Determined position dependent Mass of the Rosen-Morse potential and its Bound state

Ali R. Amani

Department of Physics, Ayatollah Amoli Branch, Islamic Azad University, Amol, Iran
P.O. Box 678, Amol, Mazandaran, Iran

E-mail: `a.r.amani@iauamol.ac.ir`

Abstract. In this paper, Schrödinger equation has been solved analytically with position dependent mass by the Rosen-Morse potential. The position dependent mass defined as the function $1/(1 - \tanh(\eta x))$. Then corresponding position dependent mass substituted into Schrödinger equation. After that the obtained equation compared with associated Jacobi differential equation. Therefore, the eigenvalue and eigenfunction have been calculated, and from there the bound state has been found in terms of quantum numbers and the coefficients of potential. Also, the wave function has been obtained in terms of the associated Jacobi equation by $\mu = -3$ and $\nu = 2$.

1. Introduction

One of the fundamental wave equations is the Schrödinger equation that one used in physics and chemistry. Solutions of Schrödinger equation for some physical potential have important applications in molecular physics, quantum chemistry, nuclear, condensed matter physics, high energy physics and particle physics. These potentials are such as, Hulthén [1], Morse [2], Rosen-Morse [3], Pseudo-harmonic [4], Mie [5], Poschl-Teller [6], Kratzer-Fues [7] and Woods-Saxon [8]. We usually solve these potentials as analytically and we can calculate energy eigenvalues and the corresponding wave functions exactly. Application of the mentioned potentials and Woods-Saxon potential are very useful to describe molecular structures and the interaction of a neutron with a heavy nucleus, respectively.

Exact solutions of Schrödinger equation for the mentioned potentials are interesting in the fields of material science and condense matter physics. We note that, there are various methods to obtain exact solutions energy eigenvalues and corresponding wave function. One of the methods is a factorization method that can factorize the Hamiltonian of corresponding system in terms of production of two first order differential operators, in which so-called lowering and raising operators. The two obtained Hamiltonian are partner of each other. This method allowed to classify the problems according to the characteristics involved in the potentials and it is closely related to the supersymmetric quantum mechanics [9, 10, 11].

In quantum mechanics, all of the analytically solvable potentials have a feature that be called as shape invariance. In fact, the shape invariance is an integrability condition and an interesting feature of supersymmetric quantum mechanics. This method is an exact and elegant technique to determine the eigenvalues and eigenfunctions of quantum mechanical problems [9].

In this paper, we are going to solve the Schrödinger equation with position dependent mass by the Rosen-Morse potential.

We have organized the present work as follows:

In section 2 we will present the general form of Schrödinger equation with position dependent mass by the Rosen-Morse potential. In section 3 we consider the position dependent mass as a function of position, and from there by using associated Jacobi polynomial we can obtain the eigenvalue and wave function for the corresponding system. In section 4 we write the corresponding Schrödinger equation as raising and lowering operators in terms of the quantum numbers of radial and orbital angular momentum, and as well as we investigate the shape invariance condition. Finally, we will express a brief report of the present work, in section 5.

2. Effective Schrödinger equation background

In order to obtain the eigenvalues and wave function with position dependent mass by the Rosen-Morse potential, the effective Hamiltonian which was first introduced by von Roos [12] is written by Weyl ordering in terms of the kinetic energy operator as follows [13],

$$
H = \frac{1}{4(\alpha+1)}\Big\{ a\Big(m^{-1}(\vec{r})\,\hat{p}^2 + \hat{p}^2\,m^{-1}(\vec{r}) \Big) + m^{\alpha}(\vec{r})\,\hat{p}\,m^{\beta}(\vec{r})\,\hat{p}\,m^{\gamma}(\vec{r})
$$
$$
+ m^{\gamma}(\vec{r})\,\hat{p}\,m^{\beta}(\vec{r})\,\hat{p}\,m^{\alpha}(\vec{r}) \Big\}, \tag{1}
$$

We should point out that several authors considered a constraint as $\alpha + \beta + \gamma = -1$ for the effective Hamiltonian [14, 15, 16, 17]. In that case, the one-dimensional Schrödinger equation be written by the corresponding constraint as the following expression,

$$
-\frac{\hbar^2}{2m(x)}\frac{d^2\psi(x)}{dx^2} + \frac{\hbar^2}{2}\left(\frac{1}{m^2(x)}\frac{dm(x)}{dx}\right)\frac{d\psi(x)}{dx} + \left[V(x) + U_{\alpha\beta\gamma a}(x) - E\right]\psi(x) = 0, \tag{2}
$$

where $U_{\alpha\beta\gamma a}$ is equal to,

$$
U_{\alpha\beta\gamma a}(x) = -\frac{\hbar^2}{4m^3(x)(a+1)}\left[(\alpha+\gamma-a)m(x)\frac{d^2m(x)}{dx^2} + 2(a-\alpha\gamma-\alpha-\gamma)\left(\frac{dm(x)}{dx}\right)^2\right]. \tag{3}
$$

It should be noted that several authors solved the Schrödinger equation with position dependent mass for different cases such as: ($a = \beta = \gamma = 0$ and $\alpha = -1$ [18]), ($a = \alpha = \gamma = 0$ and $\beta = -1$ [19]), ($a = 0$, $\alpha = \gamma = -\frac{1}{2}$ and $\beta = 0$ [20]) and ($a = \alpha = 0$ and $\beta = \gamma = -\frac{1}{2}$ [22]).

In this job, we are going to solve the corresponding system by the Rosen-Morse potential. We note that this potential describes the vibrations of diatomic molecule, and one be written as trigonometric functions in the following form,

$$
V(x) = V_1 \tanh^2(\eta x) + V_2 \tanh(\eta x) - V_1, \tag{4}
$$

where V_1 and V_2 are the depth of the potential and η is the range of the potential. The corresponding potential has a minimum in equilibrium distance or bond length i.e., $r_e = -\frac{1}{\eta}\mathrm{arctanh}(\frac{V_2}{2V_1})$.

By substituting (3) and (4) into (2), and by making a variable change as $y = \tanh(\eta x)$, the Schrödinger equation (2) be rewritten in terms of variable y as,

$$
(1-y^2)\psi''(y) + \left[-2y - (1-y^2)\frac{m'(y)}{m(y)} \right]\psi'(y) + \left[-\frac{2V_1}{\hbar^2\eta^2}\frac{y^2}{1-y^2}\,m(y) - \frac{2V_2}{\hbar^2\eta^2}\frac{y}{1-y^2}\,m(y) \right.
$$
$$
+ \frac{2(V_1+E)}{\hbar^2\eta^2}\frac{1}{1-y^2}\,m(y) + \frac{\alpha+\gamma-a}{2(a+1)}(1-y^2)\frac{m''(y)}{m(y)} - \frac{\alpha+\gamma-a}{(a+1)}y\frac{m'(y)}{m(y)}
$$
$$
\left. + \frac{a-\alpha\gamma-\alpha-\gamma}{(a+1)}(1-y^2)\frac{m'^2(y)}{m^2(y)} \right]\psi(y) = 0, \tag{5}
$$

where the prime displays the derivative with respect to variable y. We note that the coefficients α, β and γ obey the constraint $\alpha + \beta + \gamma = -1$.

In next section, we will solve the above equation, and we will obtain the eigenvalues and wave function for l-wave cases.

3. The Eigenvalues and the wave function

In this section, we are going to solve Eq. (5), the motivation of the job is based on analytically solution. In that case, we refer to use position dependent mass in the following form,

$$m(y) = \frac{1}{1-y}, \tag{6}$$

This choice will lead to an analytical solution. So that $m(y)$ is a function without singularity, because there is a definite value for $m(y)$ in all values y.

Now we are inserting Eq. (6) into (5), then the Schrödinger equation becomes,

$$(1-y^2)\psi''(y) - (3y+1)\,\psi'(y) + \left[-\frac{2\,V_1}{\hbar^2\eta^2}\frac{y^2}{(1-y)^2(1+y)} - \frac{2\,V_2}{\hbar^2\eta^2}\frac{y}{(1-y)^2(1+y)} \right.$$
$$\left. +\frac{2\,(V_1+E)}{\hbar^2\eta^2}\frac{1}{(1-y)^2(1+y)} + \frac{\alpha+\gamma-a}{(a+1)}\frac{(1+y)}{(1-y)} - \frac{\alpha+\gamma-a}{(a+1)}\frac{y}{(1-y)} + \frac{a-\alpha\gamma-\alpha-\gamma}{(a+1)}\frac{2(1+y)}{(1-y)} \right] \psi(y) = 0, \tag{7}$$

We can get wave function $\psi(y)$ by making separation of variables method as multiplication of two functions $u(y)$ and $P(y)$. Here we note that function $P(y)$ is as a special function [21]. This method helps us that we have an analytical solution, so we have,

$$\psi(y) = u(y)P(y), \tag{8}$$

by substituting Eq. (8) into Eq. (7) leads to,

$$(1-y^2)P''(y) + \left[-2\frac{u'}{u}(1-y^2) - (3y+1) \right]P'(y)$$
$$+\left[(1-y^2)\frac{u''}{u} - (3y+1)\frac{u'}{u} - \frac{2\,V_1}{\hbar^2\eta^2}\frac{y^2}{(1-y)^2(1+y)} - \frac{2\,V_2}{\hbar^2\eta^2}\frac{y}{(1-y)^2(1+y)} \right.$$
$$\left. +\frac{2\,(V_1+E)}{\hbar^2\eta^2}\frac{1}{(1-y)^2(1+y)} + \frac{\alpha+\gamma-a}{(a+1)}\frac{(1+y)}{(1-y)} - \frac{\alpha+\gamma-a}{(a+1)}\frac{y}{(1-y)} + \frac{a-\alpha\gamma-\alpha-\gamma}{(a+1)}\frac{2(1+y)}{(1-y)} \right] P(y) = 0. \tag{9}$$

On the other hand, the associated Jacobi polynomial is an orthogonal polynomial in the interval $[-1, 1]$, so one to be written as,

$$(1-y^2)P_{n,l}^{\mu,\nu\,''}(y) - [\mu - \nu + (\mu + \nu + 2)y]\,P_{n,l}^{\mu,\nu\,'}(y)$$
$$+\left[n(\mu + \nu + n + 1) - \frac{l(\mu+\nu+l+(\mu-\nu)y)}{1-y^2} \right] P_{n,l}^{\mu,\nu}(y) = 0, \tag{10}$$

where $n \geq l \geq 0$, so n and l are positive integers. The polynomial $P_{n,l}^{\mu,\nu}(y)$ is the eigenfunction of Eq. (10) and its Rodrigues formula is written in the following form,

$$P_{n,l}^{\mu,\nu}(y) = \sqrt{\frac{\Gamma(\mu+\nu+n+l+1)}{\Gamma(n-l+1)\Gamma(\mu+n+1)\Gamma(\nu+n+1)}}\,\frac{C\,(-1)^l\,2^{-n}}{(1-y)^{\mu+\frac{l}{2}}(1+y)^{\nu+\frac{l}{2}}}\frac{d^{n-l}}{dy^{n-l}}\left((1-y)^{\mu+n}(1+y)^{\nu+n} \right), \tag{11}$$

where C is an arbitrary function in terms of n and l.

By considering function $P(y)$ in Eq. (9) as the associated Jacobi polinomial, and to compare Eqs. (9) and (10) together we can obtain the parameters of eigenvalue and eigenfunction for

the Rosen-Morse potential. For this purpose, we will achieve the associated Jcobi parameters as $\mu = -3$ and $\nu = 2$. Therefore, we can obtain function $u(y)$ in the following form,

$$u(y) = \frac{(1+y)}{(1-y)^2}. \tag{12}$$

Comparing the third terms (9) and (10), we get the following relations,

$$\alpha = \frac{n^2(a+1)}{\gamma+1} - 1, \tag{13}$$

$$V_1 = \frac{\eta^2 \hbar^2}{2}\left(n^2 + 5l - 3 + \frac{\gamma\alpha}{a+1}\right), \tag{14}$$

$$V_2 = -\frac{\eta^2 \hbar^2}{2}\left(-n^2 + l^2 + 4l + \frac{4a + \alpha\gamma + \alpha + \gamma + 5}{a+1}\right), \tag{15}$$

by aforesaid equations, the energy eigenvalue becomes,

$$E_{n,l} = \frac{\eta^2 \hbar^2}{(1+\gamma)}\left[\frac{\gamma(2a-\gamma)}{2(a+1)} + (\gamma + \tfrac{1}{2})(n^2 - l(l-1) - 7) + \gamma(-\tfrac{5}{2}l + 1) + \frac{2V_2}{\eta^2 \hbar^2}\right] + 2V_1, \tag{16}$$

we can see from Eq. (16) that the obtained energy values depend on the quantum numbers, the potential coefficients and constants a and γ.

The corresponding eigenfunction will rewrite for Rosen-Morse potential with position dependent mass as the following form,

$$\psi(x) = \frac{(1 + \tanh(\eta x))}{(1 - \tanh(\eta x))^2}\, P_{n,l}^{-3,2}(\tanh(\eta x)). \tag{17}$$

4. Ladder Operators and shape invariance condition
In this section, we are going to factorize the corresponding wave function in terms of two first-order differential equations. These first-order equations so-called the raising and lowering operators [23, 24, 25]. For this purpose, we use from the associated Jocobi differential equation. Then, we will write corresponding operators with respect to quantum numbers in he following forms,

4.1. Ladder Operators in terms of n and l
The corresponding operators are [26],

$$A_{n,l}^+(y)A_{n,l}^-(y)P_{n,l}^{\mu,\nu}(y) = \frac{4(n-l)(\mu+n)(\nu+n)(\mu+\nu+n+l)}{(\mu+\nu+2n)^2}P_{n,l}^{\mu,\nu}(y), \tag{18}$$

$$A_{n,l}^-(y)A_{n,l}^+(y)P_{n-1,l}^{\mu,\nu}(y) = \frac{4(n-l)(\mu+n)(\nu+n)(\mu+\nu+n+l)}{(\mu+\nu+2n)^2}P_{n-1,l}^{\mu,\nu}(y), \tag{19}$$

where

$$A_{n,l}^+(y) = (1-y^2)\frac{d}{dy} - (\mu+\nu+n)y - \frac{(\mu-\nu)(\mu+\nu+n+l)}{\mu+\nu+2n}, \tag{20}$$

$$A_{n,l}^-(y) = -(1-y^2)\frac{d}{dy} - ny + \frac{(\mu-\nu)(n-l)}{\mu+\nu+2n}, \tag{21}$$

by comparing the raising and lowering operators with the Jacobi differential equation (10) and Schrödinger equation, we can obtain the raising and lowering operators in terms of the parameters n and l against position x in the following form,

$$A_{n,l}^{+}(x) = \frac{1}{\eta}\frac{d}{dx} - (\mu + \nu + n)\tanh(\eta x) - \frac{(\mu - \nu)(\mu + \nu + n + l)}{\mu + \nu + 2n}, \tag{22}$$

$$A_{n,l}^{-}(x) = -\frac{1}{\eta}\frac{d}{dx} - n\tanh(\eta x) + \frac{(\mu - \nu)(n - l)}{\mu + \nu + 2n}. \tag{23}$$

By inserting $\mu = -3$ and $\nu = 2$ into the above equation be yielded as,

$$A_{n,l}^{+}(x) = \frac{1}{\eta}\frac{d}{dx} - (n - 1)\tanh(\eta x) + \frac{5(n + l - 1)}{2n - 1}, \tag{24}$$

$$A_{n,l}^{-}(x) = -\frac{1}{\eta}\frac{d}{dx} - n\tanh(\eta x) - \frac{5(n - l)}{2n - 1}. \tag{25}$$

On the one hand, in order to study the shape invariance condition we have to obtain the partner Hamiltonian as multiplying between the raising and lowering operators i.e., $H_{n,l}^{+} = A_{n,l}^{+}A_{n,l}^{-}$ and $H_{n,l}^{-} = A_{n,l}^{-}A_{n,l}^{+}$. On the other hand, the shape invariance condition satisfies by condition $H_{n,l}^{+} - H_{n',l'}^{-} = $ constant (n' and l' are shifted integers n and l). Then we can obtain the corresponding condition as,

$$H_{n,l}^{+} - H_{n+1,l}^{-} = -2n + \frac{50(n^2 + l^2 - l)}{(2n - 1)^2(2n + 1)}, \tag{26}$$

thus we have obtained the shape invariance condition with respect to n and l.

4.2. Ladder Operators with quantum number l
We can factorize the associated Jacobi's differential equation by raising and lowering operators with respect to the parameter l as,

$$A_l^{+}(y)A_l^{-}(y)P_{n,l}^{\mu,\nu}(y) = (n - l + 1)(\mu + \nu + n + l)P_{n,l}^{\mu,\nu}(y), \tag{27}$$

$$A_l^{-}(y)A_l^{+}(y)P_{n,l-1}^{\mu,\nu}(y) = (n - l + 1)(\mu + \nu + n + l)P_{n,l-1}^{\mu,\nu}(y), \tag{28}$$

where

$$A_l^{+}(y) = \sqrt{1 - y^2}\frac{d}{dy} + \frac{(l - 1)y}{\sqrt{1 - y^2}}, \tag{29}$$

$$A_l^{-}(y) = -\sqrt{1 - y^2}\frac{d}{dy} + \frac{(\mu - \nu) + (\mu + \nu + l)y}{\sqrt{1 - y^2}}, \tag{30}$$

On the other hand, by comparing the above Jacobi's differential equation and corresponding Schrödinger equation we will achieve to the following expressions,

$$A_l^{+}(x) = \frac{\cosh(\eta x)}{\eta}\frac{d}{dx} + (l - 1)\sinh(\eta x), \tag{31}$$

$$A_l^{-}(x) = -\frac{\cosh(\eta x)}{\eta}\frac{d}{dx} + (\mu - \nu)\cosh(\eta x) + (\mu + \nu + l)\sinh(\eta x). \tag{32}$$

By substituting $\mu = -3$ and $\nu = 2$ into the corresponding operators, we will have

$$A_l^+(x) = \frac{\cosh(\eta x)}{\eta}\frac{d}{dx} + (l-1)\sinh(\eta x),\tag{33}$$

$$A_l^-(x) = -\frac{\cosh(\eta x)}{\eta}\frac{d}{dx} - 5\cosh(\eta x) + (l-1)\sinh(\eta x).\tag{34}$$

Similar to the above approach, the shape invariance condition obtain with the multiplicity of the raising and lowering operators as the partner Hamiltonian as $H_l^+ = A_l^+ A_l^-$ and $H_l^- = A_l^- A_l^+$. Therefore, we can obtain this condition as,

$$H_l^+ - H_{l+1}^- = 2l - 1,\tag{35}$$

this relation shows us that the shape invariance condition is established with respect to parameters l. Therefore, there is the shape invariance condition for issue of position dependent mass.

5. Conclusion

In this paper, we have discussed on the solution of the Schrödinger equation with position dependent mass by Rosen-Morse potential. In that case we have chosen a position dependent mass as functional of $\tanh(\eta x)$. This choice leads to an analytically solution for wave function and eigenvalues. By using the factorization method and comparing it with associated Jacobi differential equation, we have obtained energy eigenvalue and wave function. The raising and lowering operators have calculated with respect to parameters n and l. By computing the partner Hamiltonian of system, the shape invariance condition has been satisfied with respect to parameters n and l.

[1] W. C. Qiang, Y. Gao, and R. S. Zhou, Cent. Eur. J. Phys. 6, no. 2 (2008): 356-362. 1
[2] J. Lu, Physica Scripta 72, no. 5 (2005): 349. 1
[3] N. Rosen and P. M. Morse, Physical Review 42, no. 2 (1932): 210. 1
[4] R. Sever, C. Tezcan and M. Aktaş, Ö. Yeşiltaş, J. Math. Chem., 43, (2007) 845. 1
[5] Y. Weissman and J. Jortner, Phys. Lett. A 70, (1979) 177. 1
[6] G. Pöschl and E. Teller, Z. Phys. 83, (1933) 143. 1
[7] A. Kratzer, Z. Phys. 3, (1920) 289. 1
[8] H. Fakhri and J. Sadeghi, J., Mod. Phys. Lett. A 19, (2004) 615. 1
[9] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251 (1995) 268. 1
[10] A. R. Amani and H. Ghorbanpour, Acta Physica Polonica-Series B 43.9 (2012): 1795. 1
[11] A. R. Amani, J. Sadeghi and M. Rostami, Proc. of International Conference on Science and Technology, Universiti Teknologi MARA, Pulau Pinang, Malaysia (2008) 1620-1626. 1
[12] O. von Roos, 1993 Phys. Rev. B 27 7547. 2
[13] J. Sá Borges, L. N. Epele, H. Fanchiotti, C. A. Garcia Canal and F. R. A. Simão, Phys. Rev. A 38 (1988)3101. 2
[14] O. Rojo and J. S. Levinger, Phys. Rev. 123 (1961) 2177. 2
[15] M. Razavy, G. Field and J. S. Levinger, Phys. Rev. 125 (1962) 269. 2
[16] A. S. Dutra and C. A. S. Almeida, Phys. Lett. A 275, 25 (2000). 2
[17] A. Arda and R. Sever, Commun. Theor. Phys. 56 (2011) 51. arxiv:[math-ph]1012.1977. 2
[18] T. Gora and F. Williams, Phys. Rev. 177 (1969) 1179. 2
[19] D. J. Ben Danial and C. B. Duke, Phys. Rev. 152 (1966) 683. 2
[20] Q. G. Zhu and H. Kroemer, Phys. Rev. B 27 (1983) 3519. 2
[21] J. Sadeghi, A. Amani and A. Pourdarvish, Applied Sciences-APPS, 8(1) (2006) 146-152. 3
[22] T. Li and K. J. Kuhn, Phys. Rev. B 47 (1993) 12760. 2
[23] M. A. Jafarizadeh and H. Fakhri, Phys. Lett. A, 230, (1997) 164. 4
[24] M. A. Jafarizadeh and H. Fakhri, Annals of Physics 266, no. 1 (1998): 178-206. 4
[25] A. R. Amani and H. Ghorbanpour, Acta Phys. Pol. B 43, no. 9 (2012): 1795. 4
[26] A. R. Amani, M. A. Moghrimoazzen, H. Ghorbanpour, and S. Barzegaran, Afr. J. Math. Phys. 10 (2011): 31-37. 4