Quantum self-controlling free-falling cats

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Abstract. In the separation of rotations from internal motions in the n-body problem, there appear some gauge fields which physically represent Coriolis effects. These fields are also present in the "falling cat" problem: at the kinematical level they map changes in the cat's shape to changes in its orientation whereas at the dynamical level they show up as gauge potentials in the Hamiltonian. Classically, the vanishing angular momentum condition allows for the orientation degrees of freedom to decouple from the internal ones and the cat's re-orientation can be accounted for at the kinematical level, partially. In the quantum case the cat's re-orientation description requires to be done on dynamical grounds. In this paper we explore the quantum version of the falling cat modelled as a three body problem.

1. Introduction

As it is familiar to almost everyone from everyday life experience, a free falling cat lands on its feet more often than not, regardless of its initial orientation and in spite of the fact that it might have zero angular momentum and nothing to push against. This phenomenon is counter-intuitive as it may seem to violate conservation of angular momentum, however, as it turns out cats can change their orientation by changing its shape precisely *because* of conservation of angular momentum [1–3]. In the description of the cat's reorientation, it is particularly useful the use of geometrical methods applied to the n-body problem [4].

Now, let us imagine that our cat is somehow "promoted" to a quantum cat (*i.e.*, a cat following the laws of quantum mechanics) and that it is freely released at a given height. At this point the cat may understandably feel quite uneasy and it may feel the need to consult an expert on quantum control, say professor Bogdan Mielnik [5–8]. Can the quantum cat control its muscles so that it lands on its feet as its classical analogue?

In this work we focus on the analysis of the above question in terms of geometrical methods. The structure of the paper is as follows: in Section 2 we introduce notation and discuss the general case of n-body problem in terms of geometric language, which follows closely Ref. [9], in Section 3 we present the main results of our work through the simplified case of a body composed only of three point-like masses. We give a brief conclusion in Section 4.

2. A cat as an n-body

Let us assume the cat is described by an isolated deformable body modelled by a system of n point-like particles interacting only between themselves through a potential V. Clearly, the body's configuration space is $\mathcal{C}_{\rm tot} = \mathbb{R}^{3n}$. If $\mathbf{r}_{s,\alpha} \in \mathbb{R}^3$ represents the components of α-th particle's position vector with respect to an inertial or space frame (indicated by the s sub-index) and m_α its mass ($\alpha = 1, 2, \ldots, n$), then the Lagrangian of the system is given by:

$$L_{\rm tot} = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha |\dot{\mathbf{r}}_{s\alpha}|^2 - V(\mathbf{r}_{s,1}, \ldots, \mathbf{r}_{s,n}), \tag{1}$$

where "$\cdot$" $= d/dt$. Since there are not external forces acting on the system, it is convenient to eliminate the translational degrees of freedom by introducing relative coordinates, *e.g.*, the mass-weighted Jacobi coordinates $\boldsymbol{\rho}_{s\alpha}$, $\alpha = 1, \ldots, n-1$. In this way, the Lagrangian can be written as $L_{\rm tot} = L_{\rm CM} + L$, with $L_{\rm CM} = M|\dot{\mathbf{R}}_s|^2/2$ and L given by:

$$L = \frac{1}{2}\sum_{\alpha=1}^{n-1} |\dot{\boldsymbol{\rho}}_{s\alpha}|^2 - V(\boldsymbol{\rho}_{s,1}, \ldots, \boldsymbol{\rho}_{s,n-1}). \tag{2}$$

Likewise, in these coordinates it is clear that the total configuration space can be written as $\mathcal{C}_{\rm tot} = \mathbb{R}^3 \times \mathcal{C}$, where $\mathcal{C} = \mathbb{R}^{3n-3}$ is the translation-reduced configuration space upon which $(\boldsymbol{\rho}_{s,1}, \ldots, \boldsymbol{\rho}_{s,n-1})$ are coordinates. Since the dynamics of the center of mass is trivial, without loss of generality we can assume that $\mathbf{R}_s = 0$ for the rest of the paper and refer to $\mathcal{C}$ simply as *the* configuration space. A point in $\mathcal{C}$ is defined by an n-body's shape and an orientation. There are $3n-6$ coordinates needed to specify the body's shape q^μ, which correspond to independent functions on configuration space invariant under proper rotations, *i.e.*, $q^\mu(\boldsymbol{\rho}_{s,1}, \ldots, \boldsymbol{\rho}_{s,n-1}) = q^\mu(\mathsf{Q}\boldsymbol{\rho}_{s,1}, \ldots, \mathsf{Q}\boldsymbol{\rho}_{s,n-1})$ for all $\mathsf{Q} \in SO(3)$. The body's orientation, on the other hand, is defined by *a)* a convention of body frame given by the relations

$$\boldsymbol{\rho}_\alpha = \boldsymbol{\rho}_\alpha(q^\mu), \qquad \alpha = 1, \ldots, n-1, \tag{3}$$

and *b)* the orientation coordinates in $SO(3)$ defining a rotation $\mathsf{R}(\theta^i)$ that maps the body frame into the space frame. In the above relations we introduce the convention that a vector appearing without sub-index s is referred to the body frame. The convention of body frame is a choice of gauge, it consists of specifying the reference orientation in which both, the body and the space frames coincide, and it is defined by smoothly (and otherwise arbitrarily) attaching a frame to the body for each shape. For the non-collinear configurations considered in this work, this separation between rotations and internal motions allows us to regard $\mathcal{C}$ as an $SO(3)$ principal fibre bundle, where the base space is the "shape space" $SS = \mathbb{R}^{3n-3}/SO(3)$ and the fibre is isomorphic to $SO(3)$. In this geometric language the choice of gauge (3) defines a section $\mathcal{S}$, *i.e.*, a transversal embedding of shape space in the configuration space $\mathcal{C}$.

In terms of these shape and orientation coordinates, a point in $\mathcal{C}$ is defined according to:

$$\boldsymbol{\rho}_{s,\alpha} = \mathsf{R}(\theta^i)\boldsymbol{\rho}_\alpha(q^\mu), \qquad \alpha = 1, \ldots, n-1. \tag{4}$$

The above relation expresses the fact that given a shape of the body q^μ and a reference orientation $\boldsymbol{\rho}_\alpha(q^\mu)$, any configuration of the system can be reached through a rotation $\mathsf{R}(\theta^i)$. Similarly a velocity vector on the configuration's tangent space can be expressed in the angular velocity and shape anoholonomic basis as $v^a = (\boldsymbol{\omega}, \dot{q}^\mu)$, where $\omega^i = -\epsilon^{ijk}(\mathsf{R}^T \cdot \dot{\mathsf{R}})_{jk}/2$ is the i-th component of angular velocity of the body frame w.r.t. the space frame, referred to the body frame as indicated by the absence of subscript s. In the previous expression and in what follows we assume Einstein summation convention for repeated indices. Accordingly, the Lagrangian (2) can be recast as:

$$L = \frac{1}{2}G_{ab}v^a v^b - V(q), \tag{5}$$

where

$$G_{ab} = \begin{pmatrix} \mathsf{M} & \mathsf{M}\mathbf{A}_\nu \\ \mathbf{A}_\nu^T \mathsf{M} & g_{\mu\nu} + \mathbf{A}_\mu \cdot \mathsf{M} \cdot \mathbf{A}_\nu \end{pmatrix}, \tag{6}$$

is the metric in configuration space $\mathcal{C}$ and

$$\mathsf{M} = \sum_{\alpha=1}^{n-1} \boldsymbol{\rho}_\alpha \otimes \boldsymbol{\rho}_\alpha - |\boldsymbol{\rho}_\alpha|^2 \, \mathsf{I}, \tag{7}$$

$$\mathbf{A}_\mu = \mathsf{M}^{-1} \sum_{\alpha=1}^{n-1} \boldsymbol{\rho}_\alpha \times \frac{\partial \boldsymbol{\rho}_\alpha}{\partial q^\mu}, \tag{8}$$

$$g_{\mu\nu} = \sum_{\alpha=1}^{n-1} \frac{\partial \boldsymbol{\rho}_\alpha}{\partial q^\mu} \times \frac{\partial \boldsymbol{\rho}_\alpha}{\partial q^\nu} - \mathbf{A}_\mu \cdot \mathsf{M} \cdot \mathbf{A}_\nu, \tag{9}$$

are the inertia tensor, the Coriolis gauge potential and the metric on shape space, respectively.

A velocity vector of the form $v^a = (\boldsymbol{\omega}, 0)$ is purely rotational or *vertical* since $\dot{q}^\mu = 0$ and thus the body's shape is not changing. *A priori* there is not a canonical way to define a purely internal velocity vector, because the condition $\boldsymbol{\omega} = 0$ is not gauge independent and therefore it cannot be physically meaningful. However, we can define a vector to be purely internal or *horizontal* if it is orthogonal to all vertical vectors w.r.t. the metric G_{ab} introduced in (6). As it turns out, a vector is horizontal iff its associated angular momentum vanishes. In terms of shape and orientation coordinates, the angular momentum (referred to the body frame) can be written as:

$$\mathbf{L} = \mathsf{M} \cdot (\boldsymbol{\omega} + \mathbf{A}_\mu \dot{q}^\mu), \tag{10}$$

and thus, a vector is horizontal if $\boldsymbol{\omega} dt = -\mathbf{A}_\mu dq^\mu$. In this manner it can be observed that the gauge potential $\mathbf{A}_\mu$ maps infinitesimal changes in shape space to infinitesimal rotations under the condition of vanishing angular momentum. When the body undergoes a finite deformation $q^\mu(t)$ subjected to the condition $\mathbf{L} = 0$, the previous relation can be integrated such that the body and the space frames are related through the rotation:

$$\mathsf{R}(t) = \mathsf{R}_0 \mathcal{P} \exp \left(- \int_{q_0}^{q(t)} \mathsf{A}_\mu dq^\mu \right), \tag{11}$$

where R_0 is a rotation relating both frames at $t = 0$, $\mathcal{P}\exp$ is the path-ordered exponential and the antisymmetric matrix A_μ is associated with the gauge potential according to $(\mathsf{A}_\mu)_{ij} = -\epsilon_{ijk} A_\mu^k$. Of course, in general this rotation is gauge-dependent and not meaningful from the physical viewpoint, nevertheless for closed paths – *i.e.*, for cyclic deformations of the body's shape – it actually describes a change of the body's orientation. In fact, just as in any Yang-Mills gauge theory, it is possible to define an associated curvature form or Coriolis tensor

$$\mathbf{B}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - \mathbf{A}_\mu \times \mathbf{A}_\nu, \tag{12}$$

such that a cyclic deformation in shape space under conditions of vanishing angular momentum, represented by the infinitesimal parallelogram spanned by the vectors y^μ and z^μ, produces the gauge-invariant infinitesimal rotation $\boldsymbol{\omega} dt = -\mathbf{B}_{\mu\nu} y^\mu z^\nu$. We then notice that in order to get a non trivial rotation, we need both the enclosed area by the body's deformation path in shape space and the curvature tensor to be different from zero.

The dynamics of the system can also be described by means of the gauge-covariant Hamiltonian:

$$H = \frac{1}{2} \mathbf{L} \cdot \mathsf{M} \cdot \mathbf{L} + \frac{1}{2} (p_\mu - \mathbf{A}_\mu \cdot \mathbf{L}) g^{\mu\nu} (p_\nu - \mathbf{A}_\nu \cdot \mathbf{L}) + V(q), \tag{13}$$

where $p_\mu = g_{\mu\nu}\dot{q}^\nu + \mathbf{A}_\mu \cdot \mathbf{L}$ is the momentum conjugate to shape coordinate q^μ. The first and second terms in previous expression correspond to the vertical and horizontal contributions to the kinetic energy, respectively. From the Hamiltonian (13) we immediately note that if $\mathbf{L} = 0$, then the shape and orientation degrees of freedom decouple and we can independently solve for $q^\mu = q^\mu(t)$ and then plug it into expression (11) to obtain the orientation trajectory in $SO(3)$. Alternatively, one could assume that the body deformation is controlled externally so that by conveniently tailoring a cyclic path $q^\mu(t)$ in shape space, the body would get rotated by (11) as desired. Even though the latter scenario may be interesting for applications in quantum control theory, the system in that case is not longer isolated as we initially assumed.

Clearly the condition $\mathbf{L} = 0$ can be imposed in the classical context in a straightforward way, but in the quantum realm such a condition is too restrictive since it would imply that the system's state is the $l = 0$ angular momentum's eigenstate, in which the dispersion of the orientation angle operators is infinite. Thus, a more sensible condition in the quantum regime would be that the expectation value of $\mathbf{L}$ vanished. Therefore, if we are interested in describing a net rotation of a quantum deformable body produced by a cyclic change of its shape, we need to find the eigenvectors of the corresponding Hamiltonian operator at some point, but since M, $\mathbf{A}_\mu$ and $g^{\mu\nu}$ are operators themselves, the problem may be quite difficult. Indeed, since (13) commutes with $\mathbf{L}^2$ and L_{sz}, we know that its eigenfunctions are also eigenfunctions of $\mathbf{L}^2$ and L_{sz} with eigenvalues l and m, *i.e.*:

$$\psi_{lm}(\mathsf{R}, q^\mu) = \sum_{k=-l}^{l} \chi_k^l(q) D_{mk}^l(\mathsf{R})^*, \tag{14}$$

where $D_{km}^l(\mathsf{R})$ are the Wigner functions corresponding to the $(2l+1) \times (2l+1)$ irreducible matrix representation of $\mathsf{R} \in SO(3)$, and $\chi_k^l(q)$ is the gauge-dependent wavefuntion in the section $\mathcal{S}$. The previous expression tells us that for eigenstates of the system, its wavefunction for any orientation R can be obtained from the wavefunction in the section $\chi_k^l(q^\mu)$ through a rotation, and then in some sense it is the quantum version of relation (3). In this way, to solve the eigenvalue problem accounts to diagonalize an $(2l+1) \times (2l+1)$ matrix of operators in shape space, which is rather complicated in general. To simplify things a bit, we will consider quasi-rigid deformable bodies.

Let us now assume that the body consists of a system of n coupled harmonic oscillators and let us restrict ourselves to the analysis of small vibrations around a non-collinear equilibrium configuration q_0^μ, so that $q^\mu = q_0^\mu + \lambda x^\mu$, where λ is a small ordering parameter representing the ratio of deformation length scale to the equilibrium configuration length scale. We further choose x^μ to be both Riemann normal coordinates and normal modes of the potential, so that the potential can be written as $V(x) = \omega_{(\mu)}^2 x^\mu x_\mu / \lambda^2$, where we have implicitly assumed that $\partial_\mu^2 V = \omega_{(\mu)}^2/\lambda^4$. On the other hand, to further simplify things we use Poincaré gauge, $\mathbf{A}_\mu x^\mu = 0$, that is, in this gauge it holds $\mathbf{A}_\mu(0) = 0$. As it has been previously showed by Littlejhon and Mitchell [10], Poincaré gauge can be geometrically interpreted as choosing the gauge so that the section is the horizontal lift of radial lines emerging from the equilibrium configuration, which of course depends on the coordinates used in shape space. If Riemann coordinates are chosen, then the section is flat and it coincides with the Eckart frame. Finally, we scale $p_\mu \to p_\mu/\lambda$, $H \to \lambda^2 H$, $\mathsf{M} \to \lambda^2 \mathsf{M}$, $\mathbf{A}_\mu \to \lambda \mathbf{A}_\mu$ and $\mathbf{L} \to \mathbf{L}$, and write $R_{\mu\alpha\beta\nu}$ in terms of M and $\mathbf{B}_{\mu\nu}$ [1], so that the Hamiltonian (13) up to $O(\lambda^2)$ can be expanded as

$$H = \frac{1}{2}\left(p^\mu p_\mu + \omega_{(\mu)}^2 x^\mu x_\mu\right) + \frac{1}{2}\lambda^2 (\mathbf{L} - \mathbf{S}) \cdot \mathsf{M}^{-1} \cdot (\mathbf{L} - \mathbf{S}), \tag{15}$$

[1] Namely, $2R_{\mu\nu\sigma\tau} = \mathbf{B}_{\mu\nu} \cdot \mathsf{M} \cdot \mathbf{B}_{\sigma\tau} + \mathbf{B}_{\mu[\tau} \cdot \mathsf{M} \cdot \mathbf{B}_{\sigma]\nu}$ *cf.* relation (5.61) from [9].

where M^{-1} in the above expression is evaluated at q_0^μ and we have introduced the "internal angular momentum" $\mathbf{S}$ and "shape angular momentum" S:

$$\mathbf{S} = \frac{1}{4}\mathsf{M}\cdot\mathbf{B}_{\mu\nu}S^{\mu\nu}, \quad S^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu. \tag{16}$$

The only term in the Hamiltonian (15) that couples the shape and orientation degrees of freedom is the crossed one, which is of the form $\mathbf{L}\cdot\mathsf{M}^{-1}\cdot\mathbf{S}$, and since $\mathbf{L}$ is the generator of rotations, when acting on the orientation sector of the wavefunction, it can be interpreted as an infinitesimal rotation about the unit vector $\widehat{\mathsf{M}^{-1}\cdot\mathbf{S}}$ by an angle $|\mathsf{M}^{-1}\cdot\mathbf{S}|$, which are both operators in shape space. At this point we are ready to show that a change of its shape induces a rotation of a body as the dynamical process generated by the Hamiltonian (15).

3. Three-body case

As it turns out, to "see" the cat rotating while it changes its shape, it is enough to model it as a three-body problem. In this case the shape space has dimension $3n-6=3$ and we can define $n-1=2$ Jacobi vectors as:

$$\boldsymbol{\rho}_{s1} = \sqrt{\mu_1}\left(\mathbf{r}_{s1}-\mathbf{r}_{s3}\right), \tag{17}$$
$$\boldsymbol{\rho}_{s2} = \sqrt{\mu_2}\left(\mathbf{r}_{s2}-\mathbf{R}_{s,13}\right), \tag{18}$$

where $\mu_1 = m_1m_3/m_{13}$, $\mu_2 = m_2m_{13}/M$ and $\mathbf{R}_{s,13}$ is the center of mass of subsystem $m_{13}=(m_1+m_3)$. A convenient set of coordinates in shape space is:

$$\begin{aligned} q_1 &= |\boldsymbol{\rho}_{s1}|^2 - |\boldsymbol{\rho}_{s2}|^2,\\ q_2 &= 2\boldsymbol{\rho}_{s1}\cdot\boldsymbol{\rho}_{s2},\\ q_3 &= 2|\boldsymbol{\rho}_{s1}\times\boldsymbol{\rho}_{s2}| \geq 0, \end{aligned} \tag{19}$$

which makes it clear that the shape space is one half of $\mathbb{R}^3$. The q_3 coordinate measures the area of the body, and the plane $q_3=0$ then corresponds to collinear shapes (which have zero area) while the q_3 axis contains symmetric shapes, *i.e.*, the shapes whose inertia tensor is degenerate in the body xy plane.

For the sake of concreteness, let us adopt now the north regular gauge [9, 11] which is also an Eckart (and Poincaré) gauge for the equilibrium shape $q_0^\mu = (0,0,1)$, which corresponds to the configuration given by one mass located at each vertex of an equilateral triangle with edges of unit length. If, moreover, we consider small oscillations around equilibrium q_0^μ, *i.e.*, $q^\mu = q_0^\mu + \lambda x^\mu$, such that x^μ are Riemann and normal modes of the potential, then we can apply the the quasi-rigid bodies results obtained in the previous Section 2. Normal modes x^μ are related to the coordinates q^μ as defined in (19) according to:

$$\begin{aligned} q_1 &= -\lambda(x_1+\sqrt{2}x_2),\\ q_2 &= -\lambda(x_2-\sqrt{2}x_1),\\ q_3 &= (1-2\lambda x_3), \end{aligned} \tag{20}$$

where x_3 corresponds to the breathing mode of higher frequency $\omega_3=\sqrt{2}$ (in natural units), while x_1 and x_2 are degenerate orthogonal modes of frequency $\omega_1=\omega_2=1$.

The moment of inertia tensor for the equilibrium configuration is

$$M_{ij} = \begin{pmatrix} 1/2 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1 \end{pmatrix}, \tag{21}$$

the only non trivial component of Coriolis tensor is $\mathbf{B}_{12} = 2\hat{\mathbf{z}}$ and therefore the "internal angular momentum" reduces to $\mathbf{S} = S_{12}\hat{\mathbf{z}}$, where $S_{12} = x_1p_2 - x_2p_1$ is the angular momentum in the plane x_1x_2 of shape space. That is, the fibre bundle in this case is curved only on the plane x_1x_2, which from the physical point of view can be understood because only an out-of-phase combined excitation of modes x_1 and x_2 can generate internal angular momentum.

Then, the Hamitonian takes the form:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + x_1^2 + x_2^2\right) + \frac{1}{2}\left(p_3^2 + 2x_3^2\right) + \lambda^2\left(\mathbf{L}^2 - \frac{1}{2}L_z^2 - L_zS_{12} + \frac{1}{2}S_{12}^2\right). \tag{22}$$

As claimed, in the classical case the condition of vanishing angular momentum makes it possible to find the solutions of the equations of motion for shape coordinates independently of the orientation ones. From the previous Hamiltonian (22), it can be realized that $\dot{S}_{12} = 0$ and so S_{12} is a constant of motion. In this case $x^\mu = x^\mu(t)$ correspond to harmonic oscillations to leading order in λ:

$$x^\mu(t) = x_0^\mu \cos\omega_\mu t + \frac{p_0^\mu}{\omega_\mu}\sin\omega_\mu t, \tag{23}$$

with frequency $\omega_\mu = (1, 1, \sqrt{2})$ and $p^\mu(t) = \dot{x}^\mu(t)$. Since S_{12} is a constant of motion and the gauge field $\mathbf{A}_\mu$ is pseudo Abelian in this case, relation (11) yields:

$$R_{ij}(t) = \begin{pmatrix} \cos\left(\lambda^2 S_{12}t\right) & \sin\left(\lambda^2 S_{12}t\right) & 0 \\ -\sin\left(\lambda^2 S_{12}t\right) & \cos\left(\lambda^2 S_{12}t\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{24}$$

which represents a rotation about $\hat{\mathbf{z}}$ by an angle $\alpha(t) = -\lambda^2 S_{12}t$, such that a cyclic change of shape (23) with $x_3(0) = 0 = p_3(0)$ gives rise to $\alpha(2\pi) = -2\pi\lambda^2 S_{12}$. In Fig. 1 we show the cyclic deformation sequence that produces the rotation (24), with $S_{12}(0) = 2$ for three periods $0 \le t \le 3T = 3(2\pi/\omega_1)$. Clearly, if initial conditions are chosen such that $S_{12}(0)$'s sign is opposite, then the angle's sign of the resulting rotation is also opposite.

Coming back to the quantum case, it is convenient to define the annihilation and creation operators in shape space as usual $a_\mu = (x_\mu + ip_\mu)/\sqrt{2}$ and $a_\mu^\dagger$ (along this work we will work in units such that $\hbar = 1$), and analogously the annihilation and creation operators associated with angular excitations in the x_1x_2 plane as $a_\pm = (a_1 \mp ia_2)/\sqrt{2}$ and $a_\pm^\dagger = (a_1^\dagger \pm ia_2^\dagger)/\sqrt{2}$, where $a_+(a_+^\dagger)$ annihilates (creates) one positive angular excitation and $a_-(a_-^\dagger)$ annihilates (creates) one negative angular excitation [13]. We also define the corresponding number operators $N_\mu = a_\mu^\dagger a_\mu$ and $N_\pm = a_\pm^\dagger a_\pm$, so that the Hamiltonian (22) can be written as:

$$H = (N+1) + \omega_3\left(N_3 + \frac{1}{2}\right) + \lambda^2\left(\mathbf{L}^2 - \frac{L_z^2}{2} - L_zS + \frac{S^2}{2}\right), \tag{25}$$

with $N = N_+ + N_-$ and $S = N_+ - N_-$. The previous Hamiltonian[2] is defined in the Hilbert space $L^2(\mathcal{C}, d^3q\ d\mathsf{R})$, with $d\mathsf{R}$ being the Haar normalized measure in $SO(3)$. It is not difficult to see that a complete set of mutually compatible operators commuting with the Hamiltonian is $\{N, S, N_3, \mathbf{L}^2, L_z, L_{sz}\}$, see [14], and then it is clear that Hamiltonian eigenstates $|n, s, n_3, l, m, k\rangle$ are labelled by the corresponding quantum numbers $n_3, n, l \in \{0, 1, 2, \ldots\}$, $m, k = -l, -l+1, \ldots, l-1, l$ and $s = -n, -n+2, \ldots, n-2, n$. Now, the vanishing angular momentum condition implies that the state of the system has to be such that $l = 0$ and thus

[2] We should include the geometric potential $V_2 = D^{-1/4}\partial_\mu(g^{\mu\nu}\partial_\nu D^{1/4})/2$ with $D = \det(\mathsf{M})\det(g_{\mu\nu})$ in the quantum Hamiltonian, however in the small oscillations regime it reduces to a constant that we neglect.

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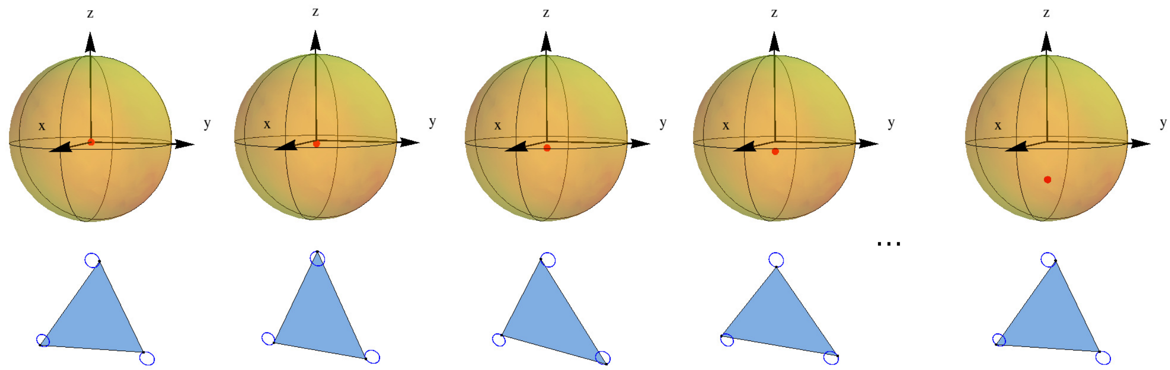

Figure 1. Sequence of the orientation change produced by a cyclic deformation in the plane x_1x_2 in shape space, for the three body problem. The figure at the top represents $SO(3)$ as the 3-ball of radius π with identified antipodes at the boundary $S^2(\pi)$, any point in this space corresponds to a rotation in the axis-angle representation, the vector joining the point to the origin corresponds to the axis of rotation while its magnitude corresponds to the angle of rotation. We present different configurations for the different values of time $t = 2\pi n/4$, with $n = 0, 1, 2, 3, \ldots, 12$, $S_{12}(0) = 2$ and $\lambda = 0.2$: at the top we show the (red) dot that represents the orientation configuration in the fibre $SO(3)$ and at the bottom the corresponding body's shape (for illustrative purposes the body's deformation was magnified by a factor of four).

$m = 0$, but since the term in the Hamiltonian that produces the rotation is $\mathbf{L}\cdot\mathsf{M}^{-1}\cdot\mathbf{S} = -\lambda^2 L_z S$, then when acting on a state satisfying the condition the only rotation it can generate is the identity. In order to see a more general rotation we shall impose the *weak* vanishing angular momentum condition $\langle\mathbf{L}\rangle = 0$.

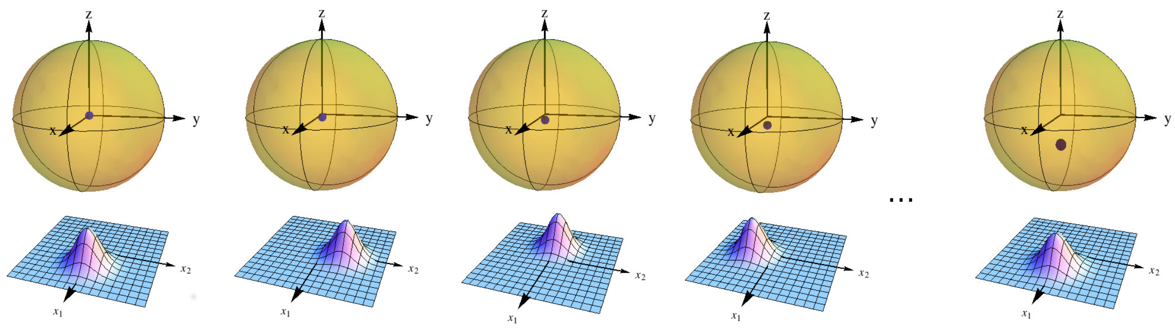

Figure 2. Sequence of the orientation change produced by a cyclic deformation in the plane x_1x_2 in shape space for the three body problem, in the quantum regime. The figure at the top represents $SO(3)$ as in Fig. 1. We present the time-evolution of the probability density associated with the initial wavefunction $\Psi(\mathsf{R}, q^\mu)$ given by (26) for the different values of time $t = 2\pi n/4$, with $n = 0, 1, 2, 3, \ldots, 12$, $\lambda = 0.2$ and $\alpha_{(+,-)} = (\sqrt{2}, 0)$. At the top we show the surface of constant (in orientation coordinates) orientation reduced probability density $P_t^O(\mathsf{R}) = .99\max(P_t^O)$ that represents the orientation configuration in the fibre $SO(3)$ and at the bottom the reduced probability in the shape space $P_t^S(x_1, x_2)$.

Let us thus consider a more general state $|\Psi\rangle$ satisfying the weak angular momentum condition

$\langle\Psi|\mathbf{L}|\Psi\rangle = 0$, with Ψ given by:

$$\Psi(\mathsf{R}, q^\mu) = \frac{1}{\sqrt{\pi}}\psi_{\alpha_+,\alpha_-}(x_1, x_2)\psi_0(x_3)\Phi(\phi), \tag{26}$$

where $\psi_{\alpha_+,\alpha_-}(x_1, x_2) = \prod_{i=1}^{2} \mathrm{e}^{-\frac{1}{2}(x_i-\langle x_i\rangle)^2}\,\mathrm{e}^{i\langle p_i\rangle\left(x_i-\frac{1}{2}\langle x_i\rangle\right)}$ is a coherent state with $\langle x_i\rangle = \sqrt{2}\,\mathrm{Re}(\alpha_i)$ and $\langle p_i\rangle = \sqrt{2}\,\mathrm{Im}(\alpha_i)$ as usual, with $\alpha_1 = (\alpha_+ + \alpha_-)/\sqrt{2}$ and $\alpha_2 = i(\alpha_+ - \alpha_-)/\sqrt{2}$ for $\alpha_\pm \in \mathbb{C}$, $\psi_0(x_3)$ is the breathing mode's ground state, and $\Phi(\phi) = \Phi(\mathsf{R}_{\hat{\mathbf{n}}}(\phi)) = N_\Phi\,\mathrm{e}^{\cos\phi/2}$ is a wavefunction in $SO(3)$ in the axis-angle representation $(\hat{\mathbf{n}}, \phi)$. The state $\Phi(\phi)$ describes a system that is most probably oriented in coincidence with the body frame.

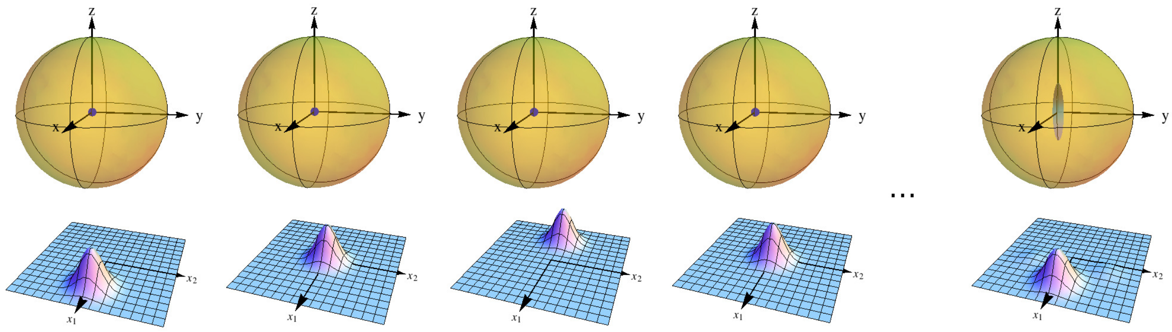

Figure 3. Sequence of the orientation change produced by a cyclic deformation in the plane x_1x_2 in the shape space for the three body problem, in the quantum regime. The figure at the top represents $SO(3)$ as in Fig. 1. We present the time-evolution of the probability density associated with the initial wavefunction $\Psi(\mathsf{R}, q^\mu)$ given by (26) for the different values of time $t = 2\pi n/4$, with $n = 0, 1, 2, 3, \ldots, 20$, $\lambda = 0.2$ and $\alpha_{(+,-)} = (\sqrt{2}, -\sqrt{2})$. At the top we show the surface of constant (in orientation coordinates) orientation reduced probability density $P_t^O(\mathsf{R}) = .99\max(P_t^O)$ that represents the orientation configuration in the fibre $SO(3)$ and at the bottom the reduced probability in the shape space $P_t^S(x_1, x_2)$.

The time evolution of the wavefunction (26) can be computed directly, though no completely trivially. The Hamiltonian (15) can be separated into three terms, $H_S = (N+1) + \omega_3(N_3 + 1/2)$ $+\lambda^2 S^2/2$ that only depends and acts on the shape degrees of freedom, $H_O = \lambda^2(\mathbf{L}^2 - L_z^2/2)$ that is basically the kinetic energy of the rotor, and $H_I = -\lambda^2 L_z S$ that couples R and q^μ. All three terms commute with each other, and thus the complete time evolution can be decomposed into the action of the three corresponding unitary transformations. The first and the second evolve the shape and orientation sectors of the wavefunction, respectively and in an independent way. The action of $\mathrm{e}^{-i\lambda^2 H_O t}$ can be computed by expanding the function $\Phi(\phi)$ in the H_O's eigenvectors $\{D^l_{km}\}$, while that of $\mathrm{e}^{-iH_S t}$ can be expressed in two steps: one corresponding to the λ-independent term which just shifts the coherent state label to $\alpha_\pm(t) = \alpha_\pm\,\mathrm{e}^{-it}$, and one associated with the λ-dependent term of H_S which does not give rise to a coherent state, but can be approximated by $\mathrm{e}^{-i\lambda^2 S^2 t/2} \approx 1 - i\lambda^2 S^2 t/2$, since $\lambda^2\langle S^2\rangle T/2$ is small. The third coupling term on the other hand is the most interesting one, when it is applied to the wavefunction it produces a coherent state labelled by $\alpha_\pm^k(t) = \alpha_\pm(t)\,\mathrm{e}^{\pm i\lambda^2 kt}$, where k is the eigenvalue of L_z, *i.e.*,

$$\Psi_t(\mathsf{R}, q^\mu) \approx \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-i\frac{\omega_3 t}{2}}\,\psi_0(x_3)\sum_{l,k} c_k^l \varphi_t^k(x_1, x_2)\,\mathrm{e}^{-i\lambda^2\left(l(l+1)-\frac{k^2}{2}\right)t}\,D^l_{kk}(\mathsf{R}), \tag{27}$$

where $\varphi_t^k(x_1, x_2) = (1 - i\lambda^2 S^2 t/2)\psi_{\alpha_+^k(t),\alpha_-^k(t)}(x_1, x_2)$ and $c^l_{k,m} = \int D^{l\,*}_{k,m}(\mathsf{R})\Phi(\mathsf{R})d\mathsf{R} = c^l_k\delta_{km}$ which turns out to vanish for $m \neq k$. To illustrate the system's change in orientation as its shape changes due to time evolution, we project the probability density $P_t(\mathsf{R}, q^\mu) = |\Psi(\mathsf{R}, q^\mu)|^2$ onto the section $\mathcal{S}$ and onto the fibre $SO(3)$, and thus we define the reduced probability density in shape space by integrating out the orientation degrees of freedom, $P_t^S(q^\mu) = \int P_t(\mathsf{R}, q^\mu) d\mathsf{R}$ and the orientation reduced probability density $P_t^O(\mathsf{R}) = \int P_t(\mathsf{R}, q^\mu)\sqrt{g} d^{3n-6}q$. In Fig. 2 we show the quantum version of the cyclic deformation and the corresponding rotation sequence that is produced by the Hamiltonian as time increases with $\alpha_{(+,-)} = (\sqrt{2}, 0)$, which implies that $\langle S \rangle = 2$, just as in the classical situation shown in Fig. 1. It can be noted that the system rotates clockwise as its shape changes in a similar fashion than its classical counterpart, however the angle of the rotation after three complete periods in SS seems to be slightly smaller than in the classical case. We can notice as well that the surface of most probable orientation in this case corresponds to an sphere whose radius grows as time increases. Both effects are related to the dispersion of the wavefunction in orientation space. And also just as in the classical case we could choose (26) such that $\alpha_{(+,-)} = (0, \sqrt{2})$, so that $\langle S \rangle = -2$ and thus the body would rotate counter clockwise. Furthermore, we could also fix as our initial state a wavefunction like (26) with $\alpha_{(+,-)} = (\sqrt{2}, \sqrt{2})$ or equivalently with $\alpha_{(1,2)} = (2, 0)$ which corresponds to a coherent state associated with a linear excitation along x_1 and therefore $\langle S \rangle = 0$. In contrast to the classical case, this system's state can produce a non vanishing probability of changing its orientation in spite of enclosing a null area, see Fig. 3. The cigar-shaped probability distribution shown in the latter figure describes a cat whose orientation is rotated, with respect to the initial one, by all angles (approximately) within the interval $[-\pi/2, \pi/2]$ about the z-axis with equal probability. So that if the cat is initially oriented, say horizontally, it would end landing on any part of its body located between its feet and its head, continuously and with equal probability, becoming a "continuous Schrödinger cat".

4. Conclusions

We presented a quantum description of the free-falling cat problem in terms of geometrical methods applied to the n-body problem. In concrete, we answered affirmatively to our initial motivating question: can the quantum cat control its muscles so that it lands on its feet as its classical analogue? The specific scenario which more closely resembles the corresponding classical analogue takes place when the cat's wavefunction is a coherent state in both, the shape and orientation sectors; furthermore, we showed that more exotic purely quantum scenarios may happen, including one describing a "continuous Schrödinger cat". This result might remind us that applying quantum mechanics without inquiring on its interpretation and foundations may have dangerous, even fatal, consequences, a message that is commonly found in professor Mielnik's work.

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