

Mielnik's evolution loop hexagon in new light

Kurt Bernardo Wolf

Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México,
Av. Universidad s/n, Cuernavaca, Morelos 62210, México

E-mail: bwolf@fis.unam.mx

Abstract. The problem of controlling quantum wavefunctions by means of potential jolts and periods of free evolution was broached by Bogdan Mielnik in 1977. This *quantum control* became a subject of great interest for the preparation of atomic and particle systems. We point out here that these manipulations are also realized in paraxial geometric and wave optics, with lenses and free spaces, and even more transparently than with matter.

PACS: 02.20.Qs, 02.30.Yy, 03.65.Vf, 42.15.Eq

Keywords: Hamiltonian systems, Paraxial optics.

1. Introduction

In his 1977 paper, Bogdan Mielnik [1] showed that the free evolution of quantum wavefunctions can be reversed in time by applying an appropriate succession of harmonic oscillator potential jolts, within the Schrödinger formalism of quantum mechanics. This he reformulated in 1986 [2] as the existence of evolution loops, namely products of operators which are equivalent to *a* (or *the*) unit transformation. Mielnik's *evolution loop hexagon*, shown in Fig. 1, appeared in this paper and was the logo for the 2014 conference on Quantum control, exact or perturbative, linear or nonlinear, held at CINVESTAV and topic of these Proceedings. Much of the context for Mielnik's hexagon was doubtless supported by a previous fundamental article of Białynicki-Birula, Mielnik and Plebański [3] that analyzes the Baker-Campbell-Hausdorff problem of exchanging the exponentials of non-commuting operators.

The purpose of this article is to present the same problems (and solutions) when the system is optical, and classical, to be compared with the quantum system. The idealized potential jolts are replaced by rather real refracting surfaces, while the time periods τ_i when the system is free become, naturally enough, spaces z_i of free propagation along the optical axis — which by tradition points horizontally to the right. Moreover, as I shall contend, optical systems are richer than the Schrödinger's model of quantum mechanics. This is so because these refracting surfaces can mimic harmonic oscillator jolts only in the *paraxial* régime, where ray or beam angles are small and we look near to the optical axis. Beyond the paraxial régime of geometric and wave optics extends the *metaxial* régime, which constitutes a series expansion involving the nonlinearities called aberrations. Finally, there is the *global* régime, where the true nature of optical systems requires rays or beams in all directions on the sphere — or circle, or hypersphere, according to the dimension of the model.

In Section 2 we present three mathematical realizations of the symplectic algebra of 2×2 matrices that describe quadratic systems: the quantum oscillator jolts and free evolution, and also the paraxial lenses and empty spaces given in Section 3. In Section 4 we point out the



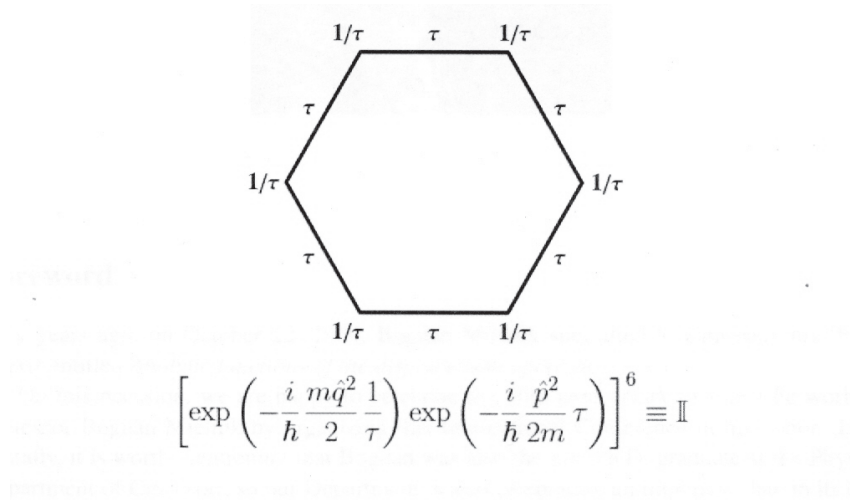


Figure 1. The Mielnik hexagon illustrates the application of six harmonic oscillator potential jolts of strength $1/\tau$, separated by six time periods τ of free evolution. This was the logotype of the conference on Quantum control, exact or perturbative, linear or nonlinear, held at CINVESTAV in October 2014.

subtle but important difference between classical and wave/quantum systems contained in the symplectic *versus* metaplectic units by means of the Bargmann parametrization of the universal cover of the symplectic groups, built out of products of lower- and upper-triangular matrices that realize lenses and free spaces. In Section 5 we collect some conclusions and point to extensions.

2. Three models under one algebra

Consider the following basis of three linearly independent functions $j_i(q, p)$ that are quadratic in q (position), and p (momentum),

$$j_+ := \frac{1}{2}p^2, \quad j_- := \frac{1}{2}q^2, \quad j_0 = \frac{1}{2}pq. \quad (1)$$

Corresponding to these we have three linear operators, built with Poisson brackets $\bar{j}_i := \{j_i, \circ\}$ with the classical coordinates of position and momentum,

$$\bar{j}_+ = -p\partial_q, \quad \bar{j}_- = q\partial_p, \quad \bar{j}_0 = \frac{1}{2}(p\partial_p - q\partial_q), \quad (2)$$

which act on *beam density* functions $\rho(q, p)$. The usual and unique Schrödinger quantization of (1) from the basic operators of position and momentum, $q \mapsto \hat{Q} = q \cdot$ and $p \mapsto \hat{P} = -i\partial_q$ yields the following three up-to-second degree operators \hat{J}_i ,

$$\hat{J}_+ = -\frac{1}{2}\partial_q^2, \quad \hat{J}_- = \frac{1}{2}q^2 \cdot, \quad \hat{J}_0 = -\frac{1}{4}i(q\partial_q + \partial_q q), \quad (3)$$

that are hermitian on the Hilbert space $\mathcal{L}^2(\mathbb{R})$ in the single variable $q \in \mathbb{R}$. Finally, we have the basis of three 2×2 real traceless matrices \mathbf{J}_i , given by

$$\mathbf{J}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{J}_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{J}_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4)$$

These matrices *present* the algebra $\mathfrak{sp}(2, \mathbb{R})$ and its similar ones, acting on the phase space coordinates that we can fit into a two-vector $\begin{pmatrix} q \\ p \end{pmatrix}$.

The common property of the four triads of \mathcal{J} 's (the notation stands for j, \bar{j}, \hat{J} or \mathbf{J}) is to realize the Lie algebra $\mathfrak{so}(2, 1) = \mathfrak{su}(1, 1) = \mathfrak{sp}(2, \mathbb{R})$, whose defining set of Lie brackets can be written as

$$\{\mathcal{J}_0, \mathcal{J}_+\} = +\iota\mathcal{J}_+, \quad \{\mathcal{J}_0, \mathcal{J}_-\} = -\iota\mathcal{J}_-, \quad \{\mathcal{J}_+, \mathcal{J}_-\} = -2\iota\mathcal{J}_0. \quad (5)$$

For the functions j_i in (1), the Lie bracket is the classical Poisson bracket $\{j_a, j_b\} \equiv \{j_a, j_b\} := (\partial_q j_a)(\partial_p j_b) - (\partial_p j_a)(\partial_q j_b)$ with $\iota = 1$. The phase space operators (2) obey (5) with the commutator Lie bracket $\{\bar{j}_a, \bar{j}_b\} \equiv [\bar{j}_a, \bar{j}_b] := \bar{j}_a \bar{j}_b - \bar{j}_b \bar{j}_a$ also with $\iota = 1$, as do the matrices in the realization (4). Only for the quantum-mechanical operators (3) the commutation relations (5) carry $\iota = i$ to preserve hermiticity in $\mathcal{L}^2(\mathbb{R})$.

Although all four sets of objects (1)–(4) realize the same Lie algebra, the exponentiation of the two operator sets, the classical (2) and the quantum (3) to Lie groups do not *quite* give the same group. While the matrix realization (4) clearly indicates that the group is that of 2×2 real matrices of unit determinant $\mathrm{SL}(2, \mathbb{R}) = \mathrm{Sp}(2, \mathbb{R})$, the quantum (and paraxial wave-optical) provides an *integral transform* realization of its two-fold cover, the *metaplectic* group $\mathrm{Mp}(2, \mathbb{R})$, as we shall elaborate below.

3. Free propagation, jolts and lenses

The formalism introduced in the previous section refers to one-dimensional models, but provides the gist of the matter. It can be made N -dimensional with the symplectic algebras $\mathfrak{sp}(2N, \mathbb{R})$; if the systems are isotropic, a *radial* $\mathfrak{sp}(2, \mathbb{R})$ algebra will arise, also satisfying (5), where the generator \mathcal{J}_+ will receive an extra ‘centrifugal’ term $\sim \mu/q^2$ that will lie, for $\mu \geq -\frac{1}{4}$, among the lower-bound discrete series D_+^k of $\mathfrak{sp}(2, \mathbb{R})$. Yet, the 2×2 matrix *presentation* of this algebra in (4) is unchanged.

Rigorous mathematics are not lacking, but let me appeal to the common understanding of Schrödinger quantum mechanics, while drawing a parallel sketch of geometric and wave paraxial optics. In this context we look first at the phase space variables as operators. The spectrum of the position operator \mathcal{Q} is the set of observation points on a line, assumed to be continuous $q \in \mathbb{R}$. Here the quantum wave function is ‘known’ at a given time τ ; in geometric optics q is the coordinate of the point where a line of light crosses a one-dimensional screen that is perpendicular to the optical axis at a given distance z ; in wave optics z plays the same role as τ in quantum mechanics, the measurement being the (complex) elongation of the wavefield on a screen placed at z .

The momentum operator \mathcal{P} is more amenable to discussion, since in all models it relates the evolution of \mathcal{Q} within the screen under infinitesimal propagation in τ or z ; thence the quantum $\hat{P} = -i\partial_q$. In geometric optics, the matter is more geometrical: see Fig. 2. A ray of position q , and inclined by an angle θ to the optical axis will, upon propagation by infinitesimal dz , go to position $q + dq$. Now, dq on the screen is perpendicular to dz on the axis, and both form a right triangle with ds , the *optical* distance (along which a wavefield would oscillate); this triangle is similar to that formed by the quantity

$$p = n \sin \theta, \quad (6)$$

which is the *optical momentum* on the screen; then, there is a quantity $p_z \equiv -h(q, p)$ that turns out to be (minus) the geometric *optical Hamiltonian*; the hypotenuse of this triangle is the refractive index n of the medium at q . This similarity leads to the Hamilton equations [4, Ch. 2]. In the *global* régime of optics, where $\theta \in \mathbb{S}^1$ (the circle), $|p| \leq n(q)$.

Indeed, the optical Hamiltonian is

$$h(q, p) = -\sqrt{n(q)^2 - p^2} \approx \frac{p^2}{2n(q)} - n(q) + \frac{p^4}{8n(q)^3} + \frac{p^6}{16n(q)^5} + \dots \quad (7)$$

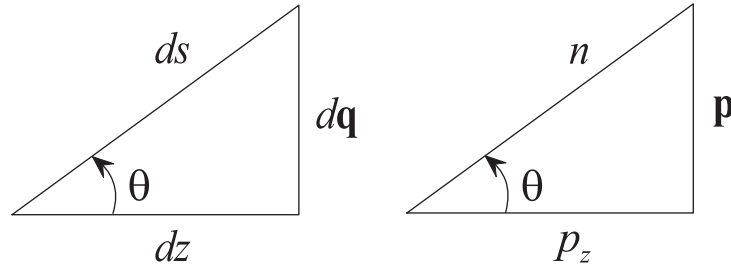


Figure 2. The left triangle, formed by infinitesimal displacements in position on the screen $d\mathbf{q}$ (in any number of dimensions) and evolution distance dz has for hypotenuse the optical distance ds . This is similar to the triangle formed by optical momentum \mathbf{p} , the (minus) Hamiltonian p_z , and the refractive index n at the point \mathbf{q} .

The paraxial approximation consists in assuming that the angle θ is near to the optical axis, so $\theta^2 \approx 0$, and so momentum (6) becomes $p = n\theta$ while the Hamiltonian (7) loses all p -terms beyond p^2 , and comes to resemble a mechanical system of variable mass [5]. But then, the *paraxial* model assumes that, notwithstanding θ is small, the momentum p can be extended to all of \mathbf{R} together with all the simplified formalism of quadratic expressions when also $n(q) \approx n_0 - \frac{1}{2}n_2 q^2$ for isotropic media and $q^2 p^2 \approx 0$; then the Hamiltonian (7) becomes $p^2/2n_0 - n_0 + \frac{1}{2}n_2 q^2$. Note that the refractive index corresponds to the *negative* of a quantum potential: $n_0 - \frac{1}{2}n_2 q^2 \sim -V(q)$, so rays ‘fall into’ regions of highest n , corresponding to the potential minimum.

With what has been said we can now broach the purely geometric free displacement (or propagation) by finite z or τ —indicate it by ζ —, generated by $\mathcal{J}_+ \sim \frac{1}{2}p^2$ and present it with the 2×2 matrices in (4), as

$$\mathcal{D}(\zeta) = \exp(-\iota\zeta\mathcal{J}_+) \quad \leftrightarrow \quad \mathcal{D}(\zeta) : \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q + \zeta p \\ p \end{pmatrix}, \quad (8)$$

where ι is 1 or i as before. The classical map is linear; it skews the phase plane and obviously conserves $p = n\theta$, the direction of light propagation. We recall that to respect the order of multiplication for group elements \mathcal{G} and their matrix presentation \mathbf{M} on phase space two-vectors, the group elements act with the *inverse* matrix, $\mathcal{G}(\mathbf{M}) \begin{pmatrix} q \\ p \end{pmatrix} = \mathbf{M}^{-1} \begin{pmatrix} q \\ p \end{pmatrix}$; because only then $\mathcal{G}(\mathbf{M}_1)\mathcal{G}(\mathbf{M}_2) = \mathcal{G}(\mathbf{M}_1\mathbf{M}_2)$.

The dynamical elements of the system are generated by $\mathcal{J}_- \sim \frac{1}{2}q^2$. In geometric optics it is clear that the effect of a lens of power g will be

$$\mathcal{L}(g) = \exp(-\iota g\mathcal{J}_-) \quad \leftrightarrow \quad \mathcal{L}(g) : \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q \\ p - gq \end{pmatrix}. \quad (9)$$

This is the *lens* transformation on classical phase space; when $g > 0$ the rays are *refracted* (i.e., broken) towards the optical axis, and a beam of rays parallel to this axis will all meet at the *focal distance* $f := 1/g$ of a convex lens; when $g < 0$, the lens is concave. In wave optics, $\mathcal{L}(g)$ acts on wavefields multiplying them by the phase $\exp(-i\frac{1}{2}gq^2)$, as a flat Fresnel lens such as we often see in pocket magnifying glasses. On the other hand, the quantum mechanical analogue of the flat lens (8) is a potential jolt. As built in [1], the evolution in the presence of a potential $V(q)/\kappa$ during a time κ is produced by the operator $\exp[-i\kappa(-\frac{1}{2}\partial_q^2 + V(q)/\kappa)]$ acting on the wavefunction. When $\kappa \rightarrow 0$ the time window narrows and the potential strength increases, so there remains the phase $\exp[-iV(q)]$. For the harmonic oscillator this is $V(q) = \frac{1}{2}\omega q^2$, where the oscillator strength is ω , which corresponds with the lens power g that we introduced above.

The problem is thus set: to produce, with free displacements and lenses, *all* linear (and *canonical*) transformations of phase space; correspondingly, with periods of free evolution interspersed with harmonic oscillator jolts, to produce all linear (and *unitary*) maps of wavefunctions in $\mathcal{L}^2(\mathbb{R})$. Of particular interest is *negative* free evolution, and thus the unit transformation that returns the beam or wavefunction to its original condition before it entered the optical setup or other such apparatus. In view of (8) and (9), the classical problem reduces to write any (and every) such transformation as a 2×2 matrix, and decompose this into products of lower- and upper-triangular matrices. Perhaps this is an unusual factorization for $\text{Sp}(2, \mathbb{R})$, which is better known through its Iwasawa, Bargmann, or Euler-type angle and boost decompositions.

Let us compose a lens (or jolt) of strength g with a free space (or time) ζ ; multiplying their matrix representatives; this ‘elementary subsystem’ we denote by

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix} \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \zeta \\ -g & 1-g\zeta \end{pmatrix}. \quad (10)$$

Two such arrangements in a row are represented by their product,

$$\mathbf{M}^2 = \begin{pmatrix} 1-g\zeta & \zeta(2-g\zeta) \\ -g(2-g\zeta) & (1-g\zeta)^2 - g\zeta \end{pmatrix}, \quad (11)$$

where we note that if $g\zeta = 2$, the transformation becomes a rotation of phase space by π , namely $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Three such elements, depicted in Fig. 3, produce

$$\mathbf{M}^3 = \begin{pmatrix} (1-g\zeta)^2 - g\zeta & \zeta(1-g\zeta)(3-g\zeta) \\ -g(1-g\zeta)(3-g\zeta) & (1-g\zeta)^3 - g\zeta(3-2g\zeta) \end{pmatrix}. \quad (12)$$

When $g\zeta = 1$ we obtain the π rotation $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, while when $g\zeta = 3$ the result is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ — the identity matrix. This last ‘identity system’ was already recognized in [2, Eq. (3.25)] to yield an output which is the input function with an overall minus sign.

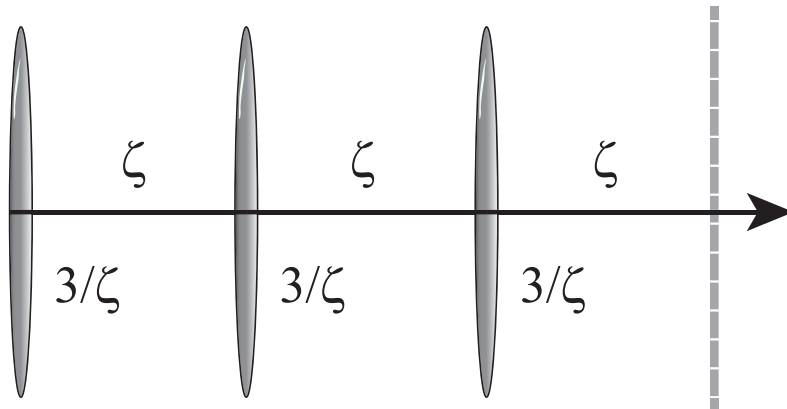


Figure 3. In geometric optics, three lenses of power $g = 3/\zeta$ and three free flights by ζ yield the identity transformation between ingoing and outgoing rays.

The above argument with matrices, which would appear to indeed simplify the Mielnik hexagon into a triangle, underscores the fact that, because $\text{Sp}(2, \mathbb{R})$ is *multiply connected*, the geometric-optical realization of the symplectic identity group element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in (12), corresponds to *two* group elements in its two-fold cover $\text{Mp}(2, \mathbb{R})$. This holds in the same way that a rotation

by 2π around any axis is the identity transformation in the three-dimensional rotation group $\text{SO}(3)$, but corresponds to *two* ‘rotations’, by 2π and by 4π , in its two-fold cover by the spin group $\text{SU}(2)$. In the next section we shall see that $\text{Sp}(2, \mathbb{R})$, as the circle \mathbb{S}^1 , is *infinitely* connected; the metaplectic group $\text{Mp}(2, \mathbb{R})$ is only its two-fold cover.

It is of interest to note, in the context of geometric optics, that the problem of realizing any linear symplectic transformation in a (two-dimensional) paraxial optical setup composed by lenses and free flights, i.e., the ‘optical’ decomposition of the three-dimensional $\text{Sp}(2, \mathbb{R})$ manifold, by means of upper- and lower-triangular 2×2 matrices, was solved in Ref. [6]. It turns out that two lenses and two free flights suffice to reach all such systems, *except* for a lower-dimensional submanifold that includes negative free flight; so, three lenses and two free flights suffice. (See also Ref. [4, Sec. 10.5].)

4. Symplectic and metaplectic units

The work of Valentin Bargmann on the symplectic groups, Ref. [7] for $\text{Sp}(2, \mathbb{R})$ and Ref. [8] for $\text{Sp}(2N, \mathbb{R})$, is as comprehensive as can be found. He uses the analogue of the characterization of complex numbers by phase and modulus, $z = e^{i \arg z} |z| \in \mathbb{C}$, which for matrices is their *polar* decomposition into the product of a unitary and a symmetric positive definite matrix. In the 2×2 case this is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \lambda + \text{Re} \mu & \text{Im} \mu \\ \text{Im} \mu & \lambda - \text{Re} \mu \end{pmatrix}, \quad (13)$$

where $ad - bc = 1$, and its *Bargmann* parameters are

$$\phi = \arg [a + d - i(b - c)] \in \mathbb{R}, \quad (14)$$

$$\mu = \frac{1}{2} e^{-i\phi} [a - d + i(b + c)] \in \mathbb{C}, \quad (15)$$

with $\lambda := +\sqrt{|\mu|^2 + 1} > |\mu|$. Note that, as in the complex number case, the ‘angle’ ϕ is now allowed to take values beyond the basic angular interval.

In terms of the Bargmann parameters, the product of two matrices $\mathbf{M} = \mathbf{M}_1 \mathbf{M}_2$ is found to be

$$\phi = \phi_1 + \phi_2 + \arg \nu, \quad (16)$$

$$\text{where } \nu := 1 + e^{-2i\phi_2} \mu_1 \mu_2^* / \lambda_1 \lambda_2, \quad (17)$$

$$\mu = e^{-i \arg \nu} (\lambda_1 \mu_2 + e^{-2i\phi_2} \mu_1 \lambda_2). \quad (18)$$

It is crucial to realize that $|\nu - 1| < 1$, so $\arg \nu \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, which implies that (16) *uniquely* defines the ‘phase’ ϕ beyond the basic angular interval, and thus parametrizes the universal covering group $\overline{\text{Sp}}(2, \mathbb{R})$. The manifold of the symplectic group $\text{Sp}(2, \mathbb{R})$ is the exterior of a one-sheeted hyperboloid, where ϕ can effect any number of turns around its waist; see for example Ref. [4, Sect. 9.4]. In the metaplectic group $\text{Mp}(2, \mathbb{R})$, ϕ is counted modulo 4π .

In (10)–(12) we drew attention to three cases: $g\zeta = 1, 2$, and 3 , which resulted in diagonal matrices, either $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ —which do not depend on g or ζ separately. With (14)–(15) and the products (16)–(18) we can know whether these form the ‘first’ or the ‘second’ metaplectic (and true) unit. For reference, we annotate that in the three cases of $g\zeta$, the matrices \mathbf{M} in (10) correspond to the following Bargmann parameters for $\zeta = 1$:

$$\begin{aligned} g\zeta = 1 : \quad \mathbf{M}_1 &:= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} & \phi &= -1.10715, \quad \mu = 0.223607 + 0.447214i, \\ g\zeta = 2 : \quad \mathbf{M}_2 &:= \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} & \phi &= -\frac{1}{2}\pi, \quad \mu = \frac{1}{2} + i, \\ g\zeta = 3 : \quad \mathbf{M}_3 &:= \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix} & \phi &= -1.81577, \quad \mu = 0.606339 + 1.697750i. \end{aligned} \quad (19)$$

Values of ζ close to 1 lead to values of ϕ, μ that are close to those listed above but, as we said, we are only interested in the cases when the powers of (19) are diagonal matrices, $\mathbf{1}$ or $-\mathbf{1}$, independent of ζ and g separately.

Since the hand computation of products in terms of Bargmann parameters is rather arduous, we resorted to symbolic and numeric computation with WOLFRAM MATHEMATICA which, after thorough checking, yielded the following parameter values for their powers:

$$\begin{aligned}
 g\zeta = 1 : \quad & \mathbf{M}_1^3 = -\mathbf{1}, & \phi = -\pi, & \mu = 0, \\
 & \mathbf{M}_1^6 = +\mathbf{1}, & \phi = -2\pi, & \mu = 0; \\
 g\zeta = 2 : \quad & \mathbf{M}_2^2 = -\mathbf{1}, & \phi = -\pi, & \mu = 0, \\
 & \mathbf{M}_2^4 = +\mathbf{1}, & \phi = -2\pi, & \mu = 0, \\
 & \mathbf{M}_2^6 = -\mathbf{1}, & \phi = -3\pi, & \mu = 0; \\
 g\zeta = 3 : \quad & \mathbf{M}_3^3 = +\mathbf{1}, & \phi = -2\pi, & \mu = 0, \\
 & \mathbf{M}_3^6 = +\mathbf{1}, & \phi = -4\pi, & \mu = 0.
 \end{aligned} \tag{20}$$

From here we can conclude that the symplectic unit can be realized with the three-lens setup \mathbf{M}_3^3 shown in Fig. 3 as a minimal arrangement (see Ref. [4, Subsec. 10.5.4]), but also by means of a \mathbf{M}_2^4 four-lens arrangement with $g\zeta = 2$, or with six lenses where $g\zeta = 1$, as shown in Figs. 1 and 4. The metaplectic unit appears when we double the setups, and thus appears only in \mathbf{M}_3^6 , with $g = 3/\zeta$.

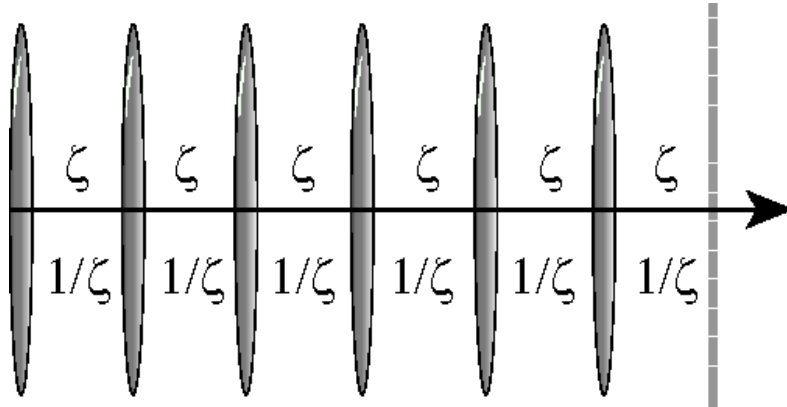


Figure 4. Six lenses of power $g = 1/\zeta$ and six free flights by ζ yield the symplectic identity transformation between ingoing and outgoing rays. The metaplectic identity is obtained by concatenating two symplectic identities each realized by the setup in the preceding Fig. 3.

5. Conclusions

When Moshinsky and Quesne defined linear canonical transforms [9] as a representation of the group of symplectic transforms that preserve the Heisenberg-Weyl algebra generated by the quantum operators \hat{Q} , \hat{P} and $\hat{1}$, realized on the space $\mathcal{L}^2(\mathbb{R})$ of quantum wavefunctions $\psi(q)$, they wrote it as an integral transform kernel [10, Ch. 9],

$$\mathcal{C}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(q, q') = \frac{e^{-i\pi/4}}{\sqrt{2\pi b}} \exp\left(i \frac{aq'^2 - 2q'q + dq^2}{2b}\right), \tag{21}$$

where $ad - bc = 1$. The square root of b in the denominator already announces a double valuation. The product of two transformations was investigated in that same article, leading

to a sign that depends on the b -elements of the two factors and the product matrices. The symplectic unit would multiply $\psi(q)$ by a minus sign while the metaplectic unit would be the true unit transform.

Although we have often written the metaplectic product law using the Bargmann parameters (13) in symbolic form, I cannot recall any paper where they have been explicitly computed for a given optical or quantum setup. In performing the exercise of analyzing the three- and six-lens optical systems in Figs. 3 and 4, as well as the product of any number of optical elements, we can also describe their quantum counterpart systems.

In the Introduction we stated that optical models are richer than quantum-mechanical ones. This is true in the metaxial (i.e., beyond the paraxial) régime because of the *aberrations* due to the non-flat nature of lenses, and because now the momentum (6) is no longer $p = n\theta$, so that now $q \mapsto q + zn \sin \theta$, and the terms $\sim p^{r+1}$ in (7) represent r^{th} order aberrations, for $r \in \{3, 5, 7, \dots\}$. As Figs. 3 and 4 suggest, as we move the screen along the z -axis, a non-flat convex lens represents a potential jolt along the two endpoints of a segment (the border of a disk in two-dimensional screens) that grows along the z (or τ) axis from zero to the size of the lens, and then decreases to a point and vanishes.

Generally metaxial systems require functions $p^u q^v$, with u, v integers and $u + v > 2$, and their corresponding Poisson or quantum operators, the latter following the Weyl correspondence rule to ensure hermiticity. The groups built in this way are $A_r \triangleleft \text{Sp}(2, \mathbb{R})$, where A_r is the normal subgroup of r^{th} order aberrations, generated by all monomials $p^u q^v$ with $2 < u + v \leq r$, plus the assumption that $p^u q^v \approx 0$ for $u + v > r$ [4, Part 4]. In this régime the primary interest is to minimize the aberrations through the best choice of the surface shape [11] and of the refractive index profile $n(q)$. If we only have constructible physical lenses and free flights, it is unclear that these optical transformations can be bent back to unity, and it is most likely that they may not.

Acknowledgments

I thank Guillermo Krötzsch for assistance with the figures and Alejandro R. Urzúa for his interest in this work, as well as the PAPIIT-UNAM support of the Óptica Matemática projects PAPIIT IN101011 and IN101115 in this period.

References

- [1] B. Mielnik, Global mobility of Schrödinger's particle, *Rep. Math. Phys.* **12**, 331–339 (1977).
- [2] B. Mielnik, Evolution loops, *J. Math. Phys.* **27**, 2290–2306 (1986).
- [3] I. Białynicki-Birula, B. Mielnik, and J. Plebański, Explicit solution of the continuous Baker-Campbell-Hausdorff problem and a new expression for the phase operator, *Ann. Phys.* **51**, 187–200 (1969).
- [4] K.B. Wolf, *Geometric Optics on Phase Space* (Springer-Verlag, Heidelberg, 2004).
- [5] S. Cruz y Cruz and O. Rosas-Ortiz, Lagrange equations and spectrum generating algebras of mechanical systems with position-dependent mass. In: Symmetry, Integrability and Geometry: Methods and Applications, *SIGMA* **9**, 004 (2013), 21 pp.
- [6] K.B. Wolf and G. Krötzsch, El problema de las tres lentes, *Rev. Mex. Fís.* **47**, 291–298 (2001).
- [7] V. Bargmann, Irreducible unitary representations of the Lorentz group, *Ann. Math.* **48**, 568–642 (1947).
- [8] V. Bargmann, Group representations in Hilbert spaces of analytic functions. In: *Analytical Methods in Mathematical Physics*, Ed. by P. Gilbert and R.G. Newton (Gordon & Breach, New York, 1970), pp. 27–63.
- [9] M. Moshinsky and C. Quesne, Linear canonical transformations and their unitary representation, *J. Math. Phys.* **12**, 1772–1780 (1971).
- [10] K.B. Wolf, *Integral Transforms in Science and Engineering* (Plenum Publ. Corp., New York, 1979).
- [11] K.B. Wolf and G. Krötzsch, Metaxial correction of fractional Fourier transformers, *J. Opt. Soc. Am. A* **16**, 821–830 (1999).