# Quantum control using genetic algorithms in quantum communication: superdense coding

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**Abstract.** We present a physical example model of how Quantum Control with genetic algorithms is applied to implement the quantum superdense code protocol. We studied a model consisting of two quantum dots with an electron with spin, including spin-orbit interaction. The electron and the spin get hybridized with the site acquiring two degrees of freedom, spin and charge. The system has tunneling and site energies as time dependent control parameters that are optimized by means of genetic algorithms to prepare a hybrid Bell-like state used as a transmission channel. This state is transformed to obtain any state of the four Bell basis as required by superdende protocol to transmit two bits of classical information. The control process protocol is equivalent to implement one of the quantum gates $\hat{X}$, $\hat{Z}$ $\hat{Y}$ in the charge subsystem. Fidelities larger than 99.5 % are achieved for the hybrid entangled state preparation and the superdense operations.

## 1. Introduction

Quantum control (QC) aims to systematically manipulate one or more control functions to drive the quantum dynamics of a system towards a specific set of desired final states and produce the quantum information processing task [1, 2]. QC has been extensively used to control dynamics of physical systems applied to quantum computation and communications [3–6]. Genetic Algorithms(GA), on the other hand, are a very active area of QC [7] and optimization resource have proven convergence without restriction of any range of possible control designs [8]. GA are an analogy to the evolution of living organisms [9] of mutation.

Quantum superdense code (QSC) is one of the most well known applications of quantum information as proposed by Bennet et. al [10], in which the information of two classical bits $\{00, 01, 10, 11\}$ in Alice's possession can be transmitted to Bob with a single qubit. This code requires the use of maximally entangled state which is possible to manipulate in order to transmit the classical two bits.

In this paper we consider a model of a hybrid system with a double quantum dot with one single electron with spin and Dresselhaus spin-orbit coupling [11, 12]. The electron spin and the occupancy form a hybrid basis, in the sense that the entangled state is formed by a qubit of charge (the occupancy) and a qubit of spin and this is the basis for the entangled channel required on the QSC protocol. Hybrid entangled state (HES) preparation will be studied first, and then the four unitary transformations needed for the QSC to transmit the two bit state will be addressed.

The QSC is treated as a control problem, where the required final states are optimized through a fidelity measure by means of genetic algorithms (GA) [13, 14]. Tunneling and site energies are time dependent tuning parameters used to design a control pulse.

### *1.1. Genetic Algorithms*

The evolution analogy of GA consists in forming chromosomes which are different possible solutions of a problem, and the genes of the chromosomes will be the values of every variable of each solution, i.e., a chromosome is a set of parameters while the genes are each of the parameters individually. We used the GA available in the PIKAIA suite [13, 15], where the solutions evolve with genetic recombination occurring at breeding and random probabilistic mutations at breeding events. GA can be understood in a quantum mechanics point of view as follows: given that a quantum state at a given time can be obtained after the actuation of a time evolution operator parameterized with a vector $\vec{b}_i$ ( control parameter, electric pulse width and amplitude etc) on initial states as $|\psi(t)\rangle = \hat{U}(\vec{b}_i,t)\,|\psi(0)\rangle$. GA generate an initial population of $i$ different $\hat{U}(\vec{b}_i,t)$ operators that will act on $|\psi(0)\rangle$. GA will test an objective function called *fitness function* for all $i$ that will be constructed as

$$f_{\rm fit}(\vec{b}_i) = |\,\langle\psi_{\rm fit}|\,\psi(t=t_m)\rangle|^2 = |\,\langle\psi_{\rm fit}|\,\hat{U}(\vec{b}_i,t)\,|\psi(0)\rangle\,|^2, \tag{1}$$

where $|\psi_{\rm fit}\rangle$ is the objective or ideal quantum state that one wants to obtain to perform the the quantum processing task. Internally, PIKAIA maximizes the objective function $f_{\rm fit}(\vec{b}_i)$ in a bounded n-dimensional space $B\,:\,\vec{b}_i \equiv (b_1, b_2, ..., b_n)$ with $b_k \in [0.0, 1.0]\forall k$. The algorithm will pick up a set of better *fitted* solutions as the parents for the next offspring. For instance, in a 2 dimensional $B$ space where the parents form the two operators $\hat{U}(\mathbf{0.105}714, \mathbf{0.542}219, t)$, $\hat{U}(0.542\mathbf{774}, 0.674\mathbf{219}, t)$, the "children" solutions could be $\hat{U}(0.105\mathbf{774}, 0.542\mathbf{219}, t)$ and $\hat{U}(0.542\mathbf{714}, 0.674\mathbf{219}, t)$ (the breeding). Mutation could happen at breeding, this makes any digit of a variable equally probable to be chosen and randomly modified, for example in the variable $0.5_1 4_2 2_3 2_4 1_5 9_6$ any digit position (from 1 to 6) could be selected, once a position is selected by a *roulette* algorithm, that digit has a probability to be substituted for any number (including the same number) For example the children solutions could *mute* to $\hat{U}(0.10\mathbf{3}774, 0.\mathbf{5}42219)$ and $\hat{U}(0.5427\mathbf{2}4, 0.6\mathbf{7}4219)$, where the bold numbers have changed in some of them. The whole process generates the new offspring that will once again pass the same process of parents selection and breeding, repeated for an explicit amount of *generations* or until a *fitness* condition is achieved.

### *1.2. Physical Model and Quantum Superdense Coding*

The studied system consists of a double lateral quantum dot shown in Fig. 1, where tunneling $t(t)$ and the site energies $E_{L,R}(t)$ are time dependent control parameters that can be tailored to accomplish QSC, and the tunneling with change of spin is given by the Dresselhaus spin-orbit (SO) interaction $t_{SO}$ which is considered constant. At $t = 0$ a spin up electron is placed in the left QD. The four dimensional computational basis, for the two qubit system is formed for the charge qubit with the presence of the electron in sites left $|\Phi_L\rangle \equiv |0\rangle$ and right $|\Phi_R\rangle \equiv |1\rangle$, and for the spin qubit with electron spin up $|\uparrow\rangle \equiv |0\rangle$ and down $|\downarrow\rangle \equiv |1\rangle$ that together form the hybrid basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \equiv \{|\Phi_L\uparrow\rangle, |\Phi_L\downarrow\rangle, |\Phi_R\uparrow\rangle, |\Phi_R\downarrow\rangle\}$. The base of this problem is occupancy and spin as follows

$$|\Psi(t)\rangle = \alpha_1\,|\Phi_L\uparrow\rangle + \alpha_2\,|\Phi_R\uparrow\rangle + \alpha_3\,|\Phi_L\downarrow\rangle + \alpha_4\,|\Phi_R\downarrow\rangle\,. \tag{2}$$

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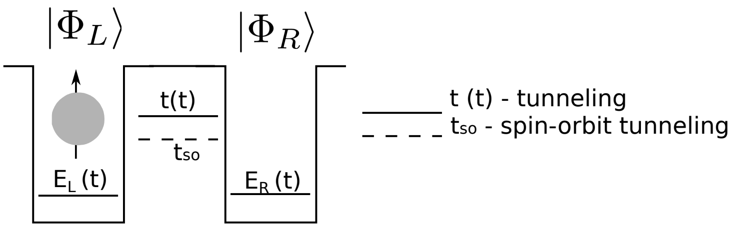


**Figure 1.** Quantum dot model for the QSD at $t = 0$ with a spin up electron in left site. Site energies and tunnelings are time dependent control parameters.

As described before, the QSC target is to transmit from Alice to Bob two classical bits of information with single qubit operations. First, Alice and Bob must share a maximally entangled state formed by two qubits (each in possession of one qubit). Let us say Alice has the charge qubit and Bob the spin qubit. QSC is achieved by unitary operations on Alice's qubit which transforms the shared state in one of the four distinguishable Bell-like states at Bob's side. In this work we consider as communication channel required in the QDC protocol, the hybrid spin-site Bell-like family of states given by

$$
\begin{aligned}
|\Phi^{\pm}_{\mathrm{LR}}\rangle &= \frac{1}{\sqrt{2}}(|\Phi_L \uparrow\rangle \pm e^{i\pi/4}\,|\Phi_R \downarrow\rangle),\\
|\Psi^{\pm}_{\mathrm{LR}}\rangle &= \frac{1}{\sqrt{2}}(|\Phi_L \downarrow\rangle \pm e^{i\pi/4}\,|\Phi_R \uparrow\rangle).
\end{aligned}
\tag{3}
$$

QSC requires the ability to achieve each of the states in (3) starting with an initial state, for instance, $|\Phi^{+}_{LR}\rangle$. Given that all operations act on Alice's qubit, the 4 transformations needed to send the two classical bits $(a1, a2), a1 = \{0, 1\}a2 = \{0, 1\}$ are summarized in Table 1, where the unitary operations represent the transformation that the initial state $|\Phi^{+}_{\mathrm{LR}}\rangle$ experiences to become any of the final Bell-like family states, as well as the required fitness functions to achieve that at a measurement time $t_m$. With $|\Phi^{+}_{LR}\rangle$ as the initial state if $a1 = 1$ then a $Z$ gate is applied to the charge qubit, similarly if $a2 = 1$ then a $X$ gate is applied. These transformations are the Pauli matrices [16] as $X = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ with $Y = ZX$ on the site subsystem. These gates are achieved with the GA control applied to our model using pulse, position, width and amplitudes as chromosomes and basis of the control parameter space. Once the transformation has been applied, Alice's qubit is sent to Bob. Bob can recover the two classical bits of information with the inverse process of entangling to decode the Bell-like state. This is achieved with the applications of a $CNOT$ and a Haddamard $H$ gates. The $CNOT = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$ is a two qubit gate, and $H = (1/\sqrt{2})(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 0|)$ creates superpositions of single qubit states.

The fitness function is composed with the fidelity of each desired state plus the concurrence $C$ added as a measure of the degree of entanglement. The fidelity is simply $|\langle\varphi|\Psi(t = t_m)\rangle|^2$, where $|\varphi\rangle$ is any of the hybrid Bell-like states. Concurrence is used as a measure of entanglement for two qubits systems [17]. In (2) basis, the concurrence can be calculated as $C = 2|\alpha_1\alpha_4 - \alpha_2\alpha_3|$. This quantity has been added to the fitness function to ensure that the final states be maximally entangled.

## 2. Quantum Control with GA

To implement QCS in the model shown in Fig. 1 all the states in (3) must be accessible. Different state preparations are needed, one for the HES, and the other four for the QSC protocol transformations. The quantum control procedure to prepare all the required states is described as 4 steps:

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**Table 1.** Quantum superdense coding to send two classical bit $(a1, a2)$, $a1 = \{0, 1\}$, $a2 = \{0, 1\}$, the unitary operations and fitness functions $f$ required for each two classical bits.

| *(a1, a2)* | Unitary operations | Final state | Fitness Function |
|---|---|---|---|
| 00 | $I_A \otimes I_B$ | $\left\|\Phi^+_{LR}\right\rangle$ | $f_{\Phi_{LR}} = \|\langle\Phi^+_{LR}\|\Psi(t_m)\rangle\|^2 + C$ |
| 01 | $X_A \otimes I_B$ | $\left\|\Psi^+_{LR}\right\rangle$ | $f = \|\langle\Psi^+_{LR}\|\Psi(t_m)\rangle\|^2 + C$ |
| 10 | $Z_A \otimes I_B$ | $\left\|\Psi^-_{LR}\right\rangle$ | $f = \|\langle\Psi^-_{LR}\|\Psi(t_m)\rangle\|^2 + C$ |
| 11 | $Y_A \otimes I_B$ | $\left\|\Phi^-_{LR}\right\rangle$ | $f = \|\langle\Phi^-_{LR}\|\Psi(t_m)\rangle\|^2 + C$ |

(i) The system and the control Hamiltonians that contain the space of parameters (chromosomes) to be optimized are defined. This two level system is simply described as,

$$H(t) = H_0 + H_S + H_C(t, b), \tag{4}$$

where $H_0$ represents the sites Hamiltonian, containing site energies and tunneling independent of spin, $H_S$ is the spin Hamiltonian with the spin-orbit effect and $H_C(t, \mathbf{b})$ is the Control Hamiltonian, which includes the $\mathbf{b}$ *chromosomes* described above as the arguments of the control functions of the pulse structure. Each Hamiltonian has the following structure,

$$H_0 = \begin{bmatrix} E_L & t_0 & 0 & 0 \\ t_0 & E_R & 0 & 0 \\ 0 & 0 & E_L & t_0 \\ 0 & 0 & t_0 & E_R \end{bmatrix}, \tag{5}$$

where $E_L$ and $E_R$ are left and right site energies respectively, and $t_0$ is a spin-independent tunneling constant,

$$H_S = \begin{bmatrix} 0 & 0 & \gamma|k|E_L e^{i\phi} & \gamma|k|t_0 e^{i\phi} \\ 0 & 0 & \gamma|k|t_0 e^{i\phi} & \gamma|k|E_R e^{i\phi} \\ \gamma|k|E_L e^{-i\phi} & \gamma|k|t_0 e^{-i\phi} & 0 & 0 \\ \gamma|k|t_0 e^{-i\phi} & \gamma|k|E_R e^{-i\phi} & 0 & 0 \end{bmatrix}, \tag{6}$$

where $\gamma$ is a material-dependent constant given by Dresselhaus spin-orbit effect and $|k|e^{i\phi}$ is the electron wave-vector as $k = k_x + ik_y$.

$$H_C(t, b) = \begin{bmatrix} \sigma^+_{LR}(t, b) & t_{LR}(t, b) & 0 & 0 \\ t^*_{LR}(t, b) & \sigma^-_{LR}(t, b) & 0 & 0 \\ 0 & 0 & \sigma^+_{LR}(t, b) & t_{LR}(t, b) \\ 0 & 0 & t^*_{LR}(t, b) & \sigma^-_{LR}(t, b) \end{bmatrix}. \tag{7}$$

$H_C(t, b)$ is formed by the control parameters defined as the time dependent pulses

$$\begin{aligned} t_{LR}(t, b) &= b_{LR} e^{-\frac{(t - t_{LR})^2}{2\tau^2_{LR}}} - t_0 \tanh \frac{(t - t_{LR})^2}{2\tau^2_{LR}}, \\ \sigma^+_{LR}(t, b) &= b\sigma_{LR} e^{-\frac{(t - t\sigma_{LR})^2}{2\tau\sigma^2_{LR}}} - t_0 \left( \tanh \frac{(t - t\sigma_{LR})^2}{2\tau\sigma^2_{LR}} - 1 \right), \\ \sigma^-_{LR}(t, b) &= -\sigma^+_{LR}(t, b). \end{aligned} \tag{8}$$

where the widths, positions and amplitudes are $\mathbf{b} = (b_{LR}, b\sigma_{LR}, t_{LR}, t\sigma_{LR}, \tau_{LR}, \tau\sigma_{LR})$ These parameters are constructed in a way that $t_{LR}(t, b)$ and $\sigma^+_{LR}(t, b)$ can change the spin independent tunneling and site energies, respectively. The control parameters that are varied by GA is the vector $\mathbf{b}$. The control pulses correspond to charge subsystem that can be controlled with electric fields directly applied to the QD configuration. This is because we are not considering magnetic fields, needed if the spin subsystem was manipulated. $t_{LR}(t,b)$ is a Gaussian-shaped tunneling control pulse that includes a hyperbolic function to cancel all transition probabilities out of control pulse action time. $\sigma^{+}_{LR}(t,b)$ is a Gaussian-shaped pulse too, where the term $t_0$ has been added just for consistency, this pulse changes the energy in sites left and right anti-symmetrically.

(ii) The fitness function, depends on the final desired state and will be explicitly defined for every step of the QSC as shown in Table 1. All the objetive or fitness function is to maximize the probability of finding the system in every hybrid Bell basis state which by redundancy has maximum concurrence,

(iii) Integration of the time dependent wave equation up to time of measurement $t_m$ and obtain $|\Psi(t_m)\rangle|$.

(iv) Genetic algorithm application to find the optimal control parameters. Some assumptions are made for the calculations: a $\gamma = 24eV\mathring{A}$ is used corresponding to GaAs [12]. Tunneling and spin-orbit parameters are defined as $t_0 = 1meV, |k| = 0.34, \phi = \pi/5, \hbar = 1$ what makes time to have arbitrary units ($au$ in the text), in what follows all energy units are considered in mili-electron volt (meV) and is used $\langle k_z^2\rangle = (\pi/\omega)2 = 24.6nm^{-2}$ [18], considering the quantum dot wide as $\omega = 20nm$. PIKAIA was set to 50 generations and default settings, which include 100 individuals, crossover probability of 0.85 and mutation variable between 0.005 and 0.25.

## 3. Results

### *3.1. Hybrid Entangled State preparation*

As a first step, the hybrid entangled state needs to be created as an information channel. The final desired state is,

$$|\Phi^{+}_{LR}\rangle = \frac{1}{\sqrt{2}}(|\Phi_L\uparrow\rangle + e^{i\pi/4}|\Phi_R\downarrow\rangle), \tag{9}$$

i.e., for the state preparation, as described in Table 1 the fitness function is $f_{\text{fit}} = |\langle\Phi^{+}_{LR}|\,\psi(t=t_m)\rangle|^2 + C$. Considering the initial state $|\psi(t=0)\rangle = |\Phi_L\uparrow\rangle$, a possible solution can be found with GA, given the defined control parameters is $\mathbf{b} = (7.209, 7.652, 310, 63, 64, 250)$ for (8) with a fidelity of 0.9972. This process can be easily understood from Fig. 2. In this figure, the time evolution of the Bell-like states probabilities is shown, together with the shape of the time dependent control pulses (inset), with the parameters described above, applied to obtain the final state (9).

Once the maximum probability of being in $|\Phi^{+}_{LR}\rangle$ is reached, we se that it is maintained, and all the others states probabilities drop to zero. By analyzing the dynamics of the probability plots, we obtain an equivalent quantum circuit of this process, which is illustrated in Fig. 3. The X and Z rotations are generated by the action of both the tunneling and site energy $t_{LR}(t,b)$ and $\sigma^{\pm}_{LR}(t,b)$, respectively. The $CNOT$ gate is accomplished by the combination of SO effects with $t_{LR}(t,b)$ together. Corresponding to the control pulses, the action of a partial Z gate is first seen by promoting a unbalance of site energies $\sigma^{\pm}_{LR}(t,b)$. In a similar way, the tunneling pulse starts at about 100 time units, and the action of an X gate with a phase could be deduced. The joint effect of $\mathrm{e}^{-i\frac{\pi Z}{4}}\,\mathrm{e}^{-i\frac{\pi X}{2}}$ is equivalent to a Hadamard gate with a phase. After that, the action of a two qubit interaction( the SO term) is depicted when control pulses are going down. All these actions represent a synthesis of the probabilities and complex amplitudes of each Bell-like state and can not be clearly seen because of the joint action of different interactions with evolving rations between them. However, the quantum circuit shows a generalization of Bell

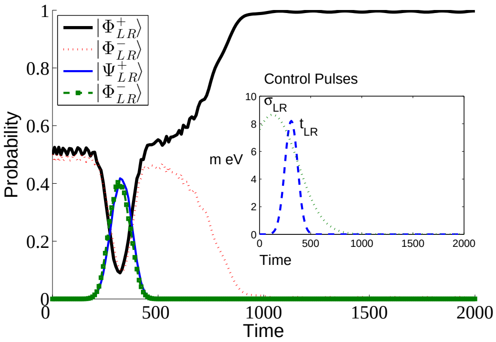


**Figure 2.** Bell-like probability evolutions for the hybrid entangled state $|\Phi^{+}_{LR}\rangle$ preparation with a fidelity of 0.997. The control pulses are presented in the inset.

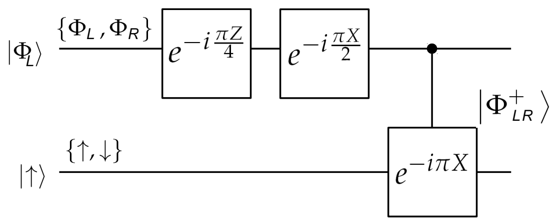

**Figure 3.** Quantum circuit for the HES $|\Phi^{+}_{LR}\rangle$ preparation with the initial condition $|\Phi_L\uparrow\rangle$

state creation operations that require only a Hadamard and a CNOT [16] but with different phases that arise because of SO effect.

### 3.2. Quantum Superdense Coding operations

We describe now, the physical process for the protocol for the QSC. After the preparation of the HES (9), the system requires the ability to achieve the 4 Bell hybrid basis states to accomplish QSDC of sending the four classical two bits. QSDC is implemented as shown in Fig. 4. This figure summarizes the operations in Table 1. The center of this figure represents the initial Bell-like HES that was previously prepared. From this initial state the four Bell-like HES can be easily accessed by proper pulse shapes. The four figures show the Bell-like time evolution probabilities through the application of the control pulses shown in the corresponding insets. Each operation is accompanied with the corresponding quantum circuit obtained from the time evolution parameters. One set of parameters is found to accomplish each of the states and are described in Table 1 by the control pulses (8). The structure and parameters of the control pulses for each two bit state to be sent are detailed as follows.

To send the two bit 00 no control pulses are needed, because the system is already in the final desired state and the quantum circuit is only an identity operation. This is shown in Fig. 4(a), as no alteration is made, fidelity is 1. To send 01, the control pulse obtained with GA is shown in Fig. 4(b). The corresponding control parameters are $\mathbf{b} = (0.7851, 0, 745, 0, 388, 0)$, and the quantum circuit shows the $X$ gate is applied with a global phase $\pi$, this is because only the tunneling control pulse is present. The fidelity achieved is of 0.998; to send 10 the control parameters obtained are $\mathbf{b} = (1.2039, 0.5766, 20, 992, 101, 496)$ as shown in Fig. 4(c) with a fidelity of 0.998. In this case the quantum circuit has a dominant part of the $Z$ gate, the control pulses show the tunneling present for a small amount of time, which promotes the phase change needed to accomplish the final state in a more complex way that only the $Z$ gate, while the site energies are dominant in the control pulses.

Finally, to send 11, the parameters of the control pulses of Fig. 4(d) are $\mathbf{b} = (1.7116, 1.9260, 997, 222, 249, 284)$ with a fidelity of 0.9838. The quantum circuit implemented shows $Z$ and $X$ rotations together that are equivalent to a $Y$ rotation, these are present because both the tunneling and the site energies are present for a time interval as seen in the control pulses in the figure. The two gates operate at different times, when site energies decrease tunneling increases in the order shown in the quantum circuit.

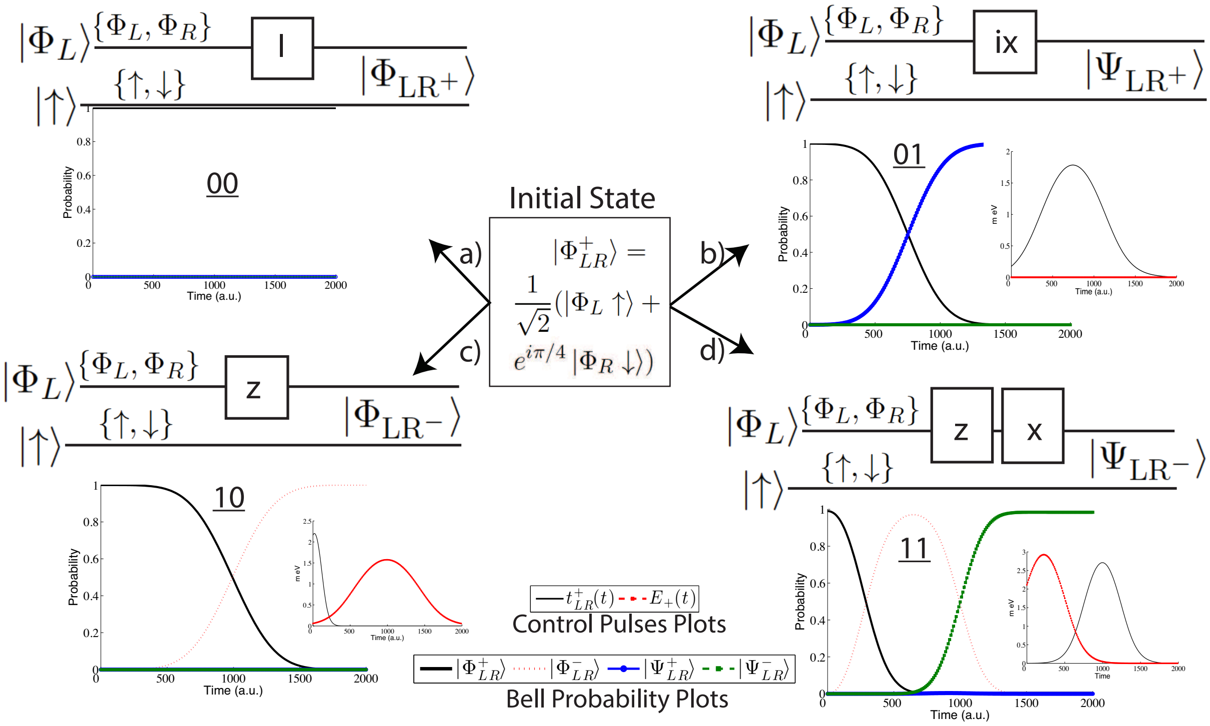

**Figure 4.** Quantum Superdense Coding scheme with the hybrid spin-charge system and control pulses with the initial $|\Phi^+_{LR}\rangle$. Each subfigure shows how to transmit the two bit basis (transform to the final hybrid Bell-like state) (a) 00 ($\Phi^+_{LR}$), only the identity operation is needed, (b) 01 ($|\Psi^+_{LR}\rangle$) with an $X$ gate with phase, obtained with the tunneling pulse shown, (c) 10 ($\Phi^-_{LR}$) the Z gate is emulated with site energies disbalance and phase adjusted with tunneling, (d) 11 ($|\Psi^-_{LR}\rangle$) Z and X gates are needed, achieved by the action of site energies unbalance and tunneling.

## 4. Conclusions

We have presented an example of how GA can be applied in quantum information processing. We implemented in a physical model the QSC protocol, with a double QD system using a spin-orbit entangled stated as a communication channel. Given the nature of the QD model with only one electron, hybrid states between charge and spin states arose naturally.

A maximally entangled hybrid Bell-like state was generated from an initial state, controlling the dynamics, using the GA approach.

The control parameters of the system were Gaussian shaped time dependent site energies and transition probabilities between sites.

The four unitary operations required to send the two bits of classical information $\{00, 01, 10, 11\}$, which transform a given Bell-like state to the other three, were performed with large fidelities or fitness. The GA parameters used allowed a fast convergence to the solution within just two minutes of calculation time. This work illustrates how straightforward a physical quantum system can be controlled using the GA approach to carry out a quantum information task.

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[1] Dong D and Petersen I R 2009 (*Preprint* `0910.2350`) URL `http://arxiv.org/abs/0910.2350`

[2] D'Alessandro D 2007 *Introduction to Quantum Control and Dynamics* (CRC Press) ISBN 1584888830
[3] Xue D, Zou J, Li J G, Chen W and Shano B 2010 *J. Phys. B* **43** 45503
[4] Delgado F 2010 *Phys. Rev. A* **81** 42317
[5] Branczyk A, Mendonça P, Gilchrist A, Doherty A and Bartlett S 2007 *Phys. Rev. A* **75** 12329
[6] Krause J, Reitze D, Sanders G, Kuznetsov A and Stanton C 1998 *Phys. Rev. B* **57** 9024–9034
[7] Giraldi G A, Portugal R and Thess R N 2004 (*Preprint* `0403003`) URL `http://arxiv.org/abs/cs/0403003`
[8] Zingg D W, Nemec M and Pulliam T H 2008 *Revue européenne de mécanique numérique* **17** 103–126 ISSN 17797179 URL `http://www.tandfonline.com/doi/abs/10.3166/remn.17.103-126`
[9] Holland J C 1975 *Adaptation in Natural and Artificial Systems* primera ed (Michigan, US: The U. of Michigan Press)
[10] Bennett C H and Wiesner S J 1992 *Phys. Rev. Lett.* **69** 2881–2884
[11] Dresselhaus G 1955 *Phys. Rev* **100** 580–586
[12] Perel V I, Tarasenko S A, YAssievich I N, Ganichev S A, Bel'kov V V and Prettl W 2003 *Phys. Rev. B* **67** 201304
[13] Carbonneau P 1995 *The Astrophysical Journal Supplement Series* **101** 309–334
[14] Navarro-Muñoz J C, Rosu H C and López-Sandoval R 2006 *Physical Review A* **74** 52308
[15] Carbonneau P and Knapp B 1995 *A user's guide to PIKAIA 1.0* 1st ed (Boulder, Colorado, US: High Altitude Observatory.National Center for Atmospheric Research)
[16] Nielsen M and Chuang I L 2000 *Quantum Computation and Quantum Information* 1st ed (Cambridge University Press)
[17] Hill S and Wootters K 1997 *Phys. Rev. Lett.* **78** 5022–5025
[18] Studer M, Walser M P, Baer S, Rusterholz H, Schön S, Schuh D, Wagscheider W, Enssin K and Salis G 2010 *Cond. Mat. Hall* 1–7