

# The homotopic mapping solutions for the generalized Schrödinger equation

Dianhen Lu <sup>1,a</sup>, Xinyu Cui <sup>1,b</sup>, Baojian Hong <sup>1,c</sup>

<sup>1</sup>Nonlinear Scientific Research Center, Faculty of Science, Jiangsu University, Zhenjiang Jiangsu, 212013, P.R.China

E-mail: <sup>a</sup>dclu@ujs.edu.cn, <sup>b</sup>2314200389@qq.com, <sup>c</sup>hbj@njit.edu.cn

**Abstract.** This work is concerned on how to find the approximation solutions of the generalized Schrödinger equation by using the homotopic mapping method. First, we give the basic idea of Homotopic mapping method. In the following, we introduce a homotopic mapping, and make the generalized Schrödinger equation do a homotopic deformation, then we get the approximation solutions in given initial conditions. Finally we give two examples about derivative Schrödinger equation and Hirota equation to illustrate efficiency of this method, and the exact solutions of these equations can be solved in the condition of selecting proper auxiliary function  $v$ .

## 1. Introduction

The investigation of solutions to the nonlinear partial differential equation (NLPDEs) play an important role in natural science, many important nonlinear phenomenon in various field such as fluid dynamic, mathematic biology and plasma physics can be described NLPDEs. Therefore, more and more scholars devote themselves to obtain the exact solutions or approximate solutions of NLPDEs. In recent years, many searching for exact solutions or approximation solutions of NLPDEs methods have also been developed and improved, including the method of homogeneous balanced [1], the method of trial function [2], the method of Adomian decomposition [3-4], the method of perturbation [5], etc. The homotopic mapping method [6-9] is a new method and well performed in searching for analytical approximate solutions of NLPDEs. For example, the variable coefficient KdV equation [10], the two-dimensional KdV-Burgers equation [11], the epidemic model [12], etc.

The object of this study is the generalized Schrödinger equation:

$$iu_t + \alpha u_{xx} + i\beta u_{xxx} + i\gamma(|u|^2 u)_x + \delta |u|^2 u = 0. \quad (1.1)$$

As  $\alpha = -1, \beta = 0, \gamma = 1, \delta = 0$ , Eq.(1.1) is the derivative Schrödinger equation:

$$u_t + iu_{xx} + (|u|^2 u)_x = 0.$$

As  $\alpha = 1, \beta = 2, \gamma = 1, \delta = 1$ , Eq.(1.1) is Hirota equation:

$$iu_t + u_{xx} + 2iu_{xxx} + i(|u|^2 u)_x + |u|^2 u = 0.$$



The generalized Schrödinger equation is a nonlinear equation which is often encountered in modern physics study, it is also a basic equation in researching light pulse transmission problem in nonlinear dispersive optical medium. For this equation, there are already a number of ways to solve the solution, such as the split-step Fourier method [13], trigonometric function expansion method [14], general mapping approach [15] etc. But these methods are just applied to some special forms of this equation. We can get the approximation solution of this equation with initial condition by using the homotopic mapping method. The object of this study is general, the consequence has a significant meaning.

## 2. Basic idea of homotopic mapping method

We consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, r \in \Omega,$$

and boundary conditions

$$B(u, \partial u / \partial n) = 0, r \in \Gamma,$$

where  $A$  is a general differential operator,  $B$  is a boundary operator,  $f(r)$  is a known analytical function,  $\Gamma$  is the boundary of the domain  $\Omega$ , and  $\partial / \partial n$  denoted differential along the normal drawn outwards from  $\Omega$ . The operator  $A$  can be divided into two parts  $L$  and  $N$ . Therefore  $A(u) - f(r) = 0$  can be rewritten as follow:

$$L(u) + N(u) - f(r) = 0$$

Now We construct a homotopic mapping

$$H(u, p) : \Omega \times [0, 1] \rightarrow R,$$

$$H(u, p) = L(u) - L(v) + p[L(v) + N(u) - f(r)], \quad (2.1)$$

where  $p$  is a parameter,  $v$  is an auxiliary function.

According to Eq.(2.1), we have

$$H(u, 0) = L(u) - L(v),$$

$$H(u, 1) = A(u) - f(r),$$

and the changing process of  $p$  from 0 to 1, is just that of  $H(u, p)$  from  $L(u) - L(v)$  to  $A(u) - f(r)$ . In topology, that is called deformation  $L(u) - L(v)$  and  $A(u) - f(r)$  are called homotopic. We can set that the solution of  $H(u, p) = 0$  can be expressed as a series in  $p$  as follow:

$$\tilde{u}(x, t, p) = \sum_{i=0}^{\infty} u_i(x, t) p^i = u_0 + pu_1 + p^2u_2 + \cdots$$

When  $p = 0$ ,  $\tilde{u}(x, t, 0) = u_0(x, t)$  is the solution of  $L(u) - L(v) = 0$ . When  $p \rightarrow 1$ , we can obtain the approximation solution of  $A(u) - f(r) = 0$  as follows

$$\tilde{u}(x, t, 1) = u_0 + u_1 + u_2 + \cdots$$

### 3. The approximation solution of the generalized Schrödinger equation

The Eq(1.1) can be rewritten as the following:

$$u_t - i\alpha u_{xx} + \beta u_{xxx} + \gamma(|u|^2 u)_x - i\delta |u|^2 u = 0. \quad (3.1)$$

In order to obtain the solution of Eq.(3.1), we constructed a homotopic mapping

$$H(u, p) : \Omega \times [0, 1] \rightarrow R, \\ H(u, p) = L(u) - L(v) + p[L(v) + N(u) - f(r)], \quad (3.2)$$

where  $R = (-\infty, +\infty)$ ,  $L(u) = u_t - i\alpha u_{xx} + \beta u_{xxx}$ ,  $v$  is an auxiliary function.

According to Homotopic mapping,  $H(u, 1) = 0$  is equal to Eq.(3.1). Therefore, as  $p \rightarrow 1$ , the solution of  $H(u, p) = 0$  is the solution of Eq.(3.1).

Set

$$\tilde{u}(x, t, p) = \sum_{i=0}^{\infty} u_i(x, t) p^i = u_0 + pu_1 + p^2 u_2 + \cdots \quad (3.3)$$

According to [16], when  $p \in [0, 1]$ , the series (3.3) is uniform convergence.

Let  $u_0 = v$ , substituting Eq.(3.3) in  $H(u, p) = 0$ , we have

$$L\left(\sum_{i=0}^{\infty} u_i p^i\right) - L(v) + p\left[L(v) + \gamma \sum_{i=0}^{\infty} A_i p^i - i\delta \sum_{i=0}^{\infty} B_i p^i\right] = 0, \quad (3.4)$$

where  $|u|^2 u = \sum_{n=0}^{\infty} B_n p^n$ ,  $(|u|^2 u)_x = \sum_{n=0}^{\infty} A_n p^n$ .

Because  $|u|^2 u$  and  $(|u|^2 u)_x$  are complex function, the corresponding polynomials are

$$\begin{aligned} |u|^2 u &= u\bar{u}u = u^2\bar{u} \\ &= (u_0 + u_1 p + u_2 p^2 + \cdots)^2 (\bar{u}_0 + \bar{u}_1 p + \bar{u}_2 p^2 + \cdots) \\ &= (C_0 + C_1 p + C_2 p^2 + C_3 p^3 + \cdots) (\bar{u}_0 + \bar{u}_1 p + \bar{u}_2 p^2 + \cdots) \\ &= B_0 + B_1 p + B_2 p^2 + B_3 p^3 + \cdots, \end{aligned}$$

where

$$C_n = \begin{cases} 2 \sum_{i=0}^k u_i u_{n-i}, & n = 2k + 1, \\ 2 \sum_{i=0}^k u_i u_{n-i} - u_k^2, & n = 2k, \end{cases} \quad k = 0, 1, 2, 3 \cdots \\ B_j = \sum_{n=0}^j C_n \bar{u}_{j-n}, \quad j = 0, 1, 2, 3 \cdots \quad (3.5)$$

According (3.5), we have

$$\begin{aligned} B_0 &= u_0^2 \bar{u}_0, \\ B_1 &= u_0^2 \bar{u}_1 + 2u_0 u_1 \bar{u}_0, \\ B_2 &= u_0^2 \bar{u}_2 + 2u_0 u_1 \bar{u}_1 + 2u_0 u_2 \bar{u}_0 + u_1^2 \bar{u}_0 \\ &\vdots \end{aligned} \quad (3.6)$$

Similarity

$$\begin{aligned} (|u|^2 u)_x &= (u\bar{u}u)_x = (u^2 \bar{u})_x \\ &= [B_0 + B_1 p + B_2 p^2 + B_3 p^3 + \cdots]_x \\ &= A_0 + A_1 p + A_2 p^2 + A_3 p^3 + \cdots, \end{aligned}$$

where

$$A_j = (B_j)_x = \left( \sum_{n=0}^j C_n \bar{u}_{j-n} \right)_x, \quad j = 0, 1, 2, 3 \dots \quad (3.7)$$

From (3.7), we have

$$\begin{aligned} A_0 &= 2u_0 u_{0x} \bar{u}_0 + u_0^2 \bar{u}_{0x}, \\ A_1 &= 2u_0 u_{0x} \bar{u}_1 + u_0^2 \bar{u}_{1x} + 2u_{0x} u_1 \bar{u}_0 + 2u_0 u_{1x} \bar{u}_0 + 2u_0 u_1 \bar{u}_{0x}, \\ A_2 &= 2u_0 u_{0x} \bar{u}_2 + u_0^2 \bar{u}_{2x} + 2u_{0x} u_1 \bar{u}_1 + 2u_0 u_{1x} \bar{u}_1 + 2u_0 u_1 \bar{u}_{1x} + 2u_{0x} u_2 \bar{u}_0 \\ &\quad + 2u_0 u_{2x} \bar{u}_0 + 2u_0 u_2 \bar{u}_{0x} + 2u_1 u_{1x} \bar{u}_0 + u_1^2 \bar{u}_{0x}, \\ &\vdots \end{aligned} \quad (3.8)$$

Arranging the coefficients of  $p$  power in (3.4), we have

$$\begin{aligned} p^0 : L(u_0) &= L(v), \\ p^1 : L(u_1) &= -L(v) - \gamma A_0 + i\delta B_0, \\ p^2 : L(u_2) &= -\gamma A_1 + i\delta B_1, \\ &\vdots \\ p^j : L(u_j) &= -\gamma A_{j-1} + i\delta B_{j-1}, \\ &\vdots \end{aligned} \quad (3.9)$$

We let  $u_0 = v$  and substitute it in (3.9),  $u_j$  can be concluded step by step. So the approximation solution of (1.1) can be expressed as follows:

$$\tilde{u} = u_0 + u_1 + u_2 + \dots$$

#### 4. The approximation solution of derivative Schrödinger equation

Consider the following derivative Schrödinger equation with initial condition:

$$\begin{cases} u_t + iu_{xx} + (|u|^2 u)_x = 0, \\ u(x, 0) = e^{\frac{1}{2}ix}. \end{cases} \quad (4.1)$$

The homotopic deformation of Eq.(4.1) is

$$L(u) - L(v) + p \left[ L(v) + \left( |u|^2 u \right)_x \right] = 0, \quad (4.2)$$

where  $L(u) = u_t + iu_{xx}$ .

Substituting  $u = u_0 + u_1 p + u_2 p^2 + \dots$  in Eq.(4.2), we can obtain

$$L \left( \sum_{i=0}^{\infty} u_i p^i \right) - L(v) + p \left[ L(v) + \sum_{i=0}^{\infty} A_i p^i \right] = 0, \quad (4.3)$$

where

$$\begin{aligned} A_0 &= 2u_0 u_{0x} \bar{u}_0 + u_0^2 \bar{u}_{0x}, \\ A_1 &= 2u_0 u_{0x} \bar{u}_1 + u_0^2 \bar{u}_{1x} + 2u_1 u_{0x} \bar{u}_0 + 2u_0 u_{1x} \bar{u}_0 + 2u_0 u_1 \bar{u}_{0x}, \\ A_2 &= 2u_0 u_{0x} \bar{u}_2 + u_0^2 \bar{u}_{2x} + 2u_1 u_{0x} \bar{u}_1 + 2u_0 u_{1x} \bar{u}_1 + 2u_0 u_1 \bar{u}_{1x} + 2u_2 u_{0x} \bar{u}_0 \\ &\quad + 2u_0 u_{2x} \bar{u}_0 + 2u_0 u_2 \bar{u}_{0x} + 2u_1 u_{1x} \bar{u}_0 + u_1^2 \bar{u}_{0x}, \\ &\vdots \end{aligned} \quad (4.4)$$

Arranging the coefficients of  $p$  power in (4.3), we have

$$\begin{aligned} p^0 : L(u_0) &= L(v), \\ p^1 : L(u_1) &= -L(v) - A_0, \\ p^2 : L(u_2) &= -A_1, \\ &\vdots \\ p^j : L(u_j) &= -A_{j-1} \\ &\vdots \end{aligned}$$

Let  $u_0 = v = e^{\frac{1}{2}i(x-ct)}$ , according (4.4) we have  $A_0 = \frac{i}{2}e^{\frac{1}{2}i(x-ct)}$ , so

$$u_{1t} + iu_{1xx} = -\left(-\frac{1}{2}ci + \frac{1}{4}i\right)e^{\frac{1}{2}i(x-ct)}.$$

If we set  $c = \frac{1}{2}$ , then  $u_1 = 0$ , and  $A_1 = 0, u_2 = 0, A_2 = 0, \dots$ , we can obtain

$$\tilde{u} = u_0 = e^{\frac{1}{2}i(x-\frac{1}{2}t)}.$$

**Remark 1:** It is easy to prove that the above  $\tilde{u}$  is the exact solution of Eq.(4.1).

## 5. The approximation solution of Hirota equation

Considering the following Hirota equation with initial condition:

$$\begin{cases} iu_t + u_{xx} + 2iu_{xxx} + i(|u|^2 u)_x + |u|^2 u = 0, \\ u(x, 0) = e^{ix}. \end{cases} \quad (5.1)$$

The homotopic deformation of Eq.(5.1) is

$$L(u) - L(v) + p \left[ L(v) + \left( |u|^2 u \right)_x - i |u|^2 u \right] = 0, \quad (5.2)$$

where  $L(u) = u_t - iu_{xx} + 2u_{xxx}$ .

Substituting  $u = u_0 + u_1p + u_2p^2 + \dots$  in Eq.(5.2), and arranging the coefficients of  $p$  power, we have

$$\begin{aligned} p^0 : L(u_0) &= L(v), \\ p^1 : L(u_1) &= -L(v) - A_0 + iB_0, \\ p^2 : L(u_2) &= -A_1 + iB_1, \\ &\vdots \\ p^j : L(u_j) &= -A_{j-1} + iB_{j-1}, \\ &\vdots \end{aligned}$$

where

$$\begin{aligned} A_0 &= 2u_0u_{0x}\bar{u}_0 + u_0^2\bar{u}_{0x}, \\ A_1 &= 2u_0u_{0x}\bar{u}_1 + u_0^2\bar{u}_{1x} + 2u_1u_{0x}\bar{u}_0 + 2u_0u_{1x}\bar{u}_0 + 2u_0u_1\bar{u}_{0x}, \\ A_2 &= 2u_0u_{0x}\bar{u}_2 + u_0^2\bar{u}_{2x} + 2u_1u_{0x}\bar{u}_1 + 2u_0u_{1x}\bar{u}_1 + 2u_0u_1\bar{u}_{1x} + 2u_2u_{0x}\bar{u}_0 \\ &\quad + 2u_0u_{2x}\bar{u}_0 + 2u_0u_2\bar{u}_{0x} + 2u_1u_{1x}\bar{u}_0 + u_1^2\bar{u}_{0x}, \\ &\vdots \end{aligned} \quad (5.3)$$

$$\begin{aligned} B_0 &= u_0^2\bar{u}_0, \\ B_1 &= u_0^2\bar{u}_1 + 2u_0u_1\bar{u}_0, \\ B_2 &= u_0^2\bar{u}_2 + 2u_0u_1\bar{u}_1 + 2u_0u_2\bar{u}_0 + u_1^2\bar{u}_0, \\ &\vdots \end{aligned} \quad (5.4)$$

Let  $u_0 = v = e^{i(x-ct)}$ , according (5.3) and (5.4), we know  $A_0 = ie^{i(x-ct)}$ ,  $B_0 = e^{i(x-ct)}$ , So

$$u_{1t} - iu_{1xx} + 2u_{1xxx} = (ci + i) e^{i(x-ct)}.$$

If we set  $c = -1$ , then  $u_1 = 0$ , and  $A_1 = 0$ ,  $B_1 = 0$ ,  $u_2 = 0$ ,  $A_2 = 0$ ,  $B_2 = 0$ ,  $\dots$ , we can obtain

$$\tilde{u} = u_0 = e^{i(x+t)}.$$

**Remark 2:** It is easy to prove that above  $\tilde{u}$  is the exact solution of Eq.(5.1).

## 6. Conclusion

From section 4 and section 5, if we choose proper auxiliary function  $v$ , a great quantity of compute work can be reduced, we even conclude the exact solution of equations. In this paper, we get the approximation solution of the generalized Schrödinger equation by using the homotopic mapping method. Then, two examples of the derivative Schrödinger equation and Hirota equation are given. Through these examples, the homotopic mapping method has advantage of brief and efficient.

## Acknowledgements

The work was supported by the National Nature Science Foundation of China (No. 61070231), and the Out standing Personnel Program in Six Fields of Jiangsu (No.2009188), and the Natural Science Foundation of Jiangsu Province (BK20140525).

## References

- [1] Fan En-gui, Zhang Hong-qing. Homogeneous balance method of nonlinear soliton equations [J]. ACTA PHYSICA SINICA. 47(3) (1998) 353-10.
- [2] Xie Yuan-xi, Tang Jia-shi. Solving a class of nonlinear partial differential equations the exact solution by the simplify trial function method [J]. Journal of Dynamics and Control. 3(1) (2005) 15-18.
- [3] Niu Hong-ling, Hao Ling, Yu Zhi-xian, Yin Jian-hua. The solutions for some fractional Nonlinear Partial Differential Equation via the Adomian Decomposition Method[J]. Journal of Liaoning Technical University (Natural Science). 2013 (2013) 010132-04.
- [4] Jin Yi, Chen Liang. The exact solution for a class of nonlinear Schrödinger equation via the Adomian decomposition Method [J]. Journal of Hangzhou Normal University(Natural Science Edition). 7(7) (2008) 256-260.
- [5] Wang Ai-hua. The Application of Perturbation Method in the Proben about Dynamic Plastic Sheet[J]. Journal of JinZhou Teachers College(Natural Sciences Edition)). 2000 (2000) 020010-04.
- [6] Zhang Ben-gong, Li Shao-yong, Liu Zheng-rong. Homotopy perturbation method for modified Camassa-Holm and Degasperis-Procesi equations[J]. Physics Letters A. 372 (2008) 1867-1872.
- [7] N.H.Sweilam, M.M.Khader. Exact solutions of some coupled nonlinear partial differential equations using the homotopy perturbation method[J]. Computers and Mathematics with Applications. 58 (2009) 213-2141.
- [8] Lin Jin. Application of variational iteration method and homotopy perturbation method to the modified Kawahara equation[J]. Mathematical and Computer Modelling 49 (2009) 573-578
- [9] J.H.He, Application of Homotopy perturbation method to nonlinear equation, Chaos, Solitons and Fractals. 26 (2005) 695-700.
- [10] Lu Dian-chen, Chen Ting-ting, Hong Bao-jian. Homotopic Approximation Solutions for the Perturbed CKdV Equation with Variable Coefficients[J]. The Scientific World Journal. 2014 (2014), Article ID 593642, 5 pages.
- [11] A.Molabahrami, F.Khani, S.Hamedi-Nezhad. Soliton solutions of the two-dimensional KdV-Burgers equation by homotopy perturbation method[J]. Physics Letters A. 370 (2007) 433-436.
- [12] M.Rafei, D.D.Ganji, H.Daniali. Solution of the epidemic model by homotopy perturbation method[J]. Applied Mathematics and Computation. 187 (2007) 1056-1062.
- [13] Zhao Lei, Sui Zhan, Zhu Qi-hua, Zhang Ying, Zuo Yan-lei. Improvement and precision analysis of the split-step Fourier method in solving the general nonlinear Schrödinger equation[J]. ACTA PHYSICA SINICA. 58(07) (2009) 4731 -4737.

- [14] Zhao Ming-zhuo, Liu Xiao-juan. Optical soliton solution to higher order dispersion modified non-linear Schrödinger equation[J]. Journal of Hunan University of Science and Technology(N atural Science Edition). 24(3) (2009) 126-128.
- [15] Fang Jian-ping. Study on the nonlinear [J]. Schrödinger system by a general mapping approach[J]. Journal of Lishui University. 27(5) (2005) 26-32.
- [16] Liao S J. Beyond Perturbation: Introduction to the Homotopy Analysis Method. New York: CRC Press. 2004.