Note on the quantum correlations of two qubits coupled to photon baths

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Abstract. The time-evolution of the quantum correlations between two qubits that are coupled to a pair of photon baths is studied. We show that conditioned transitions occurring in the entire system have influence on the time-evolution of the subsystems. Then, we show that the study of the population inversion of each of the qubits is a measure of the correlations between them that is in agreement with the notion of concurrence.

1. Introduction

Coherent superpositions in quantum mechanics give rise to correlations between the parts of a given system [1,2]. Such correlations can be classic or quantum, and both of them may coexist for a given system [3]. Entanglement was proposed as a manifestation of quantum correlations [2], though not all quantum correlations are associated to entanglement since separable states can be quantum correlated [4, 5] (see also [6, 7]). Diverse measures of entanglement have been introduced over the time, examples are the concurrence [8, 9] and the negativity [10]. More general measures, as the quantum discord [11–13], quantify quantum correlations without the requirement of entanglement.

In this contribution we analyze the time-evolution of the correlation between two qubits that are coupled to two independent photon baths. It is assumed that the systems qubit+bath are one isolated from the other; that is, they are in cavities for which no communication is allowed. In this form, each qubit interacts with its environment (the photon bath) and decoherence results. The initial correlation between the qubits is then lost and recovered in time by time because the entire system is closed. We investigate the coherences of the entire system (two qubits, two photon baths) that are missed when information of one of its subsystems (the qubits) is required by summing up (partial tracing) over the degrees of freedom of the other parts (the photon baths). Such coherences include information of conditioned transitions between the states of the entire system that is lost, in a first sight, as a consequence of looking at the subsystems. However, this information can be recovered by analyzing the state of the parts in proper form. Indeed, we shall show that the study of the population inversion of the qubits represents a measure of quantum correlations that is in agreement with the concept of concurrence.

In Section 2 we introduce notation and general properties of a bipartite system formed of two qubits. We show that the partial trace of the density matrix produces leaky information about the transitions between the states of the subsystems. The notion of concurrence is recovered from basic properties of the reduced matrices. In Section 3 we analyze the time-evolution of the qubit bipartite system when it is coupled to photon baths and discuss about the behavior

of the correlations with time. We show that the population inversions and the concurrence lead to the same information about the correlation that the qubits present at a given time. Some concluding remarks are given at the very end of the paper.

2. Qubit bipartite systems

Consider a system $\mathcal{S}$ formed of two qubits $\mathcal{S}_\ell$, $\ell = A, B$, in two separated (and isolated) cavities. Let us represent each qubit as a two level system with ground and excited states, $|-\rangle$ and $|+\rangle$ respectively, separated by an energy difference $\Delta E = \hbar \omega_q$. Using the simplest representation of the vector state basis

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1}$$

we can write the creation and annihilation operators for the qubit excitation as follows:

$$\sigma^+ = |+\rangle\langle-| \equiv X^{+-} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = |-\rangle\langle+| \equiv X^{-+} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

The dyads X^{+-} and X^{-+} are two of the four *Hubbard operators* associated to the qubit vector spaces [14], and (2) is called the *X-representation* of the operators σ^+ and σ^-. These last, together with the Pauli operator

$$\sigma_3 = [\sigma^+, \sigma^-] \equiv X^{++} - X^{--} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

are the generators of the $su(2)$ Lie algebra. Notice that the basis elements (1) are eigenvectors of this last operator $\sigma_3|\pm\rangle = \pm|\pm\rangle$. As the state space for each qubit is two-dimensional

$$\mathcal{H}_\ell = \mathrm{Span}\{|+\rangle, |-\rangle\}, \quad \ell = A, B, \tag{4}$$

the state space of the qubit bipartite system is four-dimensional

$$\mathcal{H} = \mathrm{Span}\{|+\rangle \otimes |+\rangle, |+\rangle \otimes |-\rangle, |-\rangle \otimes |+\rangle, |-\rangle \otimes |-\rangle\}. \tag{5}$$

In (5) the left (right) element in each tensor product refers to the qubit A (B). Besides this last representation we shall use the following simplified notation

$$|+\rangle \otimes |+\rangle = |++\rangle = |\varphi_1\rangle, \qquad |+\rangle \otimes |-\rangle = |+-\rangle = |\varphi_2\rangle,$$

$$|-\rangle \otimes |+\rangle = |-+\rangle = |\varphi_3\rangle, \qquad |-\rangle \otimes |-\rangle = |--\rangle = |\varphi_4\rangle. \tag{6}$$

Any state of the bipartite system $\mathcal{S}$ can be represented by a density operator of the form

$$\rho = \sum_{i,j=1}^{4} r_{ij} X^{ij}; \quad r_{ij} \in \mathbb{C}, \quad X^{ij} = |\varphi_i\rangle\langle\varphi_j|, \tag{7}$$

where the (sixteen) dyads X^{ij} are the Hubbard operators associated to the four-dimensional space (5). The matrix elements r_{ij} are defined according to the specific state ρ of the system $\mathcal{S}$ and satisfy the following equations

$$\sum_{k=1}^{4} r_{kk} = 1, \quad r_{kj} = \bar{r}_{kj} \in \mathbb{C}, \quad r_{kk} \in \mathbb{R},$$

$$0 \leq \sum_{k=1}^{4} r_{kk}^2 + 2(|r_{12}|^2 + |r_{13}|^2 + |r_{14}|^2 + |r_{23}|^2 + |r_{24}|^2 + |r_{34}|^2) \leq 1, \tag{8}$$

with $\bar{z}$ the complex conjugate of $z \in \mathbb{C}$. The diagonal elements r_{kk} are the (populations) probabilities of finding the system $\mathcal{S}$ in the state $|\varphi_k\rangle$. In turn, the off-diagonal elements $r_{k,j\neq k}$ are associated to the coherence between different states. That is, $r_{k,j\neq k}$ and its complex conjugate $\bar{r}_{k,j\neq k}$ are the probability amplitudes of the transitions

$$|\varphi_k\rangle \leftrightarrow |\varphi_j\rangle, \quad j \neq k. \tag{9}$$

No transitions between the states $|\varphi_k\rangle$ are allowed if the density matrix (7) is diagonal (this condition means the absence of interference terms in the time-depending case). Thus, the system $\mathcal{S}$ is in an *incoherent (coherent) superposition* of the basis states $|\varphi_k\rangle$ if the density matrix (7) is (not) diagonal with more than one non vanishing element. Notice, however, that the concept of *coherent superposition* depends on the representation for the density matrix (see, e.g. [15]).

2.1. Missed correlations in reduced density operators

To get information of the subsystem $\mathcal{S}_A$ we calculate a partial trace of ρ by summing up over all the degrees of freedom of $\mathcal{S}_B$. Let us rewrite (7) in the form

$$\rho = \begin{pmatrix} \overset{r_{11}}{X^{11} = X^{++} \otimes X^{++}} & \overset{r_{12}}{X^{12} = X^{++} \otimes X^{+-}} & \overset{r_{13}}{X^{13} = X^{+-} \otimes X^{++}} & \overset{r_{14}}{X^{14} = X^{+-} \otimes X^{+-}} \\ \overset{r_{21}}{X^{21} = X^{++} \otimes X^{-+}} & \overset{r_{22}}{X^{22} = X^{++} \otimes X^{--}} & \overset{r_{23}}{X^{23} = X^{+-} \otimes X^{-+}} & \overset{r_{24}}{X^{24} = X^{+-} \otimes X^{--}} \\ \overset{r_{31}}{X^{31} = X^{-+} \otimes X^{++}} & \overset{r_{32}}{X^{32} = X^{-+} \otimes X^{+-}} & \overset{r_{33}}{X^{33} = X^{--} \otimes X^{++}} & \overset{r_{34}}{X^{34} = X^{--} \otimes X^{+-}} \\ \overset{r_{41}}{X^{41} = X^{-+} \otimes X^{-+}} & \overset{r_{42}}{X^{42} = X^{-+} \otimes X^{--}} & \overset{r_{43}}{X^{43} = X^{--} \otimes X^{-+}} & \overset{r_{44}}{X^{44} = X^{--} \otimes X^{--}} \end{pmatrix}, \tag{10}$$

where we have made explicit the relationship between the Hubbard operator X^{kj} corresponding to r_{kj} and the tensor product of the qubit Hubbard operators $X^{\alpha\beta}$, $\alpha, \beta = \pm$, associated to $\mathcal{S}_A$ and $\mathcal{S}_B$. Notice that the elements in red are of the form $X^{kj} = X^{\alpha\beta} \otimes X^{\gamma\gamma}$, so that $\text{Tr}_{s_B} X^{kj} = X^{\alpha\beta}(\text{Tr}X^{\gamma\gamma}) = X^{\alpha\beta}$ will preserve the element r_{kj} in the position (α, β) of the reduced matrix ρ_{s_A}. The elements in black, in counterposition, make no contribution. Therefore, we get

$$\rho_{s_A} = \text{Tr}_{s_B}\rho = \begin{pmatrix} r_{11} + r_{22} & r_{13} + r_{24} \\ \bar{r}_{13} + \bar{r}_{24} & r_{33} + r_{44} \end{pmatrix}. \tag{11}$$

In a similar form, the terms in blue of the matrix array

$$\rho = \begin{pmatrix} \overset{r_{11}}{X^{11} = X^{++} \otimes X^{++}} & \overset{r_{12}}{X^{12} = X^{++} \otimes X^{+-}} & \overset{r_{13}}{X^{13} = X^{+-} \otimes X^{++}} & \overset{r_{14}}{X^{14} = X^{+-} \otimes X^{+-}} \\ \overset{r_{21}}{X^{21} = X^{++} \otimes X^{-+}} & \overset{r_{22}}{X^{22} = X^{++} \otimes X^{--}} & \overset{r_{23}}{X^{23} = X^{+-} \otimes X^{-+}} & \overset{r_{24}}{X^{24} = X^{+-} \otimes X^{--}} \\ \overset{r_{31}}{X^{31} = X^{-+} \otimes X^{++}} & \overset{r_{32}}{X^{32} = X^{-+} \otimes X^{+-}} & \overset{r_{33}}{X^{33} = X^{--} \otimes X^{++}} & \overset{r_{34}}{X^{34} = X^{--} \otimes X^{+-}} \\ \overset{r_{41}}{X^{41} = X^{-+} \otimes X^{-+}} & \overset{r_{42}}{X^{42} = X^{-+} \otimes X^{--}} & \overset{r_{43}}{X^{43} = X^{--} \otimes X^{-+}} & \overset{r_{44}}{X^{44} = X^{--} \otimes X^{--}} \end{pmatrix}, \tag{12}$$

lead to the reduced density operator

$$\rho_{s_B} = \text{Tr}_{s_A}\rho = \begin{pmatrix} r_{11} + r_{33} & r_{12} + r_{34} \\ \bar{r}_{12} + \bar{r}_{34} & r_{22} + r_{44} \end{pmatrix}. \tag{13}$$

Remark that (i) the anti-diagonal elements of ρ are missing in the reduced densities ρ_{s_A}, ρ_{s_B}, and (ii) all the diagonal elements of ρ are included in the diagonal of both reduced matrices. The latter property means that the populations r_{kk} represent information of the entire system $\mathcal{S}$ that can be obtained from any of the subsystems $\mathcal{S}_\ell$. In contrast, property (i) means that the probability amplitudes associated to the anti-diagonal coherences of ρ are not directly recoverable from neither $\mathcal{S}_A$ nor $\mathcal{S}_B$ since the elements r_{41}, r_{32}, and their complex conjugates, are missing in (11) and (13). According to (9), such amplitudes describe the transitions

$$|\varphi_4\rangle \leftrightarrow |\varphi_1\rangle, \qquad |\varphi_3\rangle \leftrightarrow |\varphi_2\rangle, \tag{14}$$

which in the qubit basis read as

$$|--\rangle \leftrightarrow |++\rangle, \qquad |-+\rangle \leftrightarrow |+-\rangle. \tag{15}$$

The first expression in (14)-(15) represents a transition $|-\rangle_B \leftrightarrow |+\rangle_B$ in $\mathcal{S}_B$ that is conditioned to the transition $|-\rangle_A \leftrightarrow |+\rangle_A$ in $\mathcal{S}_A$ and vice versa. Graphically,

$$|\varphi_4\rangle \leftrightarrow |\varphi_1\rangle \text{ means either } \begin{cases} |-\rangle_A \leftrightarrow |+\rangle_A & \Rightarrow & |-\rangle_B \leftrightarrow |+\rangle_B \\ & \text{or} & \\ |-\rangle_B \leftrightarrow |+\rangle_B & \Rightarrow & |-\rangle_A \leftrightarrow |+\rangle_A \end{cases} \tag{16}$$

The second expression in (14)-(15) includes the following information

$$|\varphi_3\rangle \leftrightarrow |\varphi_2\rangle \text{ means either } \begin{cases} |-\rangle_A \leftrightarrow |+\rangle_A & \Rightarrow & |+\rangle_B \leftrightarrow |-\rangle_B \\ & \text{or} & \\ |+\rangle_B \leftrightarrow |-\rangle_B & \Rightarrow & |-\rangle_A \leftrightarrow |+\rangle_A \end{cases} \tag{17}$$

Besides the anti-diagonal elements of ρ, other coherences $r_{k,j\neq k}$ are missed in ρ_{s_A} and ρ_{s_B}. Namely, the trace over the degrees of freedom of $\mathcal{S}_A$ eliminates the coherences included in the anti-diagonal 2×2 sub-matrices of ρ, see Eq. (12). That is, the probability amplitudes of the transitions $|\varphi_4\rangle \leftrightarrow |\varphi_2\rangle$ and $|\varphi_3\rangle \leftrightarrow |\varphi_1\rangle$ are also absent in ρ_{s_B}. In turn, according to (10), the tracing over $\mathcal{S}_B$ overrides the information of the transitions $|\varphi_2\rangle \leftrightarrow |\varphi_1\rangle$ and $|\varphi_4\rangle \leftrightarrow |\varphi_3\rangle$ in ρ_{s_A}.

For an arbitrary state ρ the off-diagonal elements of ρ_{s_A} have not connections a priori with the off-diagonal elements of ρ_{s_B}. However, if $\mathcal{S}$ is in a pure state, the elements of ρ_{s_A} and ρ_{s_B} are interrelated in such a manner that these last matrices have the same determinant. As we shall see, this fact is useful in recovering the information of the transitions (14)-(17) by using the information that is accessible from either ρ_{s_A} or ρ_{s_B}.

2.2. Pure and mixed states
Some relationships between ρ_{s_A} and ρ_{s_B} are achievable by considering a general pure state of the system $\mathcal{S}$, we take

$$|\psi\rangle = a|\varphi_1\rangle + b|\varphi_2\rangle + c|\varphi_3\rangle + d|\varphi_4\rangle, \tag{18}$$

with $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. The pure state $|\psi\rangle$ would be the initial condition of the qubit bipartite system $\mathcal{S}$, assuming that the coupling of the qubits to the radiation fields in the cavities is activated at $t = 0$. Using this state to construct the density operator $\rho_\psi = |\psi\rangle\langle\psi|$, one can verify that the reduced densities

$$\rho_{s_A} = \begin{pmatrix} |a|^2 + |b|^2 & a\bar{c} + b\bar{d} \\ \bar{a}c + \bar{b}d & |c|^2 + |d|^2 \end{pmatrix}, \quad \rho_{s_B} = \begin{pmatrix} |a|^2 + |c|^2 & a\bar{b} + c\bar{d} \\ \bar{a}b + \bar{c}d & |b|^2 + |d|^2 \end{pmatrix}, \tag{19}$$

have the determinant [16]:

$$\text{Det}\,\rho_{s_A} = \text{Det}\,\rho_{s_B} = |ad - bc|^2 = |ad|^2 + |cb|^2 - [ad(\overline{bc}) + (\overline{ad})bc]$$

$$= |r_{14}|^2 + |r_{32}|^2 - 2\text{Re}\left(r_{14}r_{32}\,\mathrm{e}^{i2(\theta_d - \theta_c)}\right), \quad (20)$$

where $d = |d|\,\mathrm{e}^{i\theta_d}$, $c = |c|\,\mathrm{e}^{i\theta_c}$, and θ_d, $\theta_c \in \mathbb{R}$. The first two additive terms of (20) are the probability densities associated to the transitions (14) while the third one corresponds to the interference between such transitions. Remark that the determinant (20) is not null if at least two coefficients in the superposition (18) are different from zero, provided they are such that either $ad \neq 0$ or $bc \neq 0$.

Equation (20) makes clear that, besides the populations $|a|^2$, $|b|^2$, $|c|^2$, $|d|^2$, one can obtain some other information of the entire state ρ_ψ by studying the state of any of the subsystems $\mathcal{S}_A$ and $\mathcal{S}_B$. In particular, the determinant (20) is a measure of the transitions (14)-(17) that occur in the entire system. For instance, let us assume that the transition $|-\rangle_A \leftrightarrow |+\rangle_A$ occurs at a given time t. According to (16) and (17), this last implies a transition in the state of $\mathcal{S}_B$ that is ruled by either $|-\rangle_B \leftrightarrow |+\rangle_B$ or $|+\rangle_B \leftrightarrow |-\rangle_B$. In the former case the transition in $\mathcal{S}_A$ cancels the coherence r_{32} so that $\text{Det}\,\rho_{s_{A,B}} = |r_{14}|^4 \equiv |da|^2$. In the second case the coherence r_{14} is nulled and $\text{Det}\,\rho_{s_{A,B}} = |r_{32}|^4 \equiv |bc|^2$. A similar description is obtained when either $|-\rangle_B \leftrightarrow |+\rangle_B$ or $|+\rangle_B \leftrightarrow |-\rangle_B$ activate the transition process in the entire system. Remark that these conditioned transitions in the subsystems $\mathcal{S}_A$ and $\mathcal{S}_B$ lead to maxima values of the determinant (20). Transitions in $\mathcal{S}_A$ that make no constraints in the transitions of its counterpart $\mathcal{S}_B$, and viceversa, give rise to interference phenomena in the process. Indeed, from the previous example, the (not binding) activation $|-\rangle_A \leftrightarrow |+\rangle_A$ produces a superposition state in $\mathcal{S}_B$ that includes both transitions, $|-\rangle_B \leftrightarrow |+\rangle_B$ and $|+\rangle_B \leftrightarrow |-\rangle_B$, with a probability different from zero each one. Then, for not conditioned transitions, the interference term in (20) is relevant since this can even cancel the determinant $\text{Det}\,\rho_{s_{A,B}}$. In summary, the transitions in $\mathcal{S}_{A,B}$ will be occasionally conditioned by the transitions in $\mathcal{S}_{B,A}$, so that $\text{Det}\,\rho_{s_{A,B}}$ will oscillate between minima and maxima values over the time.

As we can see, the determinant (20) represents a measure of the correlations that are lost after the partial tracing of the entire state ρ_ψ. Such correlations are represented by the conditioned transitions (16)-(17) and produce maxima values in $\text{Det}\,\rho_{s_{A,B}}$. Since this determinant takes nonnegative values, one can introduce the normalized quantity [16]:

$$C(\rho_\psi) = \kappa_0\sqrt{\text{Det}\,\rho_{s_{A,B}}} = \kappa_0|ad - bc|, \quad (21)$$

where the normalization constant κ_0 (in our case $\kappa_0 = 2$) makes $C = 1$ for the density operators associated to the Bell basis vectors:

$$|\beta_1\rangle = \tfrac{1}{\sqrt{2}}(|++\rangle + |--\rangle), \qquad |\beta_2\rangle = \tfrac{1}{\sqrt{2}}(|++\rangle - |--\rangle),$$

$$|\beta_3\rangle = \tfrac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), \qquad |\beta_4\rangle = \tfrac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle). \quad (22)$$

In the literature, the quantity $C(\rho_\psi)$ is known as *concurrence* and represents a measure of the *entanglement* between the states of $\mathcal{S}_A$ and those of $\mathcal{S}_B$ whenever the entire system $\mathcal{S}$ is in a pure state (see [8, 9] and references quoted therein). This quantity takes the values $0 \leq C \leq 1$, with $C = 1$ for fully entangled states and $C = 0$ for *factorizable* (not entangled) states. In our case, C gives information of the correlation between the states $|\varphi_4\rangle$ and $|\varphi_1\rangle$ ($|\varphi_3\rangle$ and $|\varphi_2\rangle$) of the entire system. If these are strongly correlated then, according to the cases discussed above, one has

$$C(\rho_\psi) = 2\sqrt{\text{Det}\,\rho_{s_{A,B}}} = 2|ad|. \quad (23)$$

The correlation is maximum if $|ad| = \frac{1}{2}$. Then, as the vector (18) is normalized, we can take $a = d = 1/\sqrt{2}$ to get $|\psi_{b=c=0}^{max}\rangle = |\beta_1\rangle$. That is, the Bell vector $|\beta_1\rangle$ defined in (22) corresponds to a pure state of $\mathcal{S}$ for which the correlation between the basis vectors $|\varphi_4\rangle$ and $|\varphi_1\rangle$ is maximum. Other values of this correlation are feasible by using $a = \cos\theta$ and $b = \sin\theta$, with $\theta \in \mathbb{R}$, for which

$$|\psi_{1,4}(\theta)\rangle = \cos\theta|\varphi_1\rangle + \sin\theta|\varphi_4\rangle \quad \Rightarrow \quad C(\rho_{\psi(\theta)}) = |\sin 2\theta|. \tag{24}$$

Notice that $|\psi_{1,4}(\frac{\pi}{4})\rangle = |\beta_1\rangle$ and $|\psi_{1,4}(\frac{3\pi}{4})\rangle = |\beta_2\rangle$ give $C = 1$. In similar form, for $|\varphi_3\rangle$ and $|\varphi_2\rangle$ we have

$$|\psi_{2,3}(\alpha)\rangle = \cos\alpha|\varphi_2\rangle + \sin\alpha|\varphi_3\rangle \quad \Rightarrow \quad C(\rho_{\psi(\alpha)}) = |\sin 2\alpha|, \quad \alpha \in \mathbb{R}. \tag{25}$$

In this case $|\psi_{2,3}(\frac{\pi}{4})\rangle = |\beta_3\rangle$ and $|\psi_{2,3}(\frac{3\pi}{4})\rangle = |\beta_4\rangle$ give $C = 1$.

The notion of concurrence can be generalized to include mixed states, this is accomplished by transforming the density operator ρ_ψ as follows [17]:

$$\rho_\psi^* = \sigma_2 \otimes \sigma_2 \, \rho_\psi \, \sigma_2 \otimes \sigma_2, \tag{26}$$

and calculating the trace of the product $\rho_\psi^* \rho_\psi$ to get

$$C^2(\rho_\psi) = \mathrm{Tr}(\rho_\psi^* \rho_\psi). \tag{27}$$

The product $\sigma_2 \otimes \sigma_2$ in (26) stands for the spin-flip operator [18]. Assuming that (27) holds for an arbitrary state ρ, pure or mixed, the concurrence can be written as [9]:

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \quad \lambda_j > \lambda_{j+1} \, , \, j = 1, 2, 3, \tag{28}$$

where the nonnegative numbers λ_j are the square-roots of the eigenvalues of the operator $\rho^* \rho$.

In principle, the expression (28) applies not only to pure states but also for mixed states. This is remarkable because, once the coupling between the qubits and the radiation fields is initiated, each cavity includes a system (the qubit) in an active electromagnetic environment (the photon bath). Therefore, the study of the qubit bipartite state requires a summation over the degrees of freedom of the fields and, as we have discussed, this last produces leaky in the information of the system. Hence, the description of the qubits system is necessarily done in terms of mixed states.

3. Two entangled qubits coupled to two independent quantum oscillators

Now let us assume that each of the qubits of the previous section is coupled to an independent single mode radiation field. The latter will be represented as a quantum harmonic oscillator of frequency ω_b and ladder operators $a^\dagger$, $a = (a^\dagger)^\dagger$. Using the dipole and rotating wave approximations, and considering $\omega_q \approx \omega_b$, the (dimensionless) Hamiltonian of the entire system $H = H_A + H_B$ has the components

$$H_\ell = H_{0;\ell} + H_{I;\ell}, \quad H_{0;\ell} = N_\ell + \frac{1}{2} + \frac{1}{2}\sigma_3, \quad H_{I;\ell} = \gamma_\ell(\sigma_+ a_\ell + \sigma_- a_\ell^\dagger), \quad \ell = A, B, \tag{29}$$

with $\{a_\ell, a_\ell^\dagger, N_\ell\}$ the generators of the photon algebra associated to the ℓth radiation field, γ_ℓ the coupling strength in the ℓth cavity, and $\{\sigma_3, \sigma_\pm\}$ the generators of the $su(2)$ Lie algebra associated to the excitation of the qubits in the cavities. Notice that we have dropped the sub-label ℓ from these last operators for simplicity in notation. Hereafter we shall assume that there is a bath of r (s) photons in cavity A (B), and $\gamma_A = \gamma_B = \gamma$.

The Hamiltonian H of the entire system $\mathcal{S}'$ is then formed of a free part $H_{0;A} + H_{0;B}$ and a coupling term $H_{I;A} + H_{I;B}$ (no communication between cavities A and B is allowed, so that a

coupling between them is not considered). It is straightforward to verify that $[H_{0;\ell}, H_{I;\ell'}] = 0$, so there is a common vector basis for the free and coupling parts of the Hamiltonian.

The system $\mathcal{S}'$ we consider is tetra-partite: it is formed of two independent qubits $\mathcal{S}_\ell$ coupled to two independent photon baths $\mathcal{S}_{b_\ell}$, $\ell = A, B$. Then, the state space of the entire system is the direct sum [19, 20]:

$$\mathcal{H} = \bigoplus_{n,m} \Xi_n^{(A)} \otimes \Xi_m^{(B)}, \tag{30}$$

where, provided n and m, the subspaces $\mathcal{H}_{nm} = \Xi_n^{(A)} \otimes \Xi_m^{(B)}$ are given by

$$\mathcal{H}_{nm} = \text{Span}\{|+,n\rangle \otimes |+,m\rangle, |+,n\rangle \otimes |-,m+1\rangle, |-,n+1\rangle \otimes |+,m\rangle, |-,n+1\rangle \otimes |-,m+1\rangle\}.$$

The time-evolution operator of the entire system is $U(t) = \mathrm{e}^{-iHt}$, since H is time-independent. Considering that the cavities are one independent of the other, and the commutation properties of $H_{0;\ell}$ and $H_{I;\ell}$, one can write

$$U(t) = U_A(t) \otimes U_B(t) = \mathrm{e}^{-iH_{I;A}t}\,\mathrm{e}^{-iH_{0;A}t} \otimes \mathrm{e}^{-iH_{I;B}t}\,\mathrm{e}^{-iH_{0;B}t}, \qquad U(0) = \mathbb{I}_A \otimes \mathbb{I}_B = \mathbb{I}. \tag{31}$$

In order to analyze the effect that the photon baths $\mathcal{S}_b = \mathcal{S}_{b_A} + \mathcal{S}_{b_B}$ produce in the correlations between the states of the qubits subsystem $\mathcal{S} = \mathcal{S}_A + \mathcal{S}_B$, let us assume that the initial state of $\mathcal{S}$ is the vector $|\psi_{2,3}(\alpha)\rangle$ defined in (25). That is, the bipartite qubit system $\mathcal{S}$ is initially correlated in the states $|\varphi_2\rangle$ and $|\varphi_3\rangle$ by an amount $C_0 = |\sin 2\alpha|$. As the cavities have r and s photons respectively, for $\mathcal{S}'$ we have the initial state

$$|\Phi(0)\rangle = |\psi_{2,3}(\alpha)\rangle \otimes |r\rangle \otimes |s\rangle = \cos\alpha|+,-,r,s\rangle + \sin\alpha|-,+,r,s\rangle. \tag{32}$$

To rewrite $|\Phi(0)\rangle$ in the representation of $\mathcal{H}$ given in (30) we use the fact that the direct product $A \otimes B$ is permutation equivalent to $B \otimes A$, see Theorem M4 of [14]. Therefore, up to a permutation, we have

$$|\Phi(0)\rangle = \cos\alpha|+,r\rangle \otimes |-,s\rangle + \sin\alpha|-,r\rangle \otimes |+,s\rangle. \tag{33}$$

Notice that the initial state $|\Phi(0)\rangle$ is in the subspace $\mathcal{H}_{r,s-1} \oplus \mathcal{H}_{r-1,s} \subset \mathcal{H}$. Applying (31) in (33) we get, up to a global phase $\mathrm{e}^{-i(r+s+1)t}$, the time-evolved state of the tetra-partite system

$$
\begin{aligned}
|\Phi(t)\rangle = \cos\alpha\,\big\{ &i\cos(\gamma t\sqrt{r+1})\sin(\gamma t\sqrt{s})|+,r\rangle \otimes |+,s-1\rangle \\
+ &\cos(\gamma t\sqrt{r+1})\cos(\gamma t\sqrt{s})|+,r\rangle \otimes |-,s\rangle - \sin(\gamma t\sqrt{r+1})\sin(\gamma t\sqrt{s})|-,r+1\rangle \otimes |+,s-1\rangle \\
+ &i\sin(\gamma t\sqrt{r+1})\cos(\gamma t\sqrt{s})|-,r+1\rangle \otimes |-,s\rangle \big\}
\end{aligned}
$$
$$
\begin{aligned}
+\sin\alpha\,\big\{ &i\cos(\gamma t\sqrt{s+1})\sin(\gamma t\sqrt{r})|+,r-1\rangle \otimes |+,s\rangle - \sin(\gamma t\sqrt{r})\sin(\gamma t\sqrt{s+1})|+,r-1\rangle \otimes |-,s+1\rangle \\
+ &\cos(\gamma t\sqrt{r})\cos(\gamma t\sqrt{s+1})|-,r\rangle \otimes |+,s\rangle + i\cos(\gamma t\sqrt{r})\sin(\gamma t\sqrt{s+1})|-,r\rangle \otimes |-,s+1\rangle \big\}. \tag{34}
\end{aligned}
$$

Remarkably, the time-evolved state $|\Phi(t)\rangle$ is in the subspace $\mathcal{H}_{r,s-1} \oplus \mathcal{H}_{r-1,s} \oplus \mathcal{H}_{r,s+1} \subset \mathcal{H}$; its eight components give rise to a pure state $\rho(t) = |\Phi(t)\rangle\langle\Phi(t)|$ that is expressed in terms of *sixty four* Hubbard operators [16]. Tracing over $\mathcal{S}_b$, for the qubit bipartite system $\mathcal{S}$ one gets the mixed state

$$\rho_q(\alpha, r, s; t) = \begin{pmatrix} \Omega_{11} & 0 & 0 & 0 \\ 0 & \Omega_{22} & \Omega_{23} & 0 \\ 0 & \Omega_{32} & \Omega_{33} & 0 \\ 0 & 0 & 0 & \Omega_{44} \end{pmatrix}, \tag{35}$$

where the time-dependent matrix elements $\Omega_{kj}(\alpha, r, s; t)$ are given by

$$\Omega_{11} = \cos^2\alpha \cos^2(\gamma t\sqrt{r+1})\sin^2(\gamma t\sqrt{s}) + \sin^2\alpha \cos^2(\gamma t\sqrt{s+1})\sin^2(\gamma t\sqrt{r}),$$

$$\Omega_{22} = \cos^2\alpha \cos^2(\gamma t\sqrt{r+1})\cos^2(\gamma t\sqrt{s}) + \sin^2\alpha \sin^2(\gamma t\sqrt{s+1})\sin^2(\gamma t\sqrt{r}),$$

$$\Omega_{33} = \cos^2\alpha \sin^2(\gamma t\sqrt{r+1})\sin^2(\gamma t\sqrt{s}) + \sin^2\alpha \cos^2(\gamma t\sqrt{s+1})\cos^2(\gamma t\sqrt{r}),$$

$$\Omega_{44} = \cos^2\alpha \sin^2(\gamma t\sqrt{r+1})\cos^2(\gamma t\sqrt{s}) + \sin^2\alpha \sin^2(\gamma t\sqrt{s+1})\cos^2(\gamma t\sqrt{r}),$$

$$\Omega_{23} = \Omega_{32} = \cos\alpha \sin\alpha \cos(\gamma t\sqrt{r+1})\cos(\gamma t\sqrt{r})\cos(\gamma t\sqrt{s+1})\cos(\gamma t\sqrt{s}).$$

The mixed state (35) is a coherent superposition of the basis kets $|\varphi_k\rangle$ for $\alpha \in (0, 2\pi)$, with $\alpha \neq \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and $r \neq 0$, $s \neq 0$. This has only two nonzero off-diagonal elements $\Omega_{23} = \Omega_{32} \in \mathbb{R}$ that arise from the operation

$$\text{Tr}_{s_b}(|+, r\rangle \otimes |-, s\rangle)(\langle -, r| \otimes \langle +, s|) = X^{+-} \otimes X^{-+} = X^{23}.$$

The latter is the trace over $\mathcal{S}_b$ of the outer product between the second and seventh kets of (34). Such outer product corresponds to a coherence in the entire tetra-partite state $\rho(t)$ that involves the probability amplitude of the conditioned transition

$$|+, r\rangle_{\text{cav A}} \to |-, r\rangle_{\text{cav A}} \quad \Leftrightarrow \quad |-, s\rangle_{\text{cav B}} \to |+, s\rangle_{\text{cav B}}. \tag{36}$$

As Ω_{23} is the qubit coherence involving the bipartite states $|\varphi_2\rangle$ and $|\varphi_3\rangle$, Eq. (36) means that the qubit transitions (17) are performed without a change in the number of photons in the cavities! Thus, there is a missing photon in cavity A and an additional photon in cavity B. The former is, in principle, released by $\mathcal{S}_A$ in its decaying from $|+\rangle_A$ to $|-\rangle_A$. The second is required by $\mathcal{S}_B$ in its excitation from $|-\rangle_B$ to $|+\rangle_B$. However, there is no communication between cavities A and B, so that the photon released by $\mathcal{S}_A$ is, with certainty, not the photon that increases the energy of $\mathcal{S}_B$. The puzzle is best analyzed by considering the initial state (33) of the entire system $\mathcal{S}'$. This is a linear combination of two orthogonal vectors $|+, r, -, s\rangle$ and $|-, r, +, s\rangle$ that share the same eigenvalue of the (free of interaction) energy $E_{0;A+B} = r + s + 1$. Then, at $t = 0$, the system $\mathcal{S}'$ is in the state $|+, r-, s\rangle$ with probability $\cos^2\alpha$, and in the state $|-, r, +, s\rangle$ with probability $\sin^2\alpha$. Therefore, from the very beginning, we do not know whether the system of cavity A is in the state $|+, r\rangle$ or in the state $|-, r\rangle$. A similar argument holds for the state of the system of cavity B. This fact is evidence of the non-locality of the quantum states involved since, if we avoid the coherence of $|\Psi(0)\rangle$ by making $\alpha = 0$ ($\alpha = \frac{\pi}{2}$), then we know with certainty that the system A is initially in the state $|+, r\rangle$ ($|-, r\rangle$) while the system B is in the state $|-, s\rangle$ ($|+, s\rangle$), and then the state ρ_q in (35) is an incoherent superposition of the four basis vectors (6). The transition (36) is, in this form, a consequence of the coherence Ω_{23} between the states $|+, r, -, s\rangle$ and $|-, r, +, s\rangle$, established at $t = 0$ by the degree of correlation $C_0 = |\sin 2\alpha|$ of the qubits (see Figure 1). The other non-zero coherence $\Omega_{32} = \Omega_{23}$ in (35) refers to the reversed process $|-, r, +, s\rangle \to |+, r, -, s\rangle$.

On the other hand, the straightforward calculation leads to the diagonal form of the qubit density matrix, $\rho_q = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where λ_k are its eigenvalues

$$\lambda_1 = \Omega_{11}, \quad \lambda_2 = \frac{(\Omega_{22} + \Omega_{33}) + \sqrt{(\Omega_{22} + \Omega_{33})^2 + 4(\Omega_{23}^2 - \Omega_{22}\Omega_{33})}}{2},$$

$$\lambda_3 = \Omega_{22} + \Omega_{33} - \lambda_2, \quad \lambda_4 = \Omega_{44}. \tag{37}$$

The related eigenvectors read as follows

$$|\phi_{\lambda_1}\rangle = |\varphi_1\rangle, \quad |\phi_{\lambda_2}\rangle = |\psi_{2,3}(\alpha)\rangle, \quad |\phi_{\lambda_3}\rangle = |\psi_{2,3}(-\alpha)\rangle, \quad |\phi_{\lambda_4}\rangle = |\varphi_4\rangle. \tag{38}$$
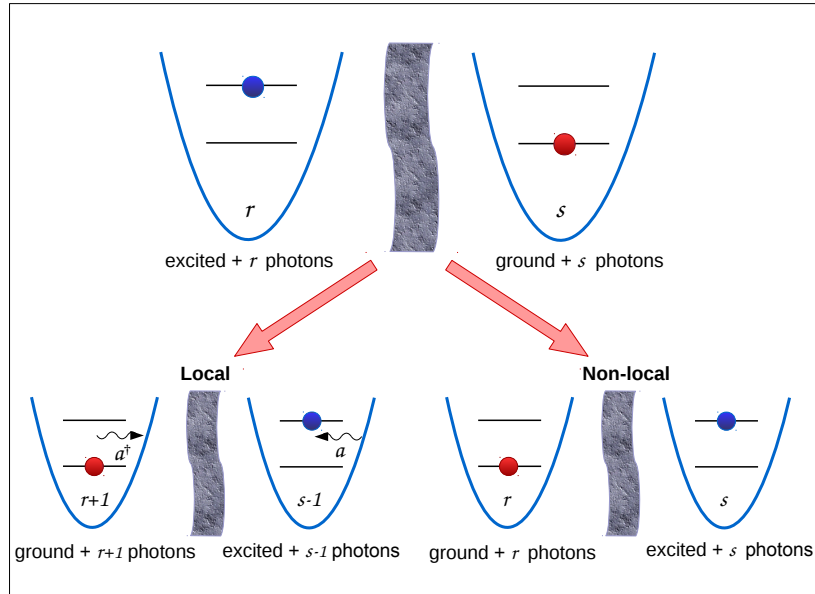
Figure 1. Schematic representation of the states transition (36), in all cases the parabolic curve represents the photon bath, blue and red dots mean respectively occupation of the excited and ground energy levels of the qubit (horizontal lines), and cavity A (left) is separated from cavity B (right) by an *impenetrable barrier*. The coherence Ω_{23} of the density operator (35) implies the change from the configuration depicted in the upper line to the one in the lower line. There are two possible forms to get such a transition. A local phenomenon (lower line, left) requires at least two photons that are not included in the initial configuration: one released by the qubit $\mathcal{S}_A$ (represented by $a^\dagger$) and one *absorbed* by the qubit $\mathcal{S}_B$ (represented by a). As the impenetrable barrier forbids communication between the cavities, the photon created in cavity A is, with certainty, not the one annihilated in cavity B. On the other hand, a non-local phenomenon (lower line, right) requires no photons of interchange. This is a consequence of the linear superposition in the initial state (33). That is, the transition (36) corresponds to transforming the configuration $|+, r, -, s\rangle$, that initially occurs with probability $\cos\alpha^2$, into the configuration $|-, r, +, s\rangle$, that occurs with probability $\sin\alpha^2$ at $t = 0$. The other not null coherence $\Omega_{32} = \Omega_{23}$ in (35) is associated to the reversed process $|-, r, +, s\rangle \rightarrow |+, r, -, s\rangle$.

The four vectors (38) are orthonormal so that ρ_q can be expressed as follows

$$\rho_q = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 X^{11} + \lambda_2 X^{22}(\alpha) + \lambda_3 X^{33}(\alpha) + \lambda_4 X^{44}$$

with $X^{22}(\alpha) = |\psi_{2,3}(\alpha)\rangle\langle\psi_{2,3}(\alpha)|$ and $X^{33}(\alpha) = |\psi_{2,3}(-\alpha)\rangle\langle\psi_{2,3}(-\alpha)|$. The diagonal form of ρ_q verifies that it is the probabilistic mixture of two disentangled pure states $|\varphi_1\rangle$, $|\varphi_4\rangle$, and two partially entangled pure states $|\psi_{2,3}(\alpha)\rangle$, $|\psi_{2,3}(-\alpha)\rangle$. It is now easy to verify that $\mathrm{Tr}(\rho_q) = 1$, and $\mathrm{Tr}(\rho_q^2) < 1$, as expected.

In order to analyze the qubits' transitions we calculate the population inversion by using $\mathcal{P}_{+-}^{(A,B)} = \mathrm{Tr}(\rho_{q_{A,B}}\sigma_3)$, with $\rho_{q_{A,B}} = \mathrm{Tr}_{s_{B,A}}\rho_q$ the trace of ρ_q over the degrees of freedom of $\mathcal{S}_B$ ($\mathcal{S}_A$). We obtain

$$\mathcal{P}_{+-}^{(A)} = \cos^2\alpha\cos(2\gamma t\sqrt{r+1}) - \sin^2\alpha\cos(2\gamma t\sqrt{r}),$$
$$\mathcal{P}_{+-}^{(B)} = \sin^2\alpha\cos(2\gamma t\sqrt{s+1}) - \cos^2\alpha\cos(2\gamma t\sqrt{s}). \tag{39}$$

At $t = 0$ one gets the inversions defined by the initial state $|\Phi(0)\rangle$ in (33), namely $\mathcal{P}_{+-}^{(A)} = \cos^2\alpha - \sin^2\alpha = -\mathcal{P}_{+-}^{(B)}$. For fully entangled qubits ($\alpha = \pi/4$ or $\alpha = 3\pi/4$) the populations in each energy level are the same and the inversion is equal to zero in both cavities (see Figure 2, left). In turn, partially entangled qubits (i.e., for other allowed values of α) produce inversions
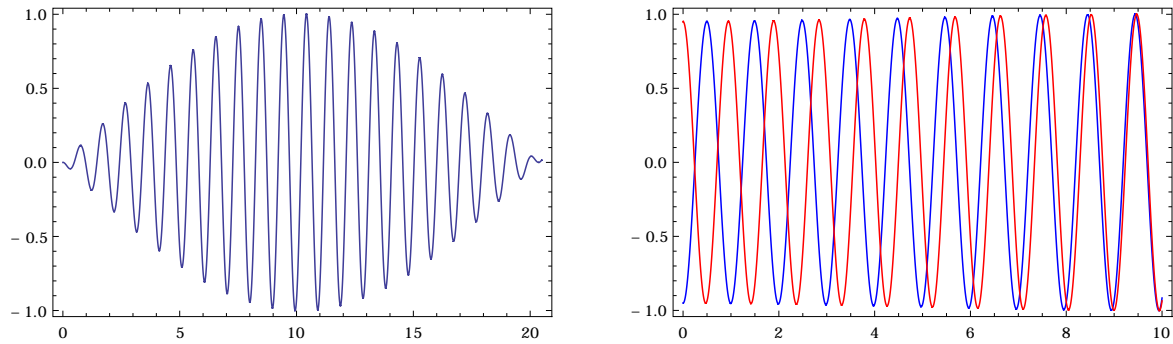
Figure 2. The time-dependence of the population inversions (39), qubit A in blue and qubit B in red. In both cavities the number of photons is the same $r = s = 10$. For fully entangled initial qubits with $\alpha = \pi/4$ (left), the time-evolution is the same in both cavities and the correlation between the qubits is smaller as larger is the absolute value of the inversion. For initial partially entangled qubits with $\alpha = 9\pi/20$ (right), the inversions have opposite sign and the correlation is smaller as larger is the absolute value of $\mathcal{P}_{+-}^{(A)} + \mathcal{P}_{+-}^{(B)}$. In both cases the initial correlation is lost in the vicinity of $t = 10$.

that differ by a sign in the cavities (Figure 2, right). The sinusoidal time-dependence of the population inversion in both cavities leads to a periodic repetition of the initial configuration. That is, the initial configuration will be lost and recovered periodically with time; the times at which it is lost are the times at which there is no correlation between the qubit in cavity A and the one in cavity B. In the case of fully entangled initial states, the correlation between the qubits is smaller as larger are the absolute values of the population inversions. For partially entangled states, the correlation is smaller as larger is the absolute value of the quantity $\mathcal{P}_{+-}^{(A)} + \mathcal{P}_{+-}^{(B)}$. In Figure 2 we have depicted the time-evolution of the population inversions (39) for a fully entangled state ($\alpha = \pi/4$) and a partially entangled state ($\alpha = 9\pi/20$) as initial condition, in both cases the number of photons is the same $r = s = 10$. Notice that the fully entangled case (Figure 2, left) departs from zero at $t = 0$ and reaches the same value at $t = 20$ (approx.). Besides, it oscillates between 1 and -1 in the vicinity of $t = 10$; in this last interval of time $t \in 10 \pm \delta$ the correlations between the qubit A and the qubit B are lost. In turn, the partially entangled case (Figure 2, right) departs from $\mathcal{P}_{+-}^{(A)} + \mathcal{P}_{+-}^{(B)} = 0$ at $t = 0$ and is such that $\mathcal{P}_{+-}^{(A)} + \mathcal{P}_{+-}^{(B)} = \pm 2$ in the vicinity of $t = 10$. As in the previous case, the correlations between the qubits are lost for $t \in 10 \pm \delta$.

The above description of the time-evolution of the correlation between the qubits is in agreement with the notion of concurrence discussed in the previous section. In Figure 3 we compare the time dependence of the population inversion and the concurrence for the fully entangled state $|\beta_3\rangle$ as initial condition. As it can be appreciated, the concurrence is minimum when the population inversion is in either its minimum (-1) or its maximum (1) value. In that case, we know that the two qubits are respectively in either the state $|+\rangle$ or $|-\rangle$, so that the initial configuration (the same probability to be in $|+\rangle$ as in $|+\rangle$) is lost. In turn, the concurrence is maximum when the population inversion cancels (the initial configuration!). Therefore, the collapses and revivals of the population inversion are a direct measure of the entanglement between the qubits, these last are maximally entangled at the time a revival occurs and they are disentangled when the inversion collapses. This oscillatory behavior of the concurrence is associated to the concepts of *sudden dead* and recovery of entanglement in the literature [21].
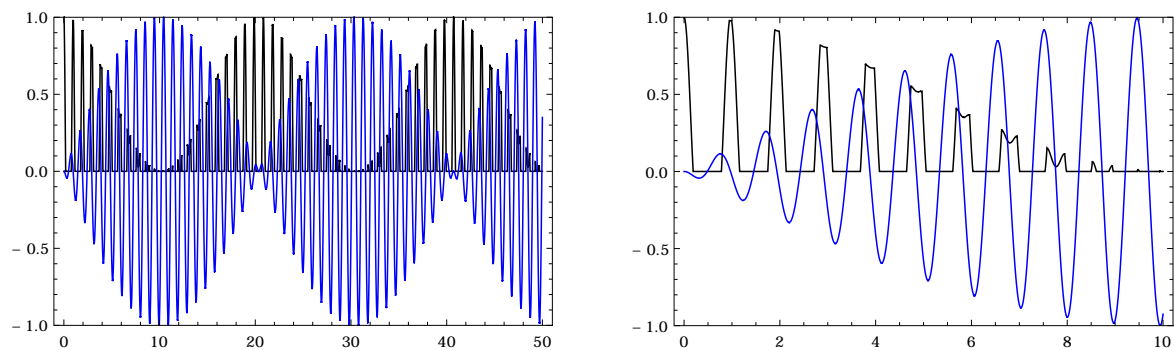
Figure 3. Time-evolution of the concurrence (28) and population inversions (39) for a pair of fully entangled qubits ($\alpha = \pi/4$) coupled to two photon baths (10 photons each one) in isolated cavities. The maxima of the absolute values of the population inversion (blue) represent the lost of correlations between the qubits and coincide with the minima of the concurrence (black). At the right we have depicted a detail of the curves in a shorter interval of time for comparison.

4. Concluding remarks

We have studied the time-evolution of the correlations between a pair of qubits that are coupled to independent photon baths. We have shown that the population inversion of each of the qubits gives the same information as the concurrence. This last because the partial trace of the density matrix associated to the entire system produces leaky information that can be recovered by analyzing the transitions between the energy states of the qubits.

Acknowledgments

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