 2qa\right]^{1/2}}, \quad (26)$$

$$\phi(k) = -\arctan \frac{2kq \cos 2qa}{(k^2 + q^2) \sin 2qa}. \quad (27)$$

The reflection and transmission coefficients are now

$$R = r^2(k) = \frac{2 \left(\frac{k-q}{k+q}\right)^2 (1 - \cos 4qa)}{1 + \left(\frac{k-q}{k+q}\right)^4 + 2 \left(\frac{k-q}{k+q}\right)^2 \cos 4qa}, \quad T = t^2(k) = \frac{\left(\frac{2k}{k+q}\right)^2 \left(\frac{2q}{k+q}\right)^2}{1 + \left(\frac{k-q}{k+q}\right)^4 + 2 \left(\frac{k-q}{k+q}\right)^2 \cos 4qa}. \quad (28)$$

Let θ_i , $i = 1, 2, 3$ be the angle between the propagation direction and the optical axis in the regions *I*, *II*, *III* respectively. Then

$$k = k_0\sqrt{2(\varepsilon + 1)} \approx k_0\theta_1, \quad q = k_0\sqrt{2n(\varepsilon + n)} \approx k_0n\theta_2.$$

This allows us to write

$$\frac{k-q}{k+q} = \frac{\theta_1 - n\theta_2}{\theta_1 + n\theta_2} = r_{12} = -r_{23}$$

$$\frac{2k}{k+q} = \frac{2\theta_1}{\theta_1 + n\theta_2} = t_{12}, \quad \frac{2q}{k+q} = \frac{2n\theta_2}{n\theta_2 + \theta_3} = t_{23},$$

where r_{ij} , t_{ij} are the Fresnel reflection and transmission coefficients, in the paraxial approximation, from region i to region j respectively [34]. Therefore we may write the reflection and transmission coefficients as

$$R = \frac{2r_{12}^2(1 - \cos 2\beta\theta_2)}{1 + r_{12}^4 - 2r_{12}^2 \cos 2\beta\theta_2}, \quad T = \frac{t_{12}^2 t_{23}^2}{1 + r_{12}^4 - 2r_{12}^2 \cos 2\beta\theta_2}, \quad (29)$$

with $\beta = 2k_0na$, which coincides with that reported for a dielectric film (see [34]). In Figure 2 we present the transmission coefficient for a homogeneous slab of refractive index $n = 1.3$. Observe the series of peaks typical of the behaviour of the transmission coefficient for particles in a short range interaction potential [21, 23]. In this case, however, the number of peaks does not only depend on the width and the refractive index of the slab, but also on the wavelength of the electromagnetic field. Figure 2 shows the transmission coefficient for (a) $k_0a = 50$ and (b) $k_0a = 10^5$. For a dielectric plate of 1cm these values correspond to wavelengths of $1256\mu\text{m}$ and 628nm respectively.

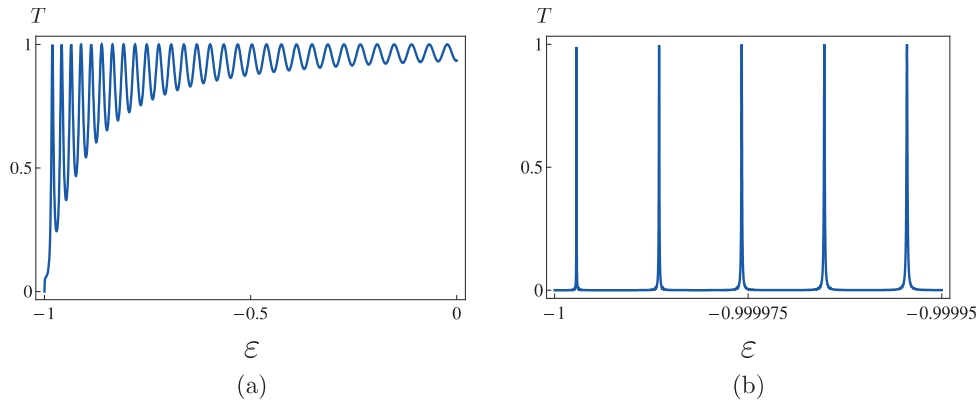


Figure 2. Transmission coefficient for a dielectric slab as a function of the eigenvalue $\varepsilon = -p_z$ for (a) $k_0a = 50$ and (b) $k_0a = 10^5$, in both cases $n = 1.3$.

3.2. The longitudinal shifts

The phases of the coefficients A_1 and B_3 give information about the longitudinal shift of a wave packet under scattering by the optical medium, this is equivalent to evaluate the reflection and transmission phase times for a particle in a rectangular well potential [9–11]. Due to the symmetry of the system this shift turns out to be the same under reflection and transmission. Consider the wave-packet scattering solution for the external electric field

$$E_I(x, z) = \int_0^\kappa dk f(k) e^{i(kx - k_0\varepsilon(k)z)} + \int_0^\kappa dk f(k) r(k) e^{i(-kx - \varepsilon(k)k_0z - 2ka + \phi(k))}, \quad (30)$$

$$E_{III}(x, z) = \int_0^\kappa dk f(k) t(k) e^{i(kx - \varepsilon(k)k_0z - 2ka + \phi(k) + \pi/2)}, \quad (31)$$

where $f(k)$ is the Fourier coefficient and κ is defined by $\varepsilon(\kappa) = 0$. At any point z , a particular component will contribute appreciable to the wave packet in the neighbourhood of the point x for which the phase of the integrand is stationary [9, 26–28]. This condition for the incident and the transmitted wave packet read, respectively, as

$$\frac{\partial}{\partial k} \left(kx - \frac{k^2}{2k_0}z + k_0z \right) = 0, \quad \frac{\partial}{\partial k} \left(kx - 2ka - \frac{k^2}{2k_0}z + k_0z + \phi(k) + \frac{\pi}{2} \right) = 0, \quad (32)$$

leading to the equations of motion for the corresponding Fourier components

$$x = \frac{k}{k_0}z, \quad x = 2a + \frac{k}{k_0}z + \frac{\partial\phi}{\partial k}. \quad (33)$$

The point z_{in} at which an incident plane wave of wave number k would reach the left wall of the optical medium is then $z_{in} = -\frac{k_0}{k}a$. In the same way the point z_t at which the reflected or transmitted wave leaves the optical medium is

$$z_t = \frac{k_0}{k} \left(-a + \frac{\partial\phi}{\partial k} \right). \quad (34)$$

Then the longitudinal shift of the reflected or transmitted wave is readily written

$$\delta z = z_t - z_{in} = \frac{k_0}{k} \frac{\partial\phi}{\partial k}. \quad (35)$$

Figure 3 shows the longitudinal shift of a Fourier component under reflection or transmission

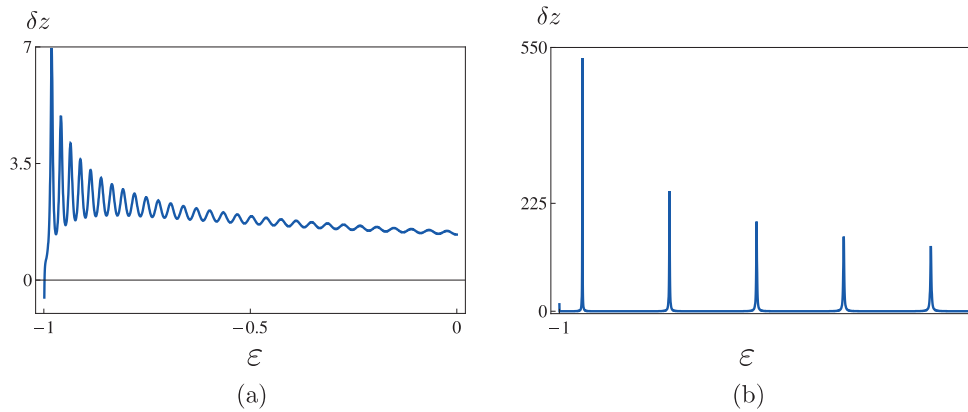


Figure 3. Reflection/transmission longitudinal shift of a Fourier component for a dielectric slab as a function of the eigenvalue $\varepsilon = -p_z$ for the product (a) $k_0 a = 50$, and (b) $k_0 a = 10^5$. In both cases $n = 1.3$.

as a function of ε for (a) $k_0 a = 50$ and for (b) $k_0 a = 10^5$. It is worthwhile to point out that the peaks of this shift coincide with the peaks of the corresponding transmission coefficient. This implies that the corresponding electromagnetic modes have maximum transmission probabilities but they travel a maximum longitudinal distance in the interaction zone before they are finally transmitted. Figure 4(a) shows the longitudinal shift as a function of $k_0 a$ for an eigenvalue $\varepsilon = -0.999$ and $n = 1.3$. As in the case of the transmission time in quantum mechanics, we can see that for some values of ε the transmission shift δz may take negative values. This negative shifts in optical systems have been already predicted for barriers as well as for rectangular wells [10, 11]. The analogy between the Schrödinger and the paraxial Helmholtz equations suggest that this phenomenon may be used to measure the negative phase time in the scattering of particles by a potential well [15]. Figure 4(b) shows the longitudinal shift surface as a function of ε and $k_0 a$ for $n = 1.3$.

4. Wave propagation in a dielectric homogeneous slab: bound and resonant states

4.1. The purely outgoing wave condition

It is also interesting to consider the solutions to the paraxial Helmholtz equation that fulfill the purely outgoing wave condition [21, 22]

$$\lim_{x \rightarrow \pm\infty} (\psi'(x) \mp ik\psi(x)) = 0, \quad k^2 = 2k_0^2(\varepsilon + 1), \quad (36)$$

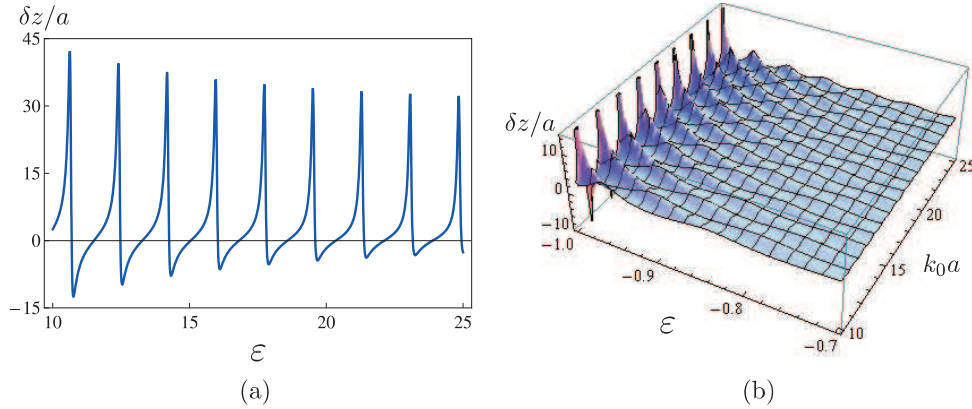


Figure 4. (a) Reflection/transmission longitudinal shift of a Fourier component for a dielectric slab as a function of k_0a with $\varepsilon = -0.999$ and $n = 1.3$. (b) Reflection/transmission longitudinal shift surface as a function of ε and k_0a with $n = 1.3$.

for ε a complex number. In this way $\varepsilon = \varepsilon_R - i\frac{\Gamma}{2}$ leading also to complex wave numbers $k = k_R + ik_I$. It is clear that the real and imaginary parts of k must obey

$$k_R^2 - k_I^2 = 2k_0^2(\varepsilon_R + 1), \quad k_R k_I = -k_0^2 \frac{\Gamma}{2}. \quad (37)$$

By applying the purely outgoing wave condition we conclude that the first terms in solution (21) for the external field must be negligible with respect to the second ones. The electric energy density $\rho_e(x, z) = \frac{\epsilon_0}{2} |E(x, z)|^2$ in regions *I* and *III* satisfies the continuity equation

$$\frac{\partial \rho_e}{\partial z} + \frac{\partial j_e}{\partial z} = 0, \quad j_e = \frac{i\epsilon_0}{4k_0} \left[\frac{\partial E^*}{\partial x} E - E^* \frac{\partial E}{\partial x} \right], \quad (38)$$

stating that the electric energy is preserved as the electromagnetic field propagates along the optical axis. The energy current density j_e^\mp in regions *I* and *III* respectively can be written as

$$j_e^\pm = \pm v \rho_e, \quad v = \frac{k_R}{k_0} \quad (39)$$

with v the flux velocity.

4.2. Bound and resonant modes

Suppose, in the first place, that ε is real. For $\varepsilon < -1$, $k^2 = 2k_0^2(\varepsilon + 1) < 0$ leading to purely imaginary values of k

$$k_\pm = \pm i\kappa, \quad \kappa = \sqrt{-2(\varepsilon + 1)}. \quad (40)$$

In this case $j_e^\pm = 0$, and the external electric field amplitudes are

$$\psi_I^\pm(x) = A_1^\pm e^{\pm k_0 \kappa x}, \quad \psi_{III}^\pm = B_3^\pm e^{\mp k_0 \kappa x}. \quad (41)$$

Observe that the solutions associated to k_+ are bounded in the external region and correspond precisely to the external vanishing fields associated to the bound modes propagating inside the slab. Therefore, even though the electric field does not disappear outside the dielectric medium, the electric energy density flux in the transversal direction is null, meaning that all the energy propagates in the direction of the optical axis. The modal fields in this case correspond to those

described in (21) with the coefficients A_1 , A_2 , B_2 , B_3 given by (23-25). The purely outgoing condition is fulfilled by demanding that $\Delta = 0$. Then, we may consider that $I \propto \Delta$ and then set $I = 0$. The eigenvalue ε is discretised, and thus, only a discrete number of modes will be able to propagate through the slab without losses. This case corresponds to the phenomenon of guided light in a waveguide by total internal reflection [35]. Since ε is related to the value of the z -component of the linear momentum \mathbf{p} , we conclude that, in order to have bound modes inside the slab, the angle between the beam and the optical axis must be discrete. The electric field modes corresponding to these eigenvalues ε_m are square integrable, showing that the transverse energy is bounded in a finite region around the optical axis. Different from the quantum mechanical case, in which the number of bound states only depends on the geometry of the potential well, in this case, the number of bound modes propagating along the slab depends on the profile of the optical medium, as well as the wavelength of the electromagnetic field.

The solutions associated to k_- are non physical since in this case the amplitude of the electric field increases exponentially as $|x| \rightarrow \infty$ (see (41)).

Consider now the case for $\varepsilon > -1$. In this case $k^2 = 2k_0^2(\varepsilon + 1) > 0$ and k is real

$$k_{\pm} = \pm k_0 \kappa, \quad \kappa = \sqrt{2(\varepsilon + 1)}. \quad (42)$$

Now we have $k_I = 0$ and $k_R = k_{\pm} = \pm k_0 \kappa$. The energy current densities outside the interaction region are

$$j_e^{\pm} = \pm k_0 \kappa \rho_e, \quad \text{for } k_+, \quad j_e^{\pm} = \mp k_0 \kappa \rho_e, \quad \text{for } k_-. \quad (43)$$

The corresponding solutions are not square integrable. There must be a net flux of electromagnetic radiation outwards the dielectric in the transversal direction. This case is discarded since we are considering no light sources or sinks inside the slab.

Finally, let us regard complex eigenvalues $\varepsilon = \varepsilon_R - i\frac{\Gamma}{2}$. Then $k = k_R + ik_I$ and

$$\varepsilon_R = \frac{k_R^2 - k_I^2}{2k_0^2}, \quad -\frac{\Gamma}{2} = \frac{k_R k_I}{k_0^2}. \quad (44)$$

The energy density flux in this case is non trivial since $k_R \neq 0$. This may correspond to an incident wave which enters into resonance with the optical medium. The wave is in this way captured inside the interaction region traveling there through a finite distance until it is finally emitted. The result is a reflected wave propagating to the left in region I , and a transmitted wave propagating to the right in region III . The correct sign for k_R is thus positive. The expression for the electric energy density fulfills

$$\lim_{x \rightarrow \pm\infty} \rho_e \propto e^{-k_0 \Gamma} |\psi(x)|^2. \quad (45)$$

From this expression it is clear that the energy density is damped as z grows whenever $\Gamma > 0$, and, as the sign of k_R is already positive, from (44) we conclude that $k_I < 0$. The external electric fields are then plane waves of increasing amplitudes. These modes are called resonances and correspond to wave numbers $k = k_R + ik_I$ in the fourth quadrant of the complex plane. The factor $1/k_0 \Gamma$ is the mean distance traveled by the beam, along the optical axis, before its intensity decays by a factor of $1/e$ of its initial intensity. In this context Γ can be interpreted as a measure of the transitory confinement of the field mode inside the medium. In the limit $\Gamma \rightarrow 0$ we will have a quasi-stationary mode propagating inside the slab almost without losses. In this limit, it is possible to establish an analytic approach to calculate the resonances and we may write

$$\frac{\Gamma/2}{\Delta\varepsilon_R} \ll 1, \quad (46)$$

where $\Delta\varepsilon_R$ is the separation between two consecutive resonances. We will also assume that $\Delta\varepsilon_R < \varepsilon_R + n$, with ε_R any resonance. Additionally to (37) we have

$$q_R^2 - q_I^2 = 2k_0^2 n(n + \varepsilon_R), \quad q_R q_I = -k_0^2 n \frac{\Gamma}{2}, \quad (47)$$

that allows us to derive an expression for q_I^{-2} [21], namely

$$q_I^{-2} = \frac{\varepsilon_R + n}{k_0^2 n (\Gamma/2)^2} \left[1 \pm \sqrt{1 + \left(\frac{\Gamma/2}{\varepsilon_R + n} \right)^2} \right]. \quad (48)$$

Since condition (46) implies that $\Gamma/2 \ll \varepsilon_R + n$ this last expression leads us to (keeping only the positive root in order to get real values of q_I)

$$q_I \approx \frac{k_0^2 n (\Gamma/2)^2}{2(\varepsilon_R + n)}. \quad (49)$$

Combining this equation with (47) we obtain

$$q_R^2 \approx 2k_0^2 n(\varepsilon_R + n), \quad q_I^2 a^2 \ll \theta^2, \quad \theta^2 = 2k_0 n(n-1)a^2. \quad (50)$$

Additionally, from (37) and (47) we may write approximate expressions for k_R and k_I in terms of q_R and q_I valid for $\theta \gg 1$

$$k_R \approx \sqrt{\frac{q_R^2 - 2k_0^2 n(n-1)}{n}}, \quad k_I = \frac{q_R q_I}{\sqrt{n(q_R^2 - 2k_0^2 n(n-1))}}. \quad (51)$$

In order to fulfill the condition $\Delta = 0$ we take the left factor of (25) equal to zero. This leads to

$$q_I \approx -\frac{\sqrt{\frac{1}{n} \left(q_R^2 + \frac{\theta^2}{a^2} \right) \cos q_R a}}{\sin q_R a + q_R a \cos q_R a}, \quad \frac{a}{q_R} \left[(n-1)q_R^2 + \frac{\theta^2}{a^2} \right] \approx -n \tan q_R a - \cot q_R a. \quad (52)$$

The values of q_R that fulfill (52) in the limit $\theta \rightarrow \infty$ are given by

$$q_R \approx m \frac{\pi}{2a} \quad (53)$$

with m an integer number. Substituting (53) into (50) we obtain a set of discrete values for the resonance eigenvalue ε_R

$$\varepsilon_R \approx \frac{1}{2k_0^2 n} \left(\frac{m\pi}{2a} \right)^2 - n. \quad (54)$$

However, since $-1 < \varepsilon_R < 0$, the number m must be bounded by some limiting values. In fact

$$\frac{2a}{\pi} k_0 \sqrt{2n(n-1)} < m < \frac{2a}{\pi} k_0 \sqrt{2n^2}. \quad (55)$$

Thus, for each combination of the parameters (a, n, k_0) , there is a finite number of resonances given by (54). The value of Γ is now determined by using (53) to obtain the values of q_I from (52) and substituting it into (51) to find $k_I \approx -\frac{1}{na}$. This result, together with (37) gives the value of Γ

$$\frac{\Gamma}{2} \approx \frac{1}{k_0 n a} \sqrt{2(\varepsilon_R + 1)}. \quad (56)$$

Similar results are obtained by setting to zero the second factor of (25).

5. Concluding remarks

We have described some propagation processes of electromagnetic waves in a homogeneous slab. In the scattering regime, the reflection and transmission coefficients were determined, which coincide with those previously reported for a dielectric film [34]. The corresponding longitudinal shifts were also obtained in the context of phase reflection and transmission time for a square well potential. It was shown that for some set of the parameters (a, n, k_0) the longitudinal shifts are negative, fact which is associated to negative group velocities of the wave packet [15]. The modal fields corresponding to guided light and resonances were presented as the eigenstates of the paraxial Helmholtz operator, corresponding to complex eigenvalues, satisfying the purely outgoing wave boundary condition. In some limiting cases this procedure allows to state analytic approaches to determine the bound modes and resonance eigenvalues. Light propagation through more general optical devices such as graded waveguides, optical fiber and multiple layered media can be studied in the same fashion. Some results on the matter are in progress.

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